0.1 Some exercises

The successor induction for $\varphi(x, \bar{z})$ is the following formula $I_{\varphi(x)} :=$

$$\forall \bar{z}[(\varphi(0, \bar{z}) \wedge \forall x(\varphi(x, \bar{z}) \Rightarrow \varphi(x+1, \bar{z}))) \Rightarrow \forall x \varphi(x, \bar{z})].$$

The order induction for $\varphi(x, \bar{z})$ is the following formula $O_{\varphi(x)} :=$

$$\forall \bar{z}[\forall x(\forall y < x \varphi(y, \bar{z}) \Rightarrow \varphi(x, \bar{z})) \Rightarrow \forall x \varphi(x, \bar{z})].$$

The least number priciple for $\varphi(x, \bar{z})$ is the formula $L_{\varphi(x)} :=$

$$\forall \bar{z}(\exists x \varphi(x, \bar{z}) \Rightarrow \exists x(\varphi(x, \bar{z}) \wedge \forall y < x \neg \varphi(y, \bar{z}))).$$

Parametr free successor induction is the successor induction but only for formulas which have only one free variable, say x. An axiom instance is of the form

$$[\varphi(0) \wedge \forall x(\varphi(x) \Rightarrow \varphi(x+1))] \Rightarrow \forall x \varphi(x).$$

PA is defined as PA^- with successor induction axioms for all formulas.

1. Show that PA^- together with least number principle for all formulas is equivalent to PA.

2. Show that PA^- together with order induction for all formulas is equivalent to PA.

3. Exhibit a model A (maybe not for PA or even PA^-, maybe in an other vocabulary but with a linear discrete ordering) such that A satisfies order induction for all formulas in the language but do not satisfies usual successor induction.

4. Is the same possible with A satisfying successor induction but not satisfying order induction?

5. How exercises two and three fit together?

6. (*?) Show that PA is equivalent to PA^- together with the parameter free induction for all formulas.

7. Let $A \models \mathrm{PA}^-$. A cut I of a model A is a subset of A such that it is closed downward and closed on the successor function. Thus, a cut is not necessarily a submodel of A. Show that each nonstandard model of PA has continuum many different cuts.

8. Let $T = \mathrm{Th}((\omega, S, 0, 1))$, where S is the successor function. How many nonisomorphic countable models for T are there? Describe their isomorphic types.

9. Prove in PA that $\forall x \exists y (x = 2y \vee x = 2y + 1)$. Prove in PA that for all a, b, c with $a \neq 0$ there are unique m, r with $r < a$ such that $b = ma+r$.

   That is: do not write a formal proof (unless you want) but simply check what induction axioms are needed and in which way.