Some PA– exercises

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1. Show that

   - $\mathrm{PA}^- \vdash \forall x \forall y (x < y \Rightarrow x + 1 \leq y)$,
   - $\mathrm{PA}^- \vdash \forall x \exists y (y + 1 = x)$,
   - for all $n \in \omega$, $\mathrm{PA}^- \vdash \forall x (x \leq \underline{n} \iff \bigvee_{i \leq n} x = \underline{i})$.

2. Define ordering on $\mathbb{Z}[X]$, the set of polynomials with coefficients from $\mathbb{Z}$ such that the set of polynomials greater of equal the zero polynomial satisfies PA^-.

   Show that $\mathrm{PA}^- \not\vdash \forall x \exists y (2y = x \vee 2y + 1 = x)$.

3. If we would define this ordering on $\mathbb{N}[X]$ instead of $\mathbb{Z}[X]$, would $\mathbb{N}[X] \models \mathrm{PA}^-$?

Let $T = \mathrm{Th}(\mathcal{N})$.

Definition 1 *An element $a \in M \models \mathrm{PA}$ codes a set*

$$X = \left\{ i \in \omega : M \models \underline{p_i} | x[a/x] \right\},$$

where p_i is the i-th prime number. We denote this set by $c_M(a)$.

If, for $X \subseteq \omega$, there exists $a \in M$ such that $X = c_M(a)$ we say that X is coded in M.

4. Show that each set $X \subseteq \omega$ is coded is some countable model of T. Conclude that there are continuum many nonisomorphic countable models of T.

5. Show that all subsets of ω are coded in $\prod_{i\in\omega}\mathcal{N}_{/\mathcal{U}}$. What is the cardinality of $\prod_{i\in\omega}\mathcal{N}_{/\mathcal{U}}$?

6. Describe all, up to isomorphism, models of $\mathrm{Th}((\omega, S, 0, 1))$, where S is the successor function.

   You may need to use Erenfeuch–Fraïsse games or elimination of quantifiers for $\mathrm{Th}((\omega, S, 0, 1))$.

7. What about models of $\mathrm{Th}((\omega, \leq, 0, 1))$?

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Definition 2 *Let $M \models \mathrm{PA}^-$. $\emptyset \neq I \subsetneq_e M$ is a cut if it is closed downward and closed on successor.*

1. Show that for $M \models \mathrm{PA}^-$, M satisfies overspill for all cuts if and only if $M \models \mathrm{PA}$.

2. (*) Show that for each model $M \models \mathrm{PA}^-$ there is a model N such that $M \prec N$ and N satisfies overspill for $\mathbb{N}$. Conlude that overspill only for $\mathbb{N}$ is weaker than induction.

3. Show that for each nonstandard $M \models \mathrm{PA}$ there is continuum many cuts of M. (It suffices to restrict only to countable models. Then, use the characterization of the order type of countable $M \models \mathrm{PA}$ as $\omega + (\omega^* + \omega)\eta$, where η is the order type of rationals.).

4. Repeat the above but with cuts closed on addition and multiplication.

5. Show that $I\Sigma_n$, $L\Sigma_n$, $I\Pi_n$, $L\Pi_n$ are equivalent for all n.

Definition 3 *A model $M \models \mathrm{PA}^-$ is ω_1–like if for all $a \in |M|$,*

$$\mathrm{card}(\{b \in |M| : M \models b \leq a\}) = \omega_0 \textit{ and } \mathrm{card}(|M|) = \omega_1.$$

6. Show that if $M \models \mathrm{PA}^-$ is ω_1–like then $M \models \mathrm{Coll}_n$, for all $n \in \mathbb{N}$.

7. (*) Show that for each model $M \models \mathrm{PA}^-$ there is N such that $\mathrm{card}(N) = \mathrm{card}(M)$ and $M \subseteq_e N$. (Use the characterization of a model M of PA^- as a positive part of a discretly ordered ring R_M. Then, consider a positive part of a ring of polynomials $R_M[X]$.)

8. (*) Show that for each countable model $M \models \mathrm{PA}^-$ there is ω_1–like $N \models \mathrm{PA}^-$ such that $M \subseteq_e N$.

9. (*) Show that $\mathrm{PA}^- \cup \bigcup_{n\in\mathbb{N}} \mathrm{Coll}_n$ does not prove $\forall x \exists y (2y = x \vee 2y + 1 = x)$ (use the fact that PA^- does not prove this).

10. Show that $I\Sigma_n \vdash B\Sigma_n$.

11. Show that $B\Sigma_{n+1} \equiv B\Pi_n$.

12. Show that $B\Sigma_{n+1} \vdash I\Sigma_n$. Also show that $\Sigma_n(B\Sigma_n)$ is closed on bounded quantification (all it was done during the lecture so there is no need to do this).

13. (*) Show that induction for parameter free formulas is equivalent to PA (Note. It is not the case that induction for parameter free Σ_n formulas is equivalent to $I\Sigma_n$.)