1 Lecture 03/10/2006

1.1 Some model theoretical background

Let $\sigma = (\{P_i\}_{i\in I}, \{f_j\}_{j\in J}, \{c_k\}_{k\in K}, \mathrm{ar})$ be a vocabulary (signature).[1] Let $A = (U_A, \{R_i^A\}_{i\in I}, \{F_j^A\}_{j\in J}, \{a_k^A\}_{k\in K})$, $B = (U_B, \{R_i^B\}_{i\in I}, \{F_j^B\}_{j\in J}, \{a_k^B\}_{k\in K})$ be two models of vocabulary σ.

Often we will use the same symbol for a model and its universe.

Definition 1 1. *A is a submodel of B (A jest podmodelem B), $A \subseteq B$, if $U_A \subseteq U_B$ and each function and relation from A is a restriction of the corresponding function and relation from B and for each $k \in K$, $a_k^A = a_k^B$.*

2. *B is an elementary extension of A (B jest elementarnym rozszerzeniem A), $A \prec B$, if $A \subseteq B$ and for each formula $\varphi(x_1, \ldots, x_n)$ and for each $a_1, \ldots, a_n \in U_A$ it holds that*

$$A \models \varphi[a_1, \ldots, a_n] \iff B \models \varphi[a_1, \ldots, a_n].$$

Definition 2 *A function $g\colon U_A \longrightarrow U_B$ is an embedding (włożeniem) of A into B if f is one-to-one and for each $i \in I$, for each $a_1, \ldots, a_{ar(R_i)} \in U_A$,*

$$(a_1, \ldots, a_{ar(R_i)}) \in R_i^A \iff (g(a_1), \ldots, g(a_{ar(R_i)})) \in R_i^B$$

and for each $j \in J$, for each $a_1, \ldots, a_{ar(f_j)} \in U_A$,

$$g(F_j^A(a_1, \ldots, a_{ar(f_j)})) = F_j^B(g(a_i), \ldots, g(a_{ar(f_j)}))$$

and for each $k \in K$, $g(a_k^A) = a_k^B$.

A function g is an isomorphism if, additionally, g is onto. If there exists an isomorphism between A and B we say that A and B are isomorphic and denote it by $A \cong B$.

Often we will use the same symbol for a model A and for its universe.

Theorem 3 *Let $\{A_i\}_{\xi<\lambda}$ be such that for all $\xi < \lambda$, $A_\xi \prec A_{\xi+1}$. Then for all $\xi < \lambda$, $A_i \prec \bigcup_{\xi<\lambda} A_i$.*

[1] ar is an arity function. Sets of predicates, functions symbols and constants should be distinct.

1.2 Ultraproducts

Let $\{A_i\}_{i\in\omega}$ be models of the same vocabulary. We define

$$\prod_{i\in\omega} A_i = \left\{ f\colon \omega \longrightarrow \bigcup_{i\in\omega} A_i : \forall i\in\omega\, f(i)\in A_i \right\}.$$

Let $\mathcal{F}$ be an ultrafilter on ω. For $f,g\in\prod_{i\in\omega} A_i$, we define $f\approx g$ if $\{i\in\omega : f(i)=g(i)\}\in\mathcal{F}$. Then define

$$\prod_{i\in\omega} A_{i/\mathcal{F}} = \left\{ [f] : f\in \prod_{i\in\omega} A_i \right\},$$

where $[f]$ is the equivalence class of f under $\approx$. and we define relations of $\prod_{i\in\omega} A_{i/\mathcal{F}}$ as

$$([f_1],\ldots,[f_{\mathrm{ar}(R_j)}]) \in R_j^{\prod_{i\in\omega} A_{i/\mathcal{F}}} \iff \left\{ i\in\omega : (f_1(i),\ldots,f_{\mathrm{ar}(R_j)}) \in R_j^{A_i} \right\}$$

and functions as

$$F_j^{\prod_{i\in\omega} A_{i/\mathcal{F}}}([f_1],\ldots,[f_{\mathrm{ar}(f_j)}]) = [G],$$

where $G(i) = F_j^{A_i}(f_1(i),\ldots,f_{\mathrm{ar}(f_j)}(i))$. Similarly, a constants are

$$a_k^{\prod_{i\in\omega} A_{i/\mathcal{F}}} = [G],$$

where $G(i)=a_k^{A_i}$.

Theorem 4 (Łoś, in polish twierdzenie Łosia) *For each* $\varphi(x_1,\ldots,x_n)$ *and for each* $f_1,\ldots,f_n\in\prod_{i\in\omega}$*,*

$$\prod_{i\in\omega} A_{i/\mathcal{F}} \models \varphi[[f_1],\ldots,[f_n]] \iff \{i\in\omega : A_i\models\varphi[f_1(i),\ldots,f_n(i)]\}\in\mathcal{F}.$$

1.3 Exercises

1. Give examples of models A and B, with $A\subseteq B$, such that $A\prec B$ and not $A\cong B$ and such that $A\cong B$ and not $A\prec B$
2. Show that the definition of $\prod_{i\in\omega} A_{i/\mathcal{F}}$ is correct and prove Łoś theorem.
3. Prove the semantic version of compactness theorem using Łoś theorem.

1.4 The standard model

Definition 5 *The standard model for arithmetic is the model $\mathcal{N}$ of the form*

$$(\omega, +, \times, <, 0, 1),$$

where ω, the set of natural numbers, is the least inductive set and $+, \times$ are usual arithmetical operations, $<$ is the ordering relation and 0 and 1 are two distinguished elements.

We also define $x \leq y$ as $x < y \vee x = y$.

Often we will make no difference between predicates in our arithmetical language and relations from a given model.

Definition 6 *Numerals For each $n \in \omega$ we define a term which is a name for it. $\underline{0} = 0$ and $\underline{1} = 1$. Then, for $n > 0$, let $\underline{n+1} = \underline{n} + 1$.*

Let us observe that we used here ambigous notation. In $\underline{0} = 0$ the first 0 is natural number and the constant from our vocabulary.

1.5 Are there other than $\mathcal{N}$ models for $\mathrm{Th}(\mathcal{N})$?

Fact 7 *There is a model $M \models T$ such that $M \ncong \mathcal{N}$.*

Proof. Let c be a new constant. Then let S be a theory

$$T \cup \{c > \underline{n} : n \in \omega\}.$$

Then, S is a consistent theory. Indeed, if S_0 is a finite subset of S then take $N = \max\{r : c > \underline{r} \in S\}$. Because S_0 is finite, N is well defined. Then, take a model

$$M = (\omega, +, \times, <, 0, 1, N+1),$$

where $N+1$ is the interpretation of c. It holds that

$$M \models T \cup S_0$$

because $N+1$ is greater than value of any term in axioms of S_0.

Thus, by compactness we deduce that S is a consistent theory and has a model. Let $B = (U, +_B, \times_B, <_B, 0_B, 1_b, c_B)$ be a model for S. It holds that $B' = (U, +_B, \times_B, <_B, 0_B, 1_b)$ is a model for T (because B is) and that $B' \ncong \mathcal{N}$.

Now, we prove the second claim. Assume, for the sake of contradiction, that $f\colon \omega \longrightarrow U$ is a isomorphism between $\mathcal{N}$ and B'. Than, one can prove by induction that for each $n \in \omega$,

$$f(n) = \underline{n}^{B'},$$

where $\underline{n}^{B'}$ is the value of a closed term $\underline{n}$ in B'. But a formula $\forall x(x > \underline{n} \Rightarrow x \neq n)$ is in T and therefore holds in B'. It follows that c_B is not in a range of f. Since f is not onto it cannot be an isomorphism. $\square$

1.6 A weak theory PA^-

Now, we present axioms for non-induction part of PA. They are universal closures of the following sentences.

Ax1 $(x + y) + z = x + (y + z)$,

Ax2 $x + y = y + x$,

Ax3 $(xy)z = x(yz)$,

Ax4 $xy = yx$,

Ax5 $(x + y)z = xz + yz$,

Ax6 $x + 0 = x$,

Ax7 $x1 = x$,

Ax8 $(x < y \wedge y < z) \Rightarrow x < z$,

Ax9 $\neg x < x$,

Ax10 $x < y \vee x = y \vee y < x$,

Ax11 $x < y \Rightarrow x + z < y + z$,

Ax12 $(0 < z \wedge x < y) \Rightarrow xz < yz$,

Ax13 $x < y \Rightarrow \exists z\, x + z = y$,

Ax14 $0 < 1 \wedge (0 < x \Rightarrow 1 \leq x)$,

Ax15 $0 \leq x$.

Definition 8 *Let $A, B \models \mathrm{PA}^-$ and $A \subseteq B$. Then B is an end extension (rozszerzeniem końcowym) of A (A is an initial segment (odcinkiem pocz atkowym) of B), $A \subseteq_e B$, if for all $a \in A$,$b \in B$ if $B \models b < a$, then $b \in A$.*

Theorem 9 *If $A \models \mathrm{PA}^-$ then there is a $B \subseteq_e A$ such that $C \cong \mathcal{N}$.*

Proof. Hint. Define an embedding f of $\mathcal{N}$ into A as follows. For each $n \in \omega$,

$$f(n) = \underline{n}^A.$$

Then it suffices to prove (by induction on k) the following: for all $n, l, k \in \omega$

1. $n \neq k \Rightarrow \mathrm{PA}^- \vdash \neg(\underline{n} = \underline{k})$,
2. $n + k = l \Rightarrow \mathrm{PA}^- \vdash \underline{n} + \underline{k} = \underline{l}$,
3. $nk = l \Rightarrow \mathrm{PA}^- \vdash \underline{nk} = \underline{l}$,
4. $n < k \Rightarrow \mathrm{PA}^- \vdash \underline{n} < \underline{k} \wedge \mathrm{PA}^- \vdash \neg(\underline{k} < \underline{n})$,
5. $\mathrm{PA}^- \vdash \forall x(x < \underline{k} \Rightarrow (x = \underline{0} \vee \ldots x = \underline{k-1}))$.

The first four points show that f is a good defined embedding. The last one, that f is onto an initial segment of A. $\square$

Let us note that for any model C such that $C \cong \mathcal{N}$ the isomorphism function is unique. Thus, we may always safely assume that any model of PA^- has as an initial segment $\mathcal{N}$.

1.7 Exercises

1. Show that the following is provable in PA^-:
   - $\forall x \forall y(x + y = x \Rightarrow y = 0)$,
   - $\forall x \forall y(x + y = 0 \Rightarrow x = 0)$,
   - $\forall x \forall y(x < y \Rightarrow x + 1 \leq y)$.
2. Show that the following is not provable in PA^-:

   $$\forall x \exists y(2y = x \vee 2y + 1 = x).$$

   Hint. Consider a set of polynomials with one variable and $\mathbb{Z}$ as coefficients, $\mathbb{Z}[X]$. Then take a model with the universe

   $$\left\{a_n X^n + a_{n-1} X^{n-1} + \ldots + a_0 \in \mathbb{Z}[X] : a_n > 0\right\} \cup 0.$$

3. Let $\mathcal{U}$ be an nonprincipial ultrafilter on ω. Let $\mathcal{N}_i = \mathcal{N}$. Show that $\prod_{i\in\omega} \mathcal{N}_{i/\mathcal{U}}$ is a nonstandard model for $\mathrm{Th}(\mathcal{N})$.