1 Lecture 10/10/2006

1.1 Adding new functions symbols and predicates – still logic

We write $\exists! y$ for a quantifer "there exists exactly one". It can be defined in the first order logic as follows: $\exists! y \varphi(y)$ is equivalent to

$$\exists y \varphi(y) \wedge \forall x \forall y ((\varphi(x) \wedge \varphi(y)) \Rightarrow x = y).$$

Let σ be a vocabulary and let P and f be a new predicate and a function symbol, respectively. Let T be a theory in the language of the vocabulary σ and let $T \vdash \forall \bar{x} \exists! y \varphi_f(\bar{x}, y)$. Then, we may extend T to T^* by two new axioms:

$$\forall \bar{x} \forall y (f(\bar{x}) = y \iff \varphi_f(\bar{x}, y))$$

and

$$\forall \bar{z} (P(\bar{z}) \iff \varphi_P(\bar{z}).$$

The relation between T and T^* can be subsume by the following exercises.

Exercises.

1. Show that for any $\varphi \in \mathcal{F}_\sigma$, a formula in the old language,

$$T \vdash \varphi \text{ if and only if } T^* \vdash \varphi.$$

2. Show that there is a (recursive) translation tr such that for any formula $\varphi \in \mathcal{F}_{\sigma \cup \{P, f\}}$,

$$T^* \vdash (\varphi \iff \mathrm{tr}(\varphi)).$$

3. In this exercise we treat Σ_n as classes of logical hierchy of formulas, not arithmetical ones.

   Show that if φ_f is a Σ_1 formula then if vp is in Σ_n (Π_n) prenex normal form then $\mathrm{tr}(\varphi)$ is also Σ_n (Π_n).

   Hint. Use the fact that the following are equivalent even in T:

$$\exists y (\varphi_f(\bar{x}, y) \wedge \psi(y)) \iff \forall y (\varphi_f(\bar{x}, y) \Rightarrow \psi(y)).$$

4. Does an analogue of the above hold for Σ_n as the arithmetical hierarchy of formulas?

1.2 Hierarchy of arithmetical formulas and relations

A quantifier in a formula φ is bounded if it is of the form $Qx \le t$ for $Q \in \{\forall, \exists\}$, where x does not appear in t. A bounded quantifier $\exists x \le t\psi$ is readed as $\exists x(x \le t \wedge \psi)$ and $\forall x \le t\psi$ as $\forall x(x \le t \Rightarrow \psi)$.

A formula φ is bounded if it has only bounded quantifiers. A class of bounded formulas is called Δ_0 or Σ_0, Π_0.

A formula φ is Σ_{n+1} (Π_{n+1}) if is of the form $\exists z_1 \ldots \exists z_k \psi$ ($\forall z_1 \ldots \forall z_k \psi$), where ψ is a Π_n (Σ_n) formula.

A relation $R \subseteq \omega^r$ is Σ_n (Π_n) if is definable in $\mathcal{N}$ by a Σ_n (Π_n) formula.

A relation R is Δ_n if it is both Σ_n and Π_n. Let us note that the last concept is semantical. However, we will say also about the formula φ that it is Δ_n in a model A (in a theory T) if it is equivalent in A (in T) to a Σ_n and to a Π_n formula.

Exercises.

1. Show that de Morgan laws hold for bounded quantifiers: $\neg\exists x \le t\psi \iff \forall x \le t\neg\psi$ and $\neg\forall x \le t\psi \iff \exists x \le t\neg\psi$. This exercise is intended mainly for pointing this fact.

1.3 What can PA$^-$ prove?

Fact 1 *Let $A, B \models \mathrm{PA}^-$ and let $A \subseteq_e B$. Then, $A \prec_{\Delta_0} B$.*

Proof. Let $A, B \models \mathrm{PA}^-$ and let $A \subseteq_e B$. We use Tarski–Vaught criterion. Let $\varphi(x, \bar{x})$ be a bounded formula, $\bar{a} \in A$ and let $B \models \exists x \le t\varphi(x, \bar{a})$. Then, there is $b \in B$ such that $B \models b \le t(\bar{a})$ and $B \models \varphi(b, \bar{a})$. Since A is a submodel of B, $t^B(\bar{a}) = t^A(\bar{a}$. Thus, $t^B(\bar{a}) \in A$ and, by $A \subseteq_e B$, $b \in A$. So, we have proved Tarski–Vaught criterion. $\Box$

Lemma 2 *Let $A, B \models \mathrm{PA}^-$ and let $A \subseteq_e B$. For any $\bar{a} \in A$ and $\varphi \in \Sigma_1$, if $A \models \varphi[\bar{a}]$, then $B \models \varphi[\bar{a}]$.*

Proof. Easily follows by fact 1. $\Box$

Now, we can estimate the strength of PA$^-$.

Proposition 3 PA$^-$ *proves all Σ_1 sentences true in $\mathcal{N}$.*

Proof. Assume that $\varphi \in \Sigma_1$ and $\mathcal{N} \models \varphi$. We show that any model $A \models \mathrm{PA}^-$ satisfies φ. Indeed, since $\mathcal{N} \subseteq_e A$, by lemma 2, $A \models \varphi$. By completeness theorem PA^- proves φ. $\square$

As we shall see PA^-, and in fact any resonable theory, falls shortly with proving all true Π_1 sentences.[1]

1.4 Scheme of induction

Definition 4 *By a successor induction for a formula $\varphi(x, \bar{z})$ we mean a formula $I_{\varphi(x,\bar{z})}$ of the form*

$$\forall \bar{z}\{[\varphi(0, \bar{z}) \wedge \forall x(\varphi(x, \bar{z}) \Rightarrow \varphi(x+1, \bar{z})] \Rightarrow \forall x \varphi(x, \bar{z})\}.$$

For a set of formulas $\mathcal{F}$ by $I\mathcal{F}$ we mean a theory consisting of induction axioms for formulas in $\mathcal{F}$.

To be super–formal in the definition of I_φ we should also include the information which variable is the main variable for an induction axiom and which are parameters. This should be always clear from the context.

In particular we will be iterested in theories like $I\Delta_0$, $I\Sigma_n$ and, of course, in Peano arithmetic, PA which is PA^- with all iduction axioms. (We assume that all considered theories include PA^- (unless stated otherwise).)

The next result often is usefull, e.g. in Parikh theorem.

Theorem 5 *$I\Delta_0$ can be axiomatized by Π_1 sentences.*

Proof. There is no problem in axiomatizing in Π_1 fashion PA^-. Thus we show that $I\Delta_0$ is equivalent to a theory, say $I\Delta_0^*$, consisting of PA^- and axioms of the form

$$\forall \bar{z}\{\forall x[\varphi(0, \bar{z}) \wedge \forall y \leq x(\varphi(y, \bar{z}) \Rightarrow \varphi(y+1, \bar{z}))] \Rightarrow \varphi(x, \bar{z})\},$$

where φ is a Δ_0 formula.

For simplicity we skip parameters. Firstly, let us assume $I\Delta_0$.

Now, let a be arbitrary and we assume the premisses of our new induction axiom: $\varphi(0)$ and $\forall y \leq a(\varphi(y) \Rightarrow \varphi(y+1))$. We want to infer that $\varphi(a)$. If $a = 0$ then we are done. Otherwise, take $I_{\psi(x)}$, for $\psi(x) := \varphi(x) \vee x > a$, where a is a parameter. Let us observe that $\psi(0)$ holds since $\varphi(0)$ holds. Now, let us assume that $\psi(x)$ holds. Then, either $x \leq a$ or $x > a$. In the latter case $x+1 > a$ holds. Thus, $\psi(x+1)$ holds too. In the former one, $\varphi(x)$ holds and $x \leq a$ so we can use our assumption that $\forall y \leq a(\varphi(y) \Rightarrow \varphi(y+1))$. So, from $\varphi(x)$ we infer that $\varphi(x+1)$ and, finally, that $\psi(x+1)$. Using $I_{\psi(x)}$

[1]Is "to fall shortly" an english expression?

we conclude that $\forall x \psi(x)$. But $\psi(a)$ is just equivalent to $\varphi(a)$. Since a was arbitrary, we have proved that

$$\forall x([\varphi(0) \wedge \forall y \leq x(\varphi(y) \Rightarrow \varphi(y+1))] \Rightarrow \varphi(x)).$$

For the other direction, let us assume $I\Delta_0^*$ and that $\varphi(0)$ and $\forall x(\varphi(x) \Rightarrow \varphi(x+1))$. We want to infer that $\forall x \varphi(x)$. Thus, take an arbitrary a. It holds that $\varphi(0)$ and $\forall x \leq a(\varphi(x) \Rightarrow \varphi(x+1))$. Thus, by an axiom of $I\Delta_0^*$ we can infer that $\varphi(a)$ what ends the proof. $\square$

1.5 Parikh theorem

Exercises.

1. Show that the order type of any countable nonstandard model of PA is isomorphic to $\omega + (\omega^* + \omega)\eta$, where α^* is the ordering which is reversed α and η is an order type of rationals.

   Hint. Use the fact, that there is only one, up to isomorphism, countable dense linear order without endpoints (the order type of $\mathbb{Q}$).