3 Lecture 17/10/2006

3.1 What are nonstandard models for?

[3] We can see arithmetic from two points. From the first one, we are mainly interested in the theory of the standard model, $\mathrm{Th}(\mathcal{N})$. In the second aproach, we are mainly interested in what can we prove about natural numbers and how. Indeed, Goedel incompleteness theorems and Tarski theorem on undefinability of truth shows that we cannot prove all sentences in $\mathrm{Th}(\mathcal{N})$ within a reasonable theory. Thus we need some aproximations like PA, which seems to the most natural one.

But now, what we investigate here are not so much natural numbers but rather first order theories about them. In other parts of mathematics one investigates groups, topological spaces and others. Our subject of mathematical studies are mathematical theories themselves. And we do this in a mathematical way. As we know no first order theory can characterize $\mathcal{N}$ up to isomorphism. Thus, if we consider arithmetic as a theory axiomatized in a first order language, it is hard to escape from nonstandard models. They constitute a very nice (and interesting itself) tool for investigating theories of natural numbers.

On the other hand, one can look at nonstandard models for PA just as a part of model theory. But extremely well structured models, since they satify PA. Sometimes it makes some things easier, often it makes them more interesting.

Finally, besides the other things, nonstandard models just ... are. They are just possible models of the world of arithmetic and we know that we cannot, using first order language, distinguish these models from the standard one.

3.2 Parikh theorem

The theorem by Parikh (in fact one of theorems named "Parikh theorem") characterizes functions which can be shown total in a Π_1 axiomatizable theory. It shows that they cannot grow too fast. In particular, any Σ_1 definable funtcion which can be proved total in $I\Delta_0$ has a growth rate bounded by polynomial.

Definition 15 *A theory T has a term domination property if for any set of terms $t_1, \ldots, t_n$ there is a term t such that for any $i \leq n$, $T \vdash \forall \bar{x} t_i \leq t$.*

[3]This is intended to be a very brief explanations. The question deserves much carefull treatment.

Let us observe that $I\Delta_0$ has the above property (and of course any $T \supseteq I\Delta_0$).

Theorem 16 (Parikh) *Let T be a Π_1 axiomatizable theory containing* PA^- *with term domination property. Let $\varphi(\bar{x}, y)$ be a Σ_1 formula such that $T \vdash \forall\bar{x}\exists y\varphi(\bar{x}, y)$. Then, there exists a term t such that $T \vdash \forall\bar{x}\exists y \leq t(\bar{x})\varphi(\bar{x}, y)$.*

Proof. For simplicity we assume that $\varphi(\bar{x}, y)$ is Δ_0 and that $\bar{x}$ is in fact a one variable x. The general case easily follows.

Let us assume that $T \vdash \forall x\exists y\varphi(x, y)$ and, for the sake of contradiction, that for all terms t $T + \exists x\forall y \leq t(x)\neg\varphi(x, y)$ is consistent. Thus, let us add a new constant c to the language and consider a theory $T^* :=$

$$T \cup \{\forall y \leq t(c)\neg\varphi(c, y) : t(x) \text{ is a term of } T\}.$$

By compactness T^* is consistent. Indeed, for any finite $T_0 \subseteq_{\mathsf{fin}} T^*$, containing k axioms of the form

$$\forall y \leq t_i(c)\neg\varphi(c, y),$$

for $i < k$, there is one fixed term t such that

$$T + \forall y \leq t(c)\neg\varphi(c, y) \vdash T_0.$$

Here we used term domination propoerty. But, by our assumption $T + \forall y \leq t(c)\neg\varphi(c, y)$ is consistent, so T_0 is consistent too.

Now, take a model $(A, a) \models T^*$ where a is an interpretation of c. Next, take a submodel B of A with the universe

$$U = \{b \in A : (A, a) \models b \leq t(c)\}.$$

It holds that $B \subseteq_e A$. Thus, $B \prec_{\Delta_0} A$ and, since Π_1 sentences are preserved downward, $B \models T^*$. But also, for any $d \in B$, $B \models \neg\varphi(a, d)$. Indeed, for any $d \in B$ there is a term $t(x)$ such that $B \models d \leq t(a)$ and $B \models \forall y \leq t(a)\neg\varphi(a, y)$. It follows that $B \models \forall y\neg\varphi(a, y)$. But this is contradictory with the provability in T of $\forall x\exists y\varphi(x, y)$. $\square$

Let us note that Parikh theorem is an upper bound. In particular there are some functions with growth rate bounded by polynomial which are not provably total in $I\Delta_0$. And for some such functions we do not know wheter we can prove totality of them in $I\Delta_0$. E.g. let $\mathrm{Prime}(x)$ be a formula $\forall z \leq x\forall w \leq xw(wz = x \Rightarrow (w = 1 \vee z = 1))$. Then, $\mathcal{N} \models \forall x\exists y \leq 2x(y \leq x \wedge \mathrm{Prime}(y)$ but we do not even know whether $I\Delta_0$ can prove that there are infinitely many primes, that is $\forall x\exists y(y \leq x \wedge \mathrm{Prime}(y)$.

Exercises.

1. Construct a function which is total, Σ_1 definable and bounded by a polynomial and for which $I\Delta_0$ does not prove that it is total.

3.3 Overspill

Now, we present a simple principle which is a kind of model counterpart of induction axioms.

Definition 17 *Let $A \models \mathrm{PA}^-$. A cut I of a model A is a subset of A which is closed downward and closed on the successor function.*

For an element $b \in A$ we write $b > I$ to indicate that for all $a \in I$, $A \models b > a$.

Lemma 18 (Overspill, lemat of przelewaniu) *Let $A \models \mathrm{PA}^-$, let $I \subsetneq A$ be a proper cut of A, let $\bar{a} \in A$ and let $\varphi(x, \bar{x}) \in \mathcal{F}_{\mathrm{PA}}$.*

1. *If for all $b \in I$, $A \models \varphi(b, \bar{a})$, then there is $c > I$ such that $A \models \forall x \leq c\varphi(x, \bar{a})$.*

2. *If for all $b \in I$ there is $d \in I$ such that $A \models d \geq b \wedge \varphi(d, \bar{a})$, then there is $c > I$ such that $A \models \varphi(c, \bar{a})$.*

Proof. We prove only the first part. Let us assume that for all $b \in I$, $A \models \varphi(b, \bar{a})$ and, for the sake of contradiction, that for all $c > I$ there is $d > I$ such that $A \models d \leq c \wedge \neg\varphi(x, \bar{a})$.

Then, we can define a cut I with a formula $\psi(x, \bar{a})$ of the form

$$\forall y \leq x\varphi(x, \bar{a}).$$

But then, induction fails for $\psi(x)$. $\square$

Overspill applied to $I = \mathbb{N}$ allows often to infer an existence of an element which has some "good" properties with respect to all standard elements. This element can e.g. code some fixed subset of $\mathbb{N}$ or be a nonstandard argument on which a given function F is defined even though we can only prove in T that F is defined on standard arguments.

Exercises.

1. Let $\varphi(x, \bar{y})$ be Δ_0, $\bar{k} \in \omega$ and let $A \models \mathrm{PA}$, A nonstandard. If for all $n \in \omega$, $A \models \varphi(n, \bar{k})$, then there is $b > \mathbb{N}$ such that $A \models \forall x \leq b\varphi(x, \bar{k})$.

2. Show that if a model $A \models \mathrm{PA}^-$ satisfies overspill for arbitrary cuts and formulas then $A \models \mathrm{PA}$.

3. (*) Show that any $A \models \mathrm{PA}^-$ has an elementary extension B such that B satifies overspill for $\mathbb{N}$. Deduce that overspill for $\mathbb{N}$ is weaker than induction scheme.

Hint. Show that for any $A \models \mathrm{PA}^-$, $\bar{a} \in A$ and $\varphi(x, \bar{z})$ there is B such that B satisfies overspill over $\mathbb{N}$ for $\varphi(x, \bar{z})$ where $\bar{a}$ are parameters substituted for $\bar{z}$. Then, iterate the construction to obtain an elementary chain of models $\{A_\xi\}_{\xi<\kappa}$ such that $A_0 = A$ and such that $\bigcup_{\xi<\kappa} A_\xi$ satisfies overspill over $\mathbb{N}$.

4. **Lemma 19 (Underspill, lemat o wlewaniu)** *Let $A \models \mathrm{PA}^-$, let $I \subsetneq A$ be a proper cut of A, let $\bar{a} \in A$ and let $\varphi(x, \bar{x}) \in \mathcal{F}_{\mathrm{PA}}$.*

   (a) *If for all $b > I$, $A \models \varphi(b, \bar{a})$, then there is $c \in I$ such that $A \models \forall x \geq c \varphi(x, \bar{a})$.*

   (b) *If for all $b > I$ there is $d > I$ such that $A \models d \geq b \wedge \varphi(d, \bar{a})$, then there is $c \in I$ such that $A \models \varphi(c, \bar{a})$.*