Central limit theorem for random Young diagrams with respect to Jack measure
(joint work with Valentin Féray)

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Young diagrams

Definition

A partition $\lambda$ is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. It can be represented by a Young diagram $\lambda$. The size of the Young diagram $\lambda$ is defined by $|\lambda| := \sum_i \lambda_i$.

Problem

We want to investigate some asymptotic properties of Young diagrams as their size is tending to infinity. How to do it?

Solution

Look on 'large Young diagrams' from a 'large perspective' and treat these discrete objects as continuous ones!
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Young diagrams as continuous objects

French convention:
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Russian convention:
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Russian convention:
A profile of a Young diagram $\lambda$ is a function $\omega(\lambda) : \mathbb{R} \rightarrow \mathbb{R}_+$ such that its graph is a profile of $\lambda$ drawn in Russian convention.
Continuous Young diagrams

Definition
A continuous Young diagram is a function $\omega : \mathbb{R} \to \mathbb{R}_+$ such that

- $\omega(x) - |x|$ has compact support;
- $|\omega(x_1) - \omega(x_2)| \leq |x_1 - x_2|$ for any $x_1, x_2 \in \mathbb{R}$.

An area of a continuous Young diagram $\omega$ is given by:

$$\text{Area}(\omega) := \frac{1}{2} \int_{\mathbb{R}} |\omega(x) - |x|| \, dx.$$ 

Remark
Let $\lambda$ - Young diagram with $|\lambda| = n$. Then

$$\text{Area}(\omega(\lambda)) = n.$$
Normalized Young diagrams

**Problem**

*How to look on the 'large Young diagrams' from 'large perspective'?*

**Solution**

*Normalize them in a way that their areas are constant.*

**Definition**

Let $\lambda$ - Young diagram with $|\lambda| = n$. We define scaled (continuous) Young diagram

$$\omega(D_{\sqrt{n}^{-1}}(\lambda))(x) := \sqrt{n}^{-1} \omega(\lambda)(\sqrt{n}x).$$

**Remark**

$$\text{Area } \left( \omega(D_{\sqrt{n}^{-1}}(\lambda)) \right) = 1.$$
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$$\text{Area} \left( \omega(D_{\sqrt{n}^{-1}}(\lambda)) \right) = 1.$$
Asymptotic shape of large Young diagrams

Let \((\lambda(n))_{n \in \mathbb{N}^+}\) - sequence of Young diagrams with \(|\lambda(n)| = n\).

**Definition**

We say that \((\lambda(n))_{n \in \mathbb{N}^+}\) has a **limit shape** \(\omega\) if

\[
\|\omega(D_{\sqrt{n}^{-1}}(\lambda(n))) - \omega\| \to 0,
\]

as \(n \to \infty\), where \(\|f\| = \sup_{x \in \mathbb{R}} |f(x)|\).

**Problem**

Let us choose \((\lambda(n))_{n \in \mathbb{N}^+}\) randomly according with some 'nice' distribution. Does it have a limit shape with a high probability? Is it unique? Can we say more about it?
Asymptotic shape of large Young diagrams

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**Definition**

We say that \((\lambda(n))_{n \in \mathbb{N}_+}\) has a limit shape \(\omega\) if

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'Nice' distribution = Plancherel distribution

\[ a(\bullet) = \text{number of boxes to the right of the given box} \]
'Nice' distribution = Plancherel distribution

\[ \ell(\bullet) = \text{number of boxes above the given box} \]
'Nice' distribution = Plancherel distribution

\[ \text{hook}^{(1)}(\bullet) := a(\bullet) + \ell(\bullet) + 1. \]
'Nice' distribution = Plancherel distribution

\[ \mathbb{P}^{(1)}_n(\lambda) = \frac{\dim(\lambda)^2}{n!}, \]

where (hook formula:)

\[ \dim(\lambda) = \frac{n!}{\prod_{\square \in \lambda} \text{hook}(\square)}. \]
'Nice' distribution = Plancherel distribution

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\[ \mathbb{P}^{(1)}_n(\lambda) = \frac{n!}{\prod_{\square \in \lambda} (\text{hook}(\square))^2}. \]

Plancherel measure \( \mathbb{P}^{(1)}_n \) is a probability measure on the set \( \mathbb{Y}_n \) of Young diagrams of size \( n \).
**Figure:** Scaled random Young diagram of size 100 distributed according with Plancherel measure
Figure: Scaled random Young diagram of size 1000 distributed according with Plancherel measure
Figure: Scaled random Young diagram of size 5000 distributed according with Plancherel measure
\( \Omega(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi} \left( x \cdot \arcsin \frac{x}{2} + \sqrt{4 - x^2} \right) & \text{otherwise.} \end{cases} \)
First order asymptotic = 'law of large numbers'

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**Theorem (Vershik-Kerov, Logan-Shepp '77)**

Let \( \lambda_{(n)} \) be a random Young diagram of size \( n \) distributed with Plancherel measure \( \mathbb{P}^{(1)}_n \). Then, in probability, as \( n \to \infty \)

\[ \left\| \omega \left( D_1 / \sqrt{n}(\lambda_{(n)}) \right) - \Omega \right\| \to 0. \]
Second order asymptotic

**Problem**

*Can we describe the second order asymptotic? What does it really mean?*

**Solution**

*We should look on the fluctuations around the limit shape.*

Let $\lambda_n$ - random Young diagram distributed according with $\mathbb{P}_n^{(1)}$.

- We know that $\|\omega(D_{1/\sqrt{n}}(\lambda_n)) - \Omega\| \to 0$ in probability.
- We would like to investigate behaviour of random variables:

  $$m_k(\lambda_n) := \int_\mathbb{R} x^k \Delta(\lambda_n)(x) \, dx,$$

  where

  $$\Delta(\lambda)(x) := \sqrt{n} \frac{\omega(D_{1/\sqrt{n}}(\lambda))(x) - \Omega(x)}{2}.$$
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Second order asymptotic = ’central limit theorem’

- \( u_k(x) = U_k(x/2) = \sum_{0 \leq j \leq \lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} x^{k-2j}; \)
- \( u_k(2 \cos(\theta)) = \frac{\sin((k+1)\theta)}{\sin(\theta)}; \)
- \( u_k(\lambda) = \int_{\mathbb{R}} u_k(x) \Delta(\lambda)(x) \, dx. \)

**Theorem (Kerov, 1993)**

Choose a sequence \((\Xi_k)_{k=2,3,...}\) of independent standard Gaussian random variables and let \(\lambda(n)\) be a random Young diagram of size \(n\) distributed with Plancherel measure. As \(n \to \infty\), we have:

\[
\left( u_k(\lambda(n)) \right)_{k=1,2,...} \xrightarrow{d} \left( \frac{\Xi_{k+1}}{\sqrt{k+1}} \right)_{k=1,2,...}.
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\]
Let us recall that

$$\text{hook}(\square) = a(\square) + \ell(\square) + 1.$$ 

Let $\alpha \in \mathbb{R}_+$. $\alpha$-deformations of hook length:

- $\text{hook}^{(\alpha)}(\square) := \sqrt{\alpha}(a(\square) + 1) + \sqrt{\alpha^{-1}}\ell(\square)$,
- $\left(\text{hook}^{(\alpha)}\right)'(\square) := \sqrt{\alpha}a(\square) + \sqrt{\alpha^{-1}}(\ell(\square) + 1)$.

**Definition**

is a probability measure $\mathbb{P}_n$ on the set $\mathcal{Y}_n$ defined by

$$\mathbb{P}_n(\lambda) := \frac{n!}{\prod_{\square \in \lambda} \text{hook}(\square) \left(\text{hook}\right)'(\square)}.$$
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Let us recall that
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\]

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**Definition**

**Jack measure** is a probability measure \( P_n^{(\alpha)} \) on the set \( \mathcal{Y}_n \) defined by

\[
P_n^{(\alpha)} (\lambda) := \frac{n!}{\prod_{\square \in \lambda} (\text{hook}^{(\alpha)} (\square) (\text{hook}^{(\alpha)})' (\square))},
\]

where \( \alpha \in \mathbb{R}_+ \).
Let us recall that
\[ \text{hook}(\square) = a(\square) + \ell(\square) + 1. \]

Let \( \alpha \in \mathbb{R}_+ \). \( \alpha \)-deformations of hook length:
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**Definition**

**Plancherel measure** is a probability measure \( \mathbb{P}_{n}^{(1)} \) on the set \( \mathcal{Y}_n \) defined by
\[
\mathbb{P}_{n}^{(1)}(\lambda) := \frac{n!}{\prod_{\square \in \lambda} \left( \text{hook}^{(1)}(\square) \left( \text{hook}^{(1)} \right)'(\square) \right)} = \frac{n!}{\prod_{\square \in \lambda} \left( \text{hook}(\square) \right)^2}.
\]

- for \( \alpha = 1 \) Jack measure \( \equiv \) Plancherel measure.
Let $\lambda$ be a Young diagram.

**Definition**

$\alpha$-anisotropic Young diagram $A_{\alpha}(\lambda)$ (for $\alpha \in \mathbb{R}_+$) - continuous Young diagram obtained from $\lambda$ (considered in French convention) by a horizontal stretching of ratio $\sqrt{\alpha}$ and a vertical stretching of ratio $\sqrt{\alpha}^{-1}$. 

$\lambda \mapsto A_{\alpha}(\lambda)$
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α-anisotropic Young diagrams

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First order asymptotic = 'law of large numbers'

**Theorem (D., Féray)**

Let \( \lambda_{(n)} \) be a random Young diagram of size \( n \) distributed with *Jack measure* \( \mathbb{P}_n^{(\alpha)} \). Then, in probability, as \( n \to \infty \)

\[
\| \omega\left( D_1/\sqrt{n}(A_{\alpha}(\lambda_{(n)})) \right) - \Omega \| \to 0.
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**Remark**

*Plugging* \( \alpha = 1 \) *we recover Vershik-Kerov, Logan-Shepp limit shape for Plancherel measure.*
Second order asymptotic = 'central limit theorem'

**Theorem (D. Féray)**

Choose a sequence \((\Xi_k)_{k=2,3,...}\) of independent standard Gaussian random variables and let \(\lambda(n)\) be a random Young diagram of size \(n\) distributed with Jack measure. As \(n \to \infty\), we have:

\[
\left( u_k^{(\alpha)}(\lambda(n)) \right)_{k=1,2,...} \xrightarrow{d} \left( \frac{\Xi_{k+1}}{\sqrt{k+1}} - \frac{\gamma}{k+1} \left[ k \text{ is odd} \right] \right)_{k=1,2,...},
\]

where \(u_k^{(\alpha)}(\lambda) = \int_{\mathbb{R}} u_k(x) \Delta(A_\alpha(\lambda))(x) \, dx\), \(\gamma := \sqrt{\alpha} - \sqrt{\alpha}^{-1}\), and we use the usual notation [condition] for the indicator function of the corresponding condition.

**Remark**

*Plugging \(\alpha = 1\) we recover central limit theorem of Kerov for Plancherel measure.*
Jack polynomials and Jack characters

Jack polynomials $J_{\lambda}^{(\alpha)}$:

- symmetric functions introduced by Jack;
- generalization of Schur symmetric function (for $\alpha = 1$);
- special case of Macdonald polynomials

Expand Jack polynomial in power-sum symmetric basis:

$$J_{\lambda}^{(\alpha)} = \sum_{\substack{\rho: \ |ho| = |\lambda|}} \theta_{\rho}^{(\alpha)}(\lambda) \ p_{\rho}.$$ 

We call quantities $\theta_{\rho}^{(\alpha)}(\lambda)$ Jack characters (for $\alpha = 1$ they coincide with the irreducible characters of the symmetric groups up to some normalization constant).
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Characterization of Jack measure

\[ \mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(\theta_{\mu}^{(\alpha)}) = \begin{cases} 1 & \text{if } \mu = 1^n, \\ 0 & \text{otherwise.} \end{cases} \]

Let \( \lambda \in \mathbb{Y}_n \).

**Proposition**

\[ \int_{\mathbb{R}} x^k \Delta(A_{\alpha}(\lambda))(x) \, dx \] can be expressed as a function of

\[ \theta_{(1^n)}^{(\alpha)}, \theta_{(2,1^{n-2})}^{(\alpha)}, \ldots, \theta_{(k-1,1^{n-k+1})}^{(\alpha)}. \]

**Corollary**

*Our central limit theorem has equivalent, algebraic version!*
Choose a sequence \((\Xi_k)_{k=2,3,...}\) of independent standard Gaussian random variables. As \(n \to \infty\), we have:

\[
\left( \frac{\sqrt{k} \theta^{(\alpha)}_{(k,1^n_{n-k})}(\lambda(n))}{n^{k/2}} \right)_{k=2,3,...} \xrightarrow{d} (\Xi_k)_{k=2,3,...},
\]

where the distribution of \(\lambda(n)\) is Jack measure of size \(n\) and where \(\xrightarrow{d}\) means convergence in distribution of the finite-dimensional law.

We can prove this theorem using algebraic methods (Jack characters after normalization span a very nice algebra)!
Polynomials functions

We define

\[
Ch_{\mu}^{(\alpha)}(\lambda) = \begin{cases} 
\alpha - \frac{|\mu| - \ell(\mu)}{2} \left( |\lambda| - |\mu| + m_1(\mu) \right) z_{\mu}^2 \theta_{\mu,1|\lambda| - |\mu|}(\lambda) & \text{if } |\lambda| \geq |\mu|; \\
0 & \text{if } |\lambda| < |\mu|, 
\end{cases}
\]

where

- \( z_{\mu} = \mu_1 \mu_2 \cdots m_1(\mu)!m_2(\mu)! \cdots \),
- \( m_i(\mu) \) - number of parts of \( \mu \) equal to \( i \).

**Theorem (Lassalle, 2009)**

The family \( \left( Ch_{\mu}^{(\alpha)} \right)_{\mu} \) span linearly an algebra \( \Lambda_\alpha \) of \( \alpha \)-shifted symmetric functions.
What do we have and what do we miss?

In order to prove our main theorem:

- We want to estimate mixed moments of Jack characters;
- Expectation of the Jack characters is easy to compute;
- Suitably normalized Jack characters span linearly some nice algebra \( \Lambda_\star^{(\alpha)} \);
- We want to expand a product:

\[
\text{Ch}_\mu^{(\alpha)} \text{Ch}_\nu^{(\alpha)} = \sum_{\rho} g_{\mu,\nu;\pi}^{(\alpha)} \text{Ch}_{\pi}^{(\alpha)}
\]

as a linear combination of suitably normalized Jack characters.

Problem

*What can we say about \( g_{\mu,\nu;\pi}^{(\alpha)} \)?*
Main result for structure constants

**Theorem (D., Féray)**

Let

\[
\mathrm{Ch}_\mu^{(\alpha)} \mathrm{Ch}_\nu^{(\alpha)} = \sum_{\rho} g_{\mu,\nu;\pi}^{(\alpha)} \mathrm{Ch}_\pi^{(\alpha)}.
\]

Then, **structure constants** \(g_{\mu,\nu;\pi}^{(\alpha)}\) are polynomials in \(\gamma := \alpha^{1/2} - \alpha^{-1/2}\) of degree less than

\[
\min_{i=1,2,3} (n_i(\mu) + n_i(\nu) - n_i(\pi)),
\]

with rational coefficients, where \(n_i(\lambda)\) - natural valued function of \(\lambda\).

- It is crucial for proving central limit theorem;
- It is applicable to different problems.
Projection on the set of Young diagrams of a fixed size

Let $\mu, \nu, \pi \in \mathbb{Y}_n$.

$$\theta^{(\alpha)}_\mu(\lambda) \theta^{(\alpha)}_\nu(\lambda) = \sum_{|\pi|=n} c^{(\alpha)}_{\mu, \nu; \pi} \theta^{(\alpha)}_\pi.$$

Hence

$$c^{(\alpha)}_{\mu, \nu; \pi} = \frac{\alpha^{d(\mu, \nu; \pi)/2}}{z_{\tilde{\mu}} z_{\tilde{\nu}}} \sum_{0 \leq i \leq m_1(\pi)} g^{(\alpha)}_{\tilde{\mu}, \tilde{\nu}; \tilde{\pi}^{\text{st}}} \cdot z_{\tilde{\pi}^{\text{st}}} \cdot i! \cdot \binom{n - |\tilde{\pi}|}{i},$$

where

- $\tilde{\mu}$ is created from $\mu$ by removing all parts equal to 1,
- $d(\mu, \nu; \pi) = |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| - \ell(\pi)).$
Let $\mathbb{C}[S_n] := \{ f : f : S_n \rightarrow \mathbb{C} \}$ be a group algebra of the symmetric group. This is algebra with the multiplication defined by:

$$f \cdot g(\sigma) := \sum_{\sigma_1 \sigma_2 = \sigma} f(\sigma_1)g(\sigma_2).$$

Let

$$Z(\mathbb{C}[S_n]) := \{ f \in \mathbb{C}[S_n] : \forall g \in \mathbb{C}[S_n], fg = gf \}$$

be the center of that algebra. It has a basis $(f_\mu)_{|\mu|=n}$, where

$$f_\mu(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ has cycle type } \mu, \\ 0 & \text{otherwise}. \end{cases}$$
\( \alpha = 1 - \text{Structure constants of the } Z(\mathbb{C}[\mathfrak{S}_n]) \)

Let

\[
f_\mu f_\nu = \sum_{|\rho|=n} c_{\mu,\nu;\rho} f_\rho.
\]

**Lemma**

The structure constant \( c_{\mu,\nu;\rho} \) is equal to the number of pairs of permutation \((\sigma_1, \sigma_2)\) such that

- \( \sigma_1 \) has cycle type \( \mu \),
- \( \sigma_2 \) has cycle type \( \nu \),
- \( \sigma_1 \sigma_2 = \sigma \), where \( \sigma \) is a fixed permutation of the cycle-type \( \rho \).
$\alpha = 1$ - Structure constants of the $\mathbb{Z}({\mathbb{C}[S_n]})$

One has a following relation:

$$c^{(1)}_{\mu,\nu;\rho} = c_{\mu,\nu;\rho}.$$

**Remark**

*From the previous theorem and a relation between $c^{(\alpha)}$ and $g^{(\alpha)}$ one can deduce a classical result of Farahat and Higman: $c_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\rho 1^{n-|\rho|}}$ is a polynomial in $n$.***
$\alpha = 2$ - Structure constants of the Hecke algebra of $(S_{2n}, H_n)$

Let $S_{2n}$ acts on the set $X_n := \{1, \bar{1}, \ldots, n, \bar{n}\}$ by permutations and let

$$S_{2n} \supset H_n := \{\sigma \in S_{2n} : \forall i \in X_n \sigma(i) = \sigma(i)\}$$

be a hyperoctahedral subgroup.

Hecke algebra $\mathbb{C}[H_n \backslash S_{2n}/H_n] < \mathbb{C}[S_{2n}]$ of the pair $(S_{2n}, H_n)$ is defined by:

$$\mathbb{C}[H_n \backslash S_{2n}/H_n] := \{x \in \mathbb{C}[S_{2n}] : h x h' = x \forall h, h' \in H_n\}.$$

Double-cosets: equivalence classes for the relation $x \sim h x h'$ (for $x \in S_{2n}$ and $h, h' \in H_n$)

- naturally indexed by partitions of size $n$;
- $F_\mu = \sum_{\text{type } \mu} \delta_x$ - linear basis of $\mathbb{C}[H_n \backslash S_{2n}/H_n]$. 
\( \alpha = 2 \) - Structure constants of the Hecke algebra of \((\mathfrak{S}_{2n}, H_n)\)

Let

\[
F_{\mu} F_{\nu} = \sum_{|\rho|=n} h_{\mu, \nu; \rho} F_{\rho}.
\]

Then

\[
c^{(2)}_{\mu, \nu; \rho} = \frac{h_{\mu, \nu; \rho}}{2^n n!}.
\]

**Remark**

*From the previous theorem and a relation between \(c^{(\alpha)}\) and \(g^{(\alpha)}\) one can deduce a result of Tout (2013):*

\[
\frac{h_{\mu 1^{n-|\mu|}, \nu 1^{n-|\nu|}; \pi 1^{n-|\pi|}}}{n! \ 2^n}
\]

*is a polynomial in \(n\).*
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