A curious identity between the orthogonal Brezin–Gross–Witten integral and Schur symmetric functions via b-deformed monotone Hurwitz numbers

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joint work with

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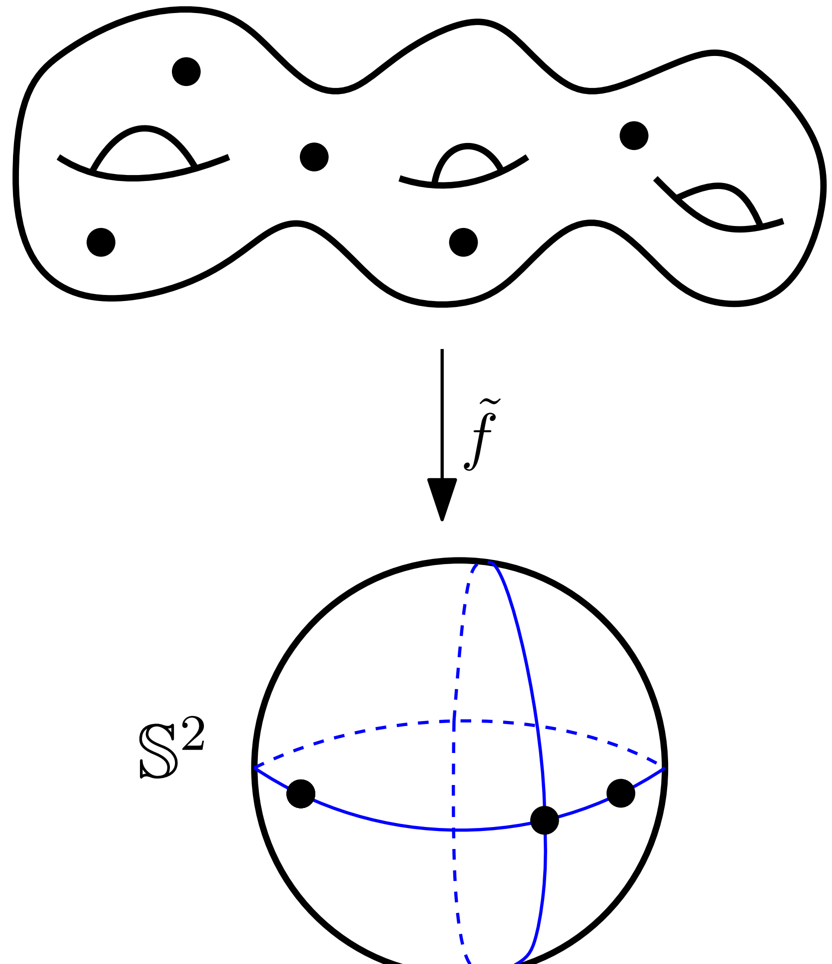
Guillaume Chapuy, CNRS & IRIF, Université de Paris

Algebraic and Enumerative Combinatorics - University of Waterloo, 10th February 2022.

I. Hurwitz numbers and their b-deformation

Hurwitz's problem

Problem: Classify/count all the branched coverings of the sphere $\mathbb{S}^2$



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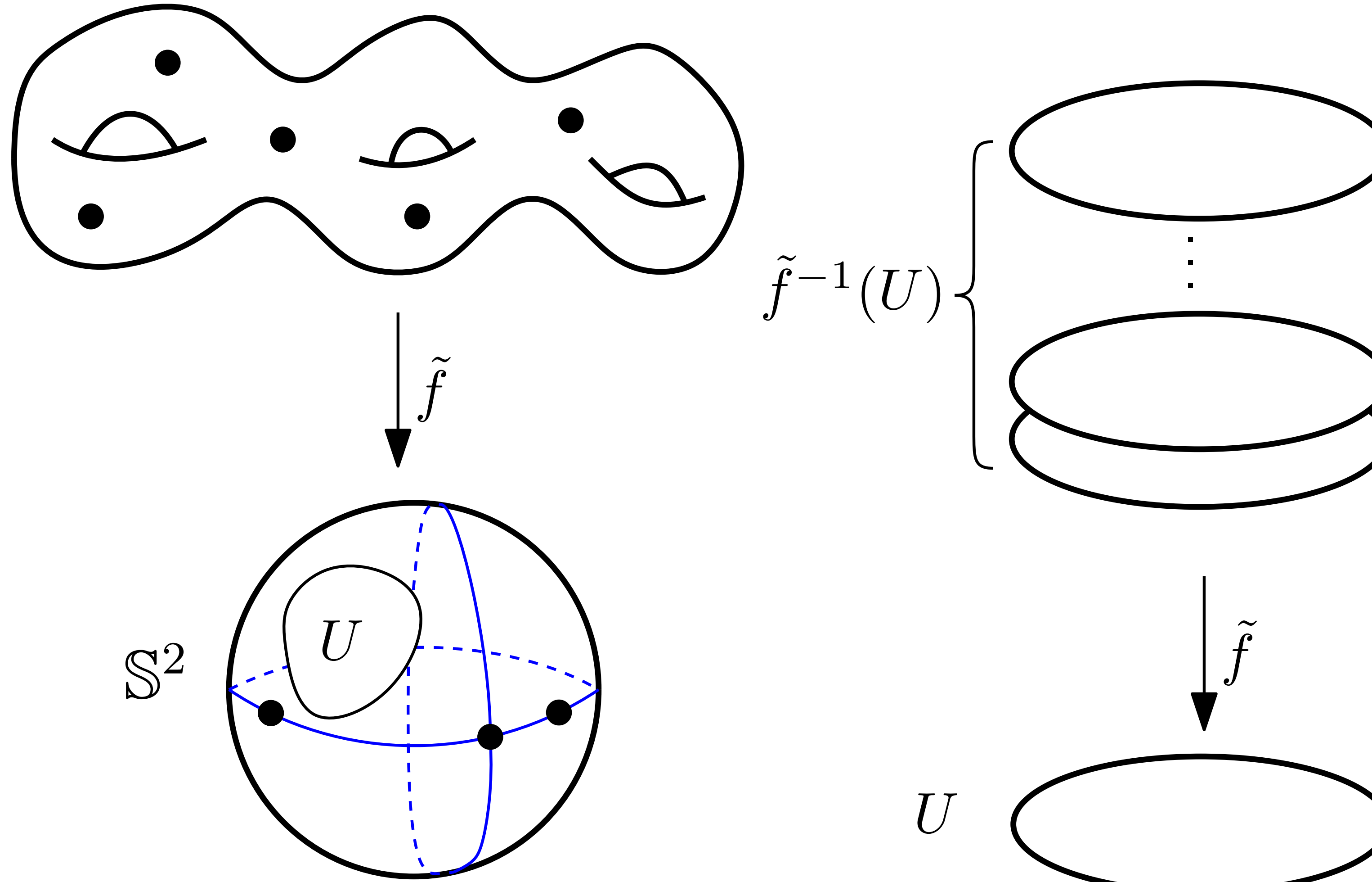

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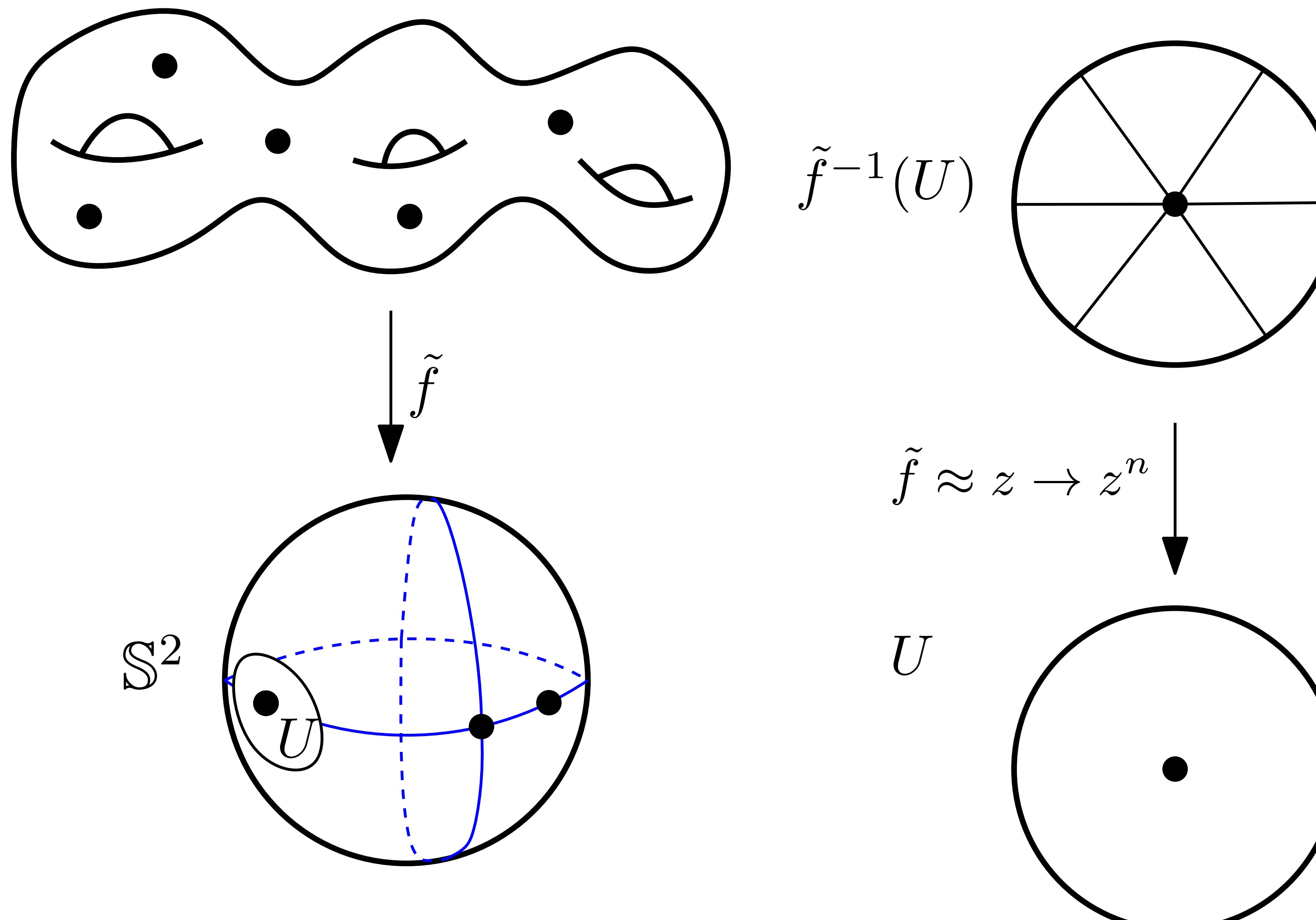

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$$\tau_k^{(0)} := \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma_1 \cdots \sigma_{k+2} = \mathrm{id} \in \mathfrak{S}(n)} \prod_{i=1}^{k+2} \mathbf{p}^{(i)}(\sigma_i)$$

where $\mathbf{p}^i(\sigma) := \prod_{c:\text{ cycle in }\sigma} p^{(i)}_{\ell(c)}$

Example: $(1245)(3) \cdot (1)(23)(4)(5) \cdot (54321) = \mathrm{id}$

$$\mathbf{p}^{(1)}((1245)(3))\mathbf{p}^{(2)}(1)(23)(4)(5)\mathbf{p}^{(3)}(54321) = p_1^{(1)} p_4^{(1)} (p_1^{(2)})^3 p_2^{(2)} p_5^{(3)}$$

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Example:

$$[\prod_{i=1}^{k+1} p_2^{(i)} (p_1^{(i)})^k \cdot p_{(k+2)}^{(k+2)}]\tau_k^{(0)} \overset{???}{=}$$

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Example:

$$[\prod_{i=1}^{k+1} p_2^{(i)}(p_1^{(i)})^k \cdot p^{(k+2)}_{(k+2)}]\tau_k^{(0)} = \frac{(k+1)!}{(k+2)!}\#\{\tau_1\cdots\tau_{k+1} = (1,2,\ldots,k+2)\}$$

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Example:

Cayley's theorem:

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$\frac{td}{dt}\log\tau_k^{(0)}$ - g.f. of transitive k-factorizations modulo conjugation $\equiv$ g.f. of branched coverings

Branched coverings vs. maps

Example:

$$(34)(12)(24) = (1234)$$

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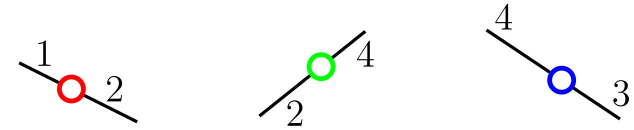

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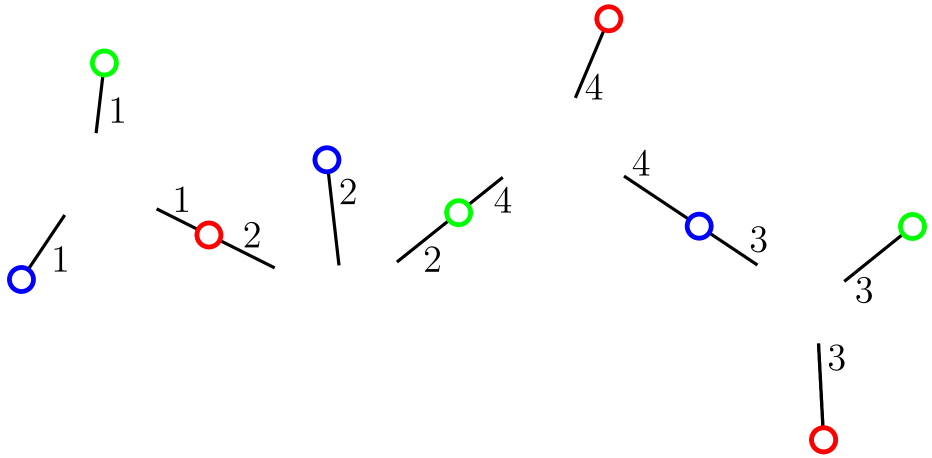

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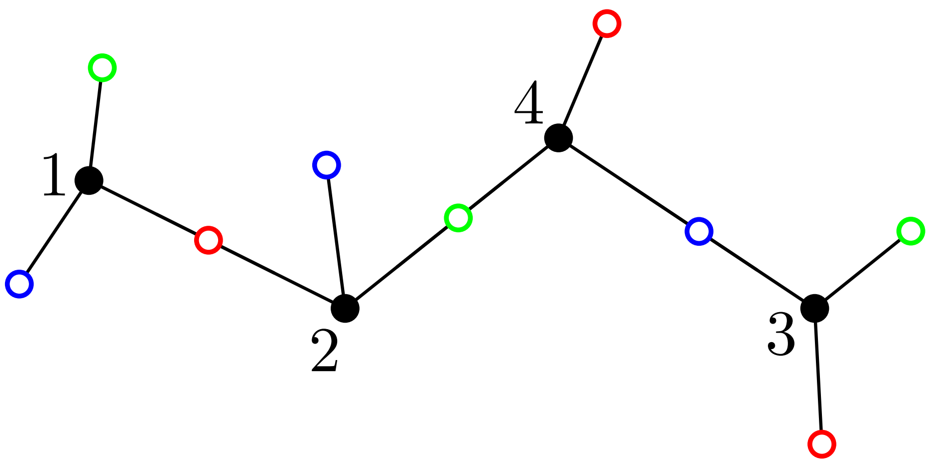

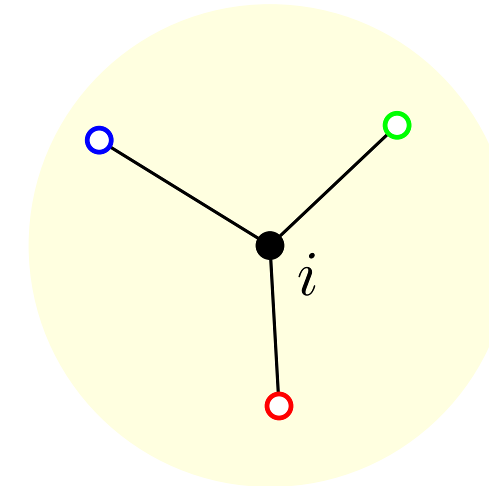

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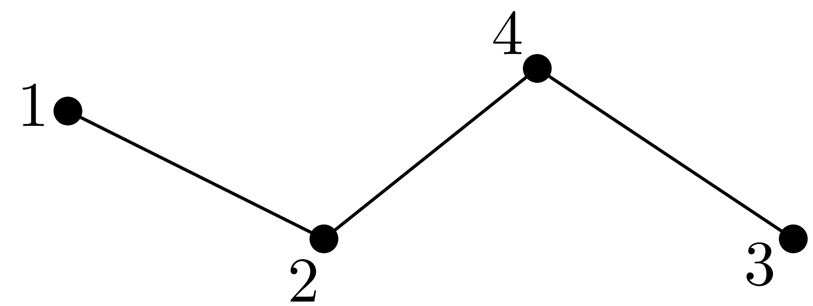

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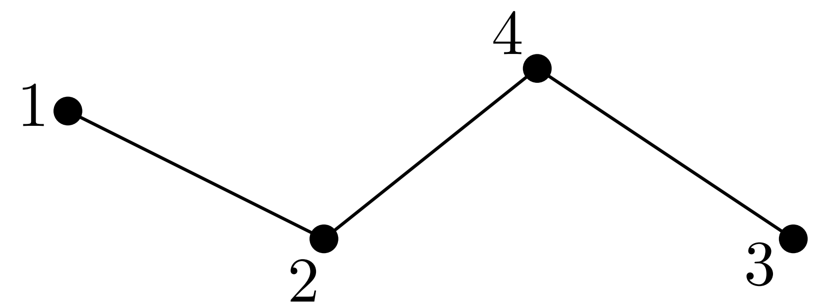

Special case of a tau function: **Grothendieck dessins d'enfants**

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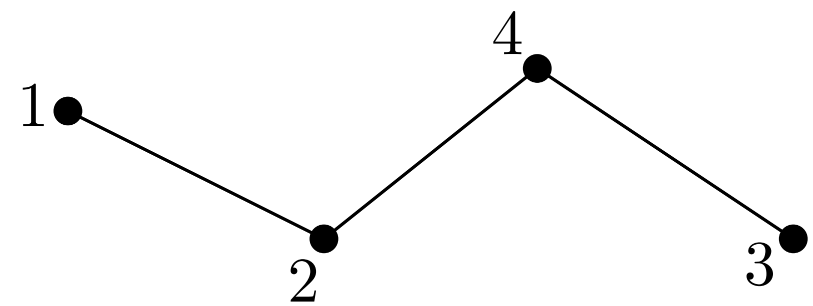

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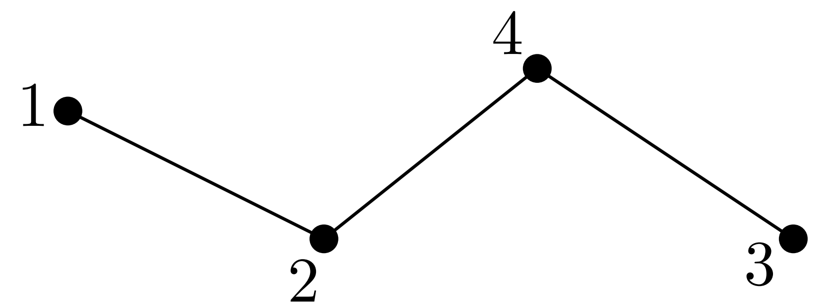

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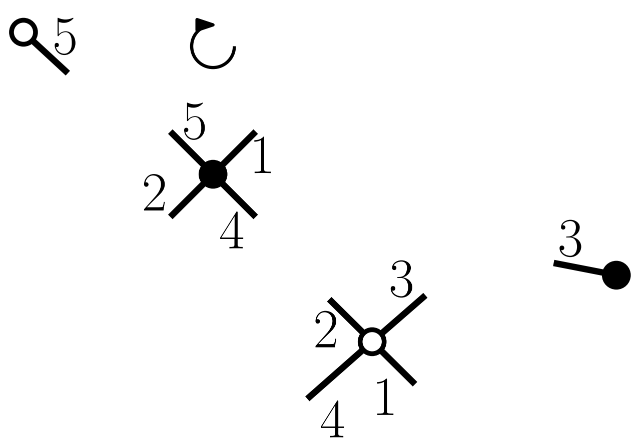

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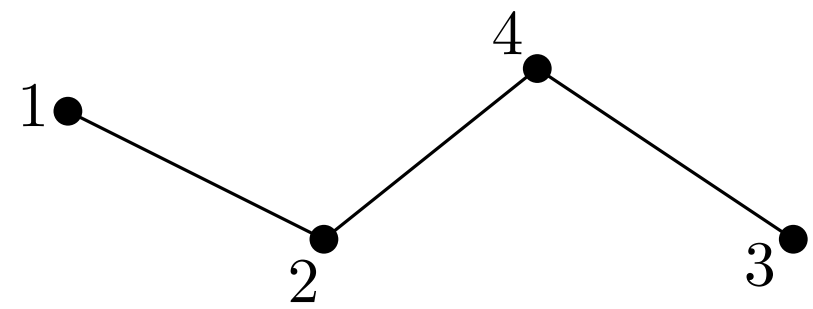

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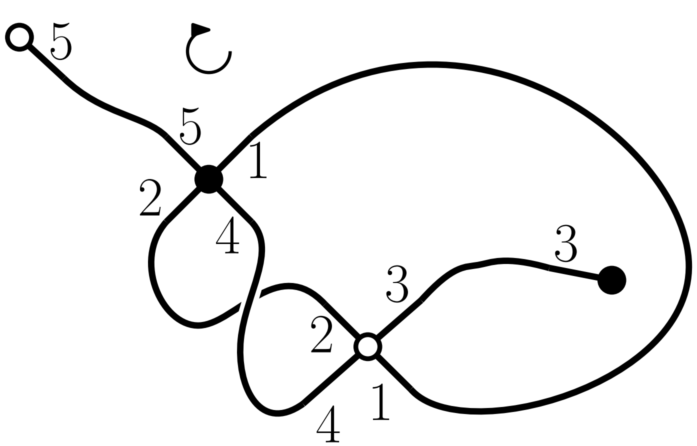

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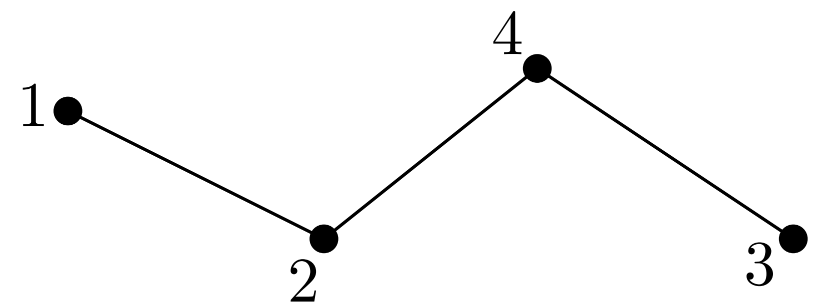

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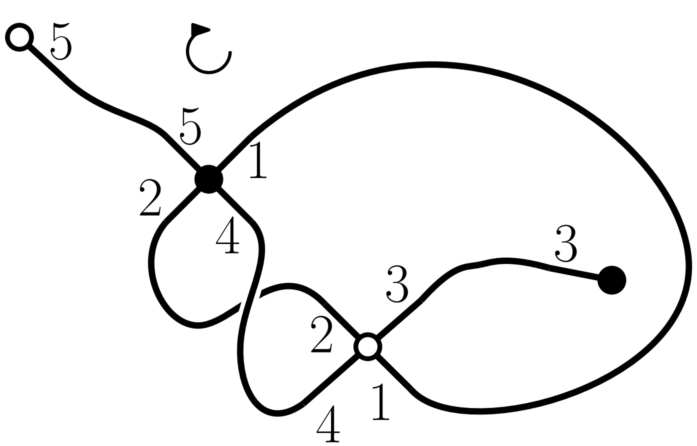

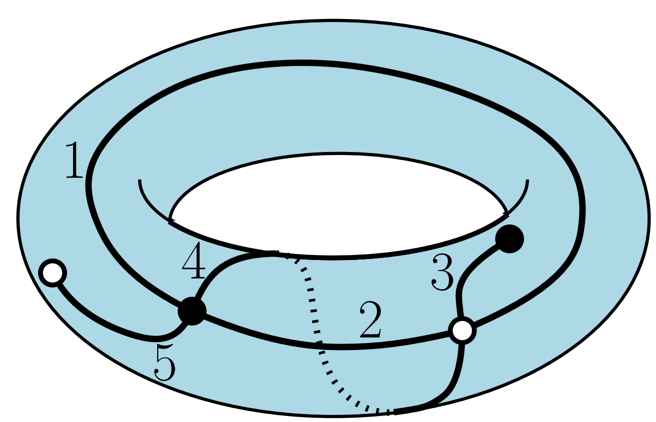

Labeled map

Map $\equiv$ graph embedded into a surface, such that it cuts this surface into simply connected pieces

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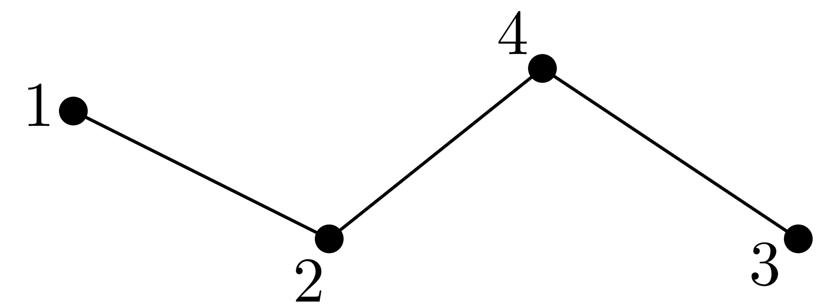

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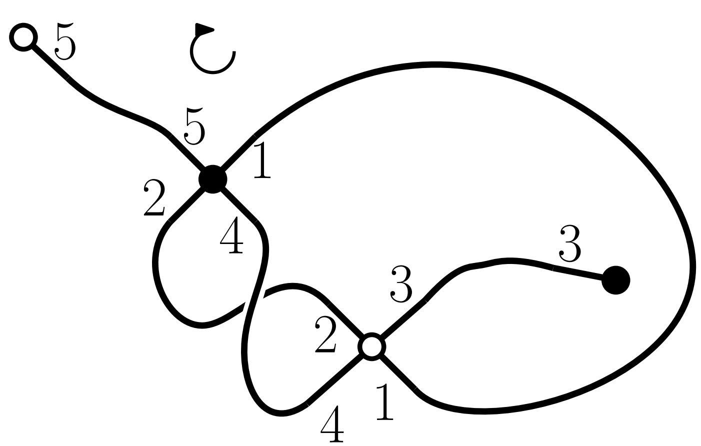

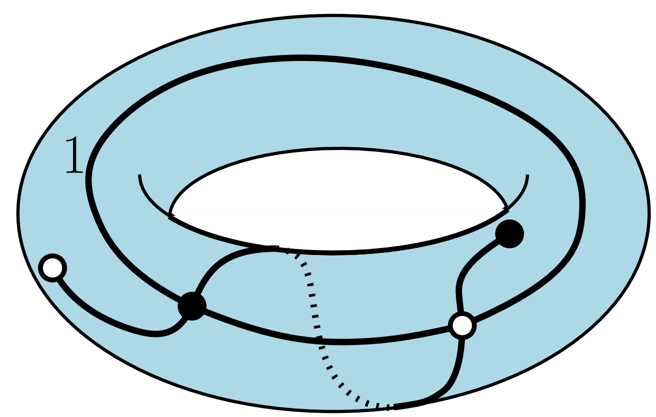

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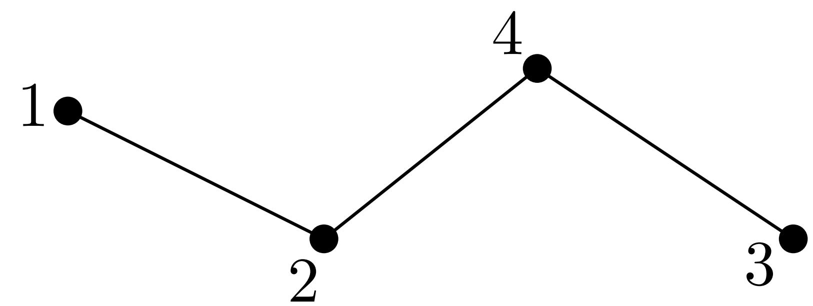

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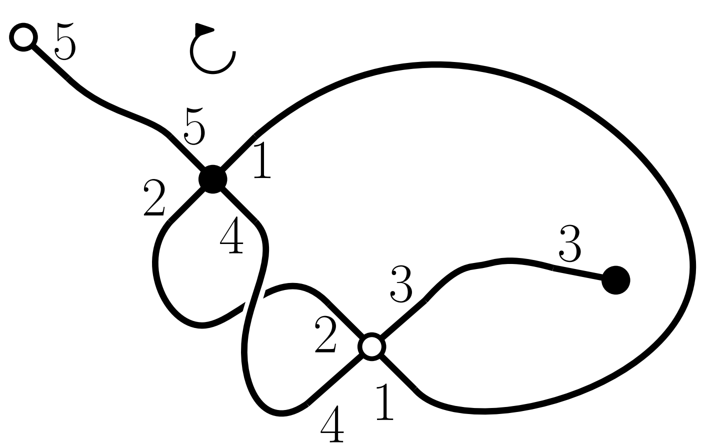

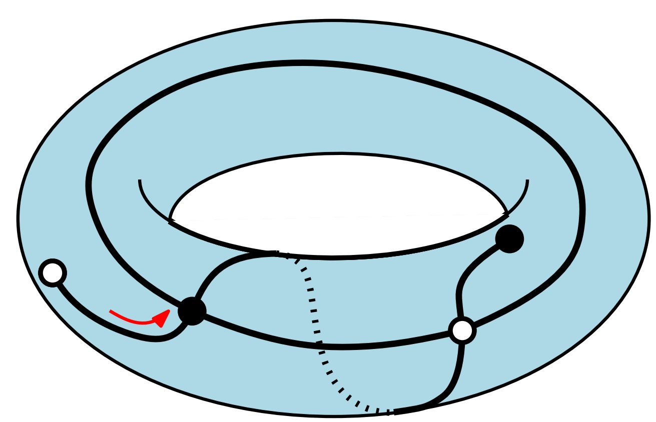

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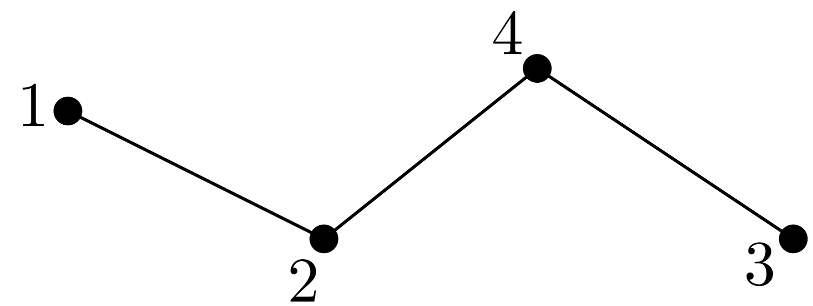

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Summary:

$$\tau_1^{(0)}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \sum_M \frac{t^{e(M)}}{e(M)!} \prod_{v_\bullet \in V_\bullet(M)} p_{\deg(v_\bullet)} \prod_{v_\circ \in V_\circ(M)} q_{\deg(v_\circ)} \prod_{f \in F(M)} r_{\deg(f)/2}$$

sum over **orientable**, labeled and possibly disconnected maps

$$\frac{td}{dt} \log \tau_1^{(0)}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \sum_M t^{e(M)} \prod_{v_\bullet \in V_\bullet(M)} p_{\deg(v_\bullet)} \prod_{v_\circ \in V_\circ(M)} q_{\deg(v_\circ)} \prod_{f \in F(M)} r_{\deg(f)/2}$$

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Branched coverings & symmetric functions

Power-sum and Schur symmetric functions: Recall that:

- $p_i := \sum_j x_j^i$ - power-sum symmetric function
- χ_λ - character of the irreducible repr. ρ_λ of the symmetric group $\equiv \mathrm{Tr}(\rho_\lambda(\cdot))$
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Frobenius character formula:

- C_μ - conjugacy class of permutatios of a cycle type μ, i.e. $\mathbf{p}(\sigma) = \prod_{i=1} p_{\mu_i}$.
- $c_\mu = \sum_{\sigma \in C_\mu} \sigma$

$$[\mathrm{id}] c_{\mu^1} \cdots c_{\mu^k} = \frac{1}{n!} \sum_\lambda \frac{\chi_\lambda(c_{\mu^1}) \cdots \chi_\lambda(c_{\mu^k})}{\dim(\rho_\lambda)^{k-2}}$$

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Corollary (cool formula):

$$\tau_k^{(0)} = \sum_{n \geq 0} t^n \sum_{\lambda \vdash n} \frac{\dim(\rho_\lambda)}{n!}^2 \prod_{i=1}^{k+2} \tilde{s}_\lambda(\mathbf{p}^{(i)})$$ where $\tilde{s}_\lambda := \frac{n!}{\dim(\rho_\lambda)} s_\lambda$

Proof: Definition + Frobenius formula

Non-oriented maps - representation theory & symmetric functions

Question: Can we encode non-oriented bipartite maps in an analogous way (representation theory/symmetric functions theory)?

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symmetric function	normalized Schur $\tilde{s}_\lambda$	Zonal polynomials Z_λ [Goulden, Jackson '96]

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sum over **non-oriented**, rooted and connected maps

Then $\tau_1^{(1)}(\mathbf{p},\mathbf{q},\mathbf{r}) := \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{\dim(\rho_{2\lambda})}{(2n)!} Z_\lambda(\mathbf{p})Z_\lambda(\mathbf{q})Z_\lambda(\mathbf{r})$

Jack symmetric functions and the b-conjecture

So far we know that:

- $\frac{td}{dt}\log\tau_1^{(0)} := \frac{td}{dt}\log\sum_{n\geq 0} t^n \sum_{\lambda\vdash n}\frac{\dim(\rho_\lambda)^2}{(n)!^2}\tilde{s}_\lambda(\mathbf{p})\tilde{s}_\lambda(\mathbf{q})\tilde{s}_\lambda(\mathbf{r})$ g.f. of **orientable** maps
- $2\frac{td}{dt}\log\tau_1^{(1)} := 2\frac{td}{dt}\log\sum_{n\geq 0} t^n \sum_{\lambda\vdash n}\frac{\dim(\rho_{2\lambda})}{(2n)!}Z_\lambda(\mathbf{p})Z_\lambda(\mathbf{q})Z_\lambda(\mathbf{r})$ g.f. of **non-oriented** maps

Jack symmetric functions and the b-conjecture

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If you are an expert in symmetric functions theory you can recognise that:

- $\tilde{s}_\lambda = J_\lambda^{(1)}, \qquad \|J_\lambda^{(1)}\|^2_{(1)} = \frac{\dim(\rho_\lambda)^2}{(n)!^2}$
- $Z_\lambda = J_\lambda^{(2)}, \qquad \|J_\lambda^{(2)}\|^2_{(2)} = \frac{\dim(\rho_{2\lambda})}{(2n)!}$

where $J_\lambda^{(\alpha)}$ is a **Jack polynomial**

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Two-lines crash-course on Jack polynomials:

- $J_\lambda^{(\alpha)} = \text{hook}_\alpha(\lambda) m_\lambda + \sum_{\mu<\lambda} a_\mu^\lambda(\alpha) m_\mu,\ a_\mu^\lambda(\alpha) \in \mathbb{Q}(\alpha)$ (uppertriangularity)
- $\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)}\rangle_{(\alpha)} = \delta_{\mu,\lambda}\, \underline{\text{hook}_\alpha(\lambda)}\ \underline{\text{hook}_\alpha(\lambda)'}$ (orthogonality)
  where $\langle p_\lambda, p_\mu\rangle_{(\alpha)} := \delta_{\mu,\lambda}|C_\lambda|\alpha^{\ell(\lambda)}$

α-deformations of classical hook products

Think: Jack polynomials are symmetric functions obtained by applying Gram-Schmidt orthogonalization process to the monomial basis

Jack symmetric functions and the b-conjecture

So far we know that:

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Conjecture (the b-conjecture) [Goulden,Jackson '96]

Let

$$\tau_1^{(b)} := \sum_{n\geq 0} t^n \sum_{\lambda\vdash n}\frac{J_\lambda^{(1+b)}(\mathbf{p})J_\lambda^{(1+b)}(\mathbf{q})J_\lambda^{(1+b)}(\mathbf{r})}{\|J_\lambda^{(1+b)}\|^2}$$

There exists a statistic MON (Measure Of Non-orientability) such that $\mathrm{MON}(M)\in\mathbb{Z}_{\geq 0}$ and $\mathrm{MON}(M)=0$ if and only if M is orientable and $\tau_1^{(b)}$ is the generating series of non-oriented maps:

$$(1+b)\frac{td}{dt}\log\tau_1^{(b)} = \sum_M t^{e(M)} b^{\mathrm{MON}(M)} \prod_{v_\bullet\in V_\bullet(M)} p_{\deg(v_\bullet)} \prod_{v_\circ\in V_\circ(M)} q_{\deg(v_\circ)} \prod_{f\in F(M)} r_{\deg(f)/2}$$

Jack symmetric functions and the b-conjecture

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Many special cases proved: [Burchardt, D., Feŕay, Goulden, Jackson, Kanunnikov, La Croix, Promyslov, Vassilieva,Visentin], Still wide open in general...

b-deformed tau function

The most fundamental function in the field: many beautiful properties; natural appearence in enumerative geometry, probability, mathematical physics [Eynard, Harnad, Okounkov, Orlov, Pandharipande, many others...]

Keep first two infinite set of variables $\to$ **tau function of the Toda hierarchy**

$$\tau_k^{(0)} = \sum_{n\geq 0} t^n \frac{\dim(\rho_\lambda)}{n!}^2 \sum_{\lambda\vdash n} \tilde{s}_\lambda(\mathbf{p})\tilde{s}_\lambda(\mathbf{q}) \prod_{i=1}^k \tilde{s}_\lambda(\underline{u_i}),$$ where $\underline{u_i} = (u_i, u_i, \dots)$

i.e. $p_1^{(i)} = p_2^{(i)} = p_3^{(i)} = \cdots = u_i$

$$\frac{td}{dt} \log \tau_k^{(0)} = \sum_{\mathbf{M}_k} t^{|\mathbf{M}_k|} \prod_{f\in F(\mathbf{M}_k)} p_{\deg(f)} \prod_{v\in V_0(\mathbf{M}_k)} q_{\deg(v)} \prod_{i=1}^k u_i^{v_i(\mathbf{M}_k)},$$

rooted orientable k-constellations

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Theorem [Chapuy, D. '20]

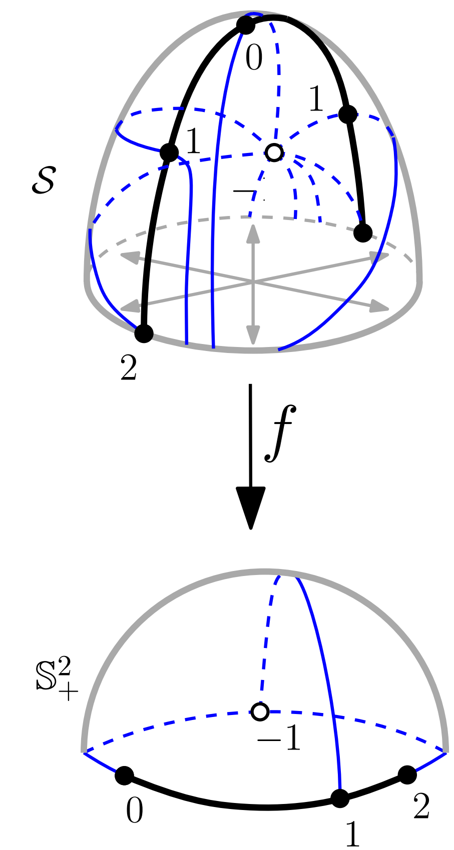
$$(1+b)\frac{td}{dt}\log\tau_k^{(b)} = \sum_{f:\mathcal{S}\to\mathbb{S}_+} \kappa(f) t^{|f|} b^{\mathrm{MON}(f)},$$

rooted generalized branched coverings f of the sphere $\mathbb{S}$ by a connected compact surface, orientable or not, with $k+2$ ramification points

$$\kappa(f) = p_{\lambda^{-1}(f)} q_{\lambda^0(f)} u_1^{v_1(f)} \dots u_k^{v_k(f)}$$

ramification profile of the first point

ramification profile of the second point



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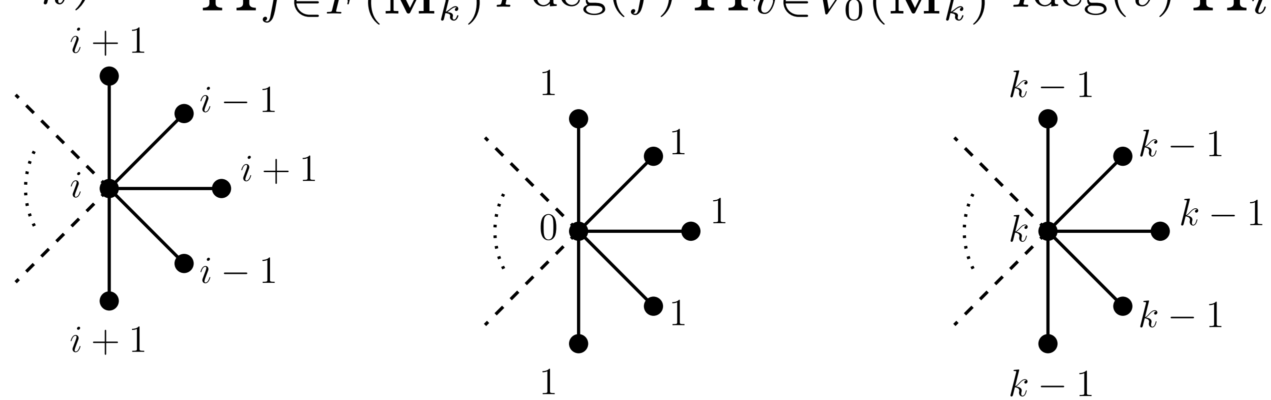
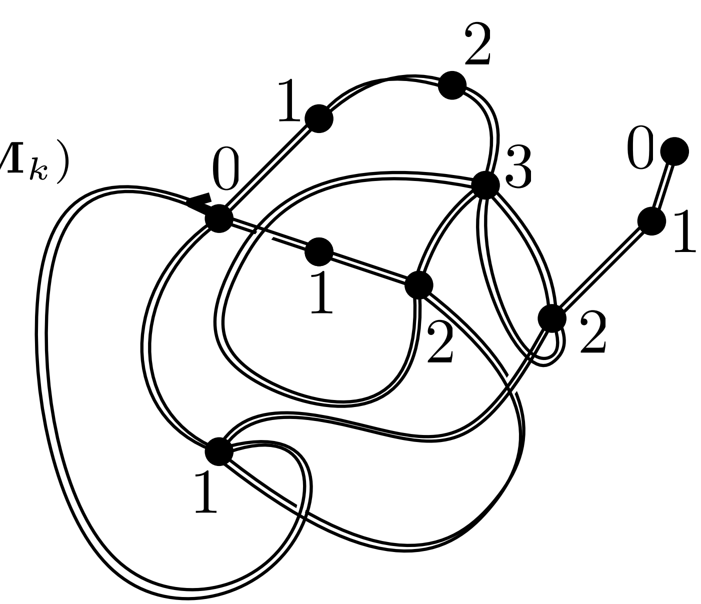
rooted k-constellations, orientable or not

$$\kappa(\mathbf{M}_k) := \prod_{f\in F(\mathbf{M}_k)} p_{\deg(f)} \prod_{v\in V_0(\mathbf{M}_k)} q_{\deg(v)} \prod_{i=1}^k u_i^{v_i(\mathbf{M}_k)}$$



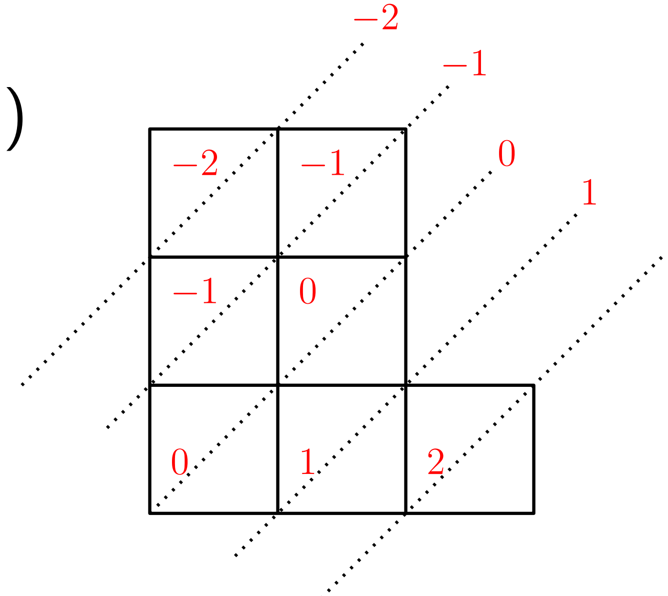


Weighted Hurwitz numbers

Observation: (essentially Weyl's dimension formula)

$$\tilde{s}_\lambda(\underline{u}) = \prod_{\square\in\lambda}(u + c(\square))$$

content of a box:



Corollary:

$$\tau_k^{(0)} = \sum_{n\geq 0} t^n \frac{\dim(\rho_\lambda)}{n!}^2 \sum_{\lambda\vdash n} \tilde{s}_\lambda(\mathbf{p})\tilde{s}_\lambda(\mathbf{q}) \prod_{i=1}^{k} \prod_{\square\in\lambda}(u_i + c(\square))$$

Weighted Hurwitz numbers

Definition: [Harnad, Guay-Paquet '15]

Let $G(z) = \prod_{i=1}^{k}(1 + a_i \cdot z)\prod_{i=1}^{l}(1 - b_i \cdot z)^{-1}$. We define the tau function of G-weighted Hurwitz numbers as:

$$\tau_G^{(0)} = \sum_{n\geq 0} t^n \frac{\dim(\rho_\lambda)}{n!}^2 \sum_{\lambda\vdash n} \tilde{s}_\lambda(\mathbf{p})\tilde{s}_\lambda(\mathbf{q}) \prod_{i=1}^{k} \prod_{\square\in\lambda} G(\hbar c(\square))$$

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Examples: (cover important classical cases)

- $G(z) = \exp(z)$

double Hurwitz numbers with ℓ simple branch points

$$\log \tau_G^{(0)} = \sum_{n\geq 1} t^n \sum_{\lambda,\mu\vdash n}\sum_{\ell\geq 0} H^\ell(\lambda,\mu)\frac{\hbar^\ell}{\ell!} p_\lambda q_\mu$$

- $G(z) = (1+z)$

Bipartite maps

$$\log \tau_G^{(0)} = \sum_{n\geq 1} t^n \sum_{\lambda,\mu\vdash n}\sum_{g\geq 0} H_g^{\mathrm{Bip}}(\lambda,\mu)\hbar^{\ell(\lambda)+\ell(\mu)+2g-2} p_\lambda q_\mu$$

- $G(z) = (1-z)^{-1}$

fixed in C_λ fixed in C_μ

monotone: $\tau_i = (a_i, b_i), \tau_{i+1} = (a_{i+1}, b_{i+1})$, $a_i < b_i, b_i < b_{i+1}$

double monotone Hurwitz numbers $= \#\{\sigma_1 \cdot \sigma_2 \cdot \tau_1 \cdots \tau_\ell = \mathrm{id}\}$

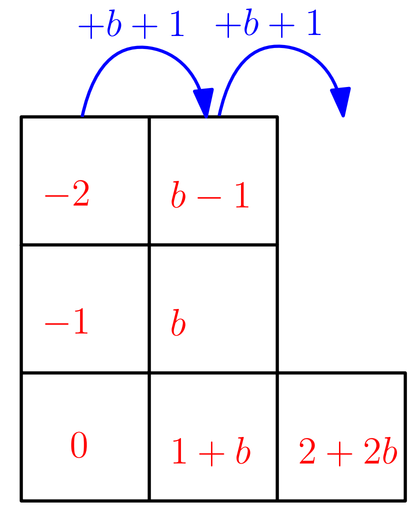
$$\log \tau_G^{(0)} = \sum_{n\geq 1} t^n \sum_{\lambda,\mu\vdash n}\sum_{\ell\geq 0} H_\ell^{\mathrm{mon}}(\lambda,\mu)\frac{\hbar^\ell}{\ell!} p_\lambda q_\mu$$

Weighted Hurwitz numbers

Theorem: [Stanley '89]

$$J_\lambda^{(1+b)}(\underline{u}) = \prod_{\square\in\lambda}(u + c_b(\square))$$

b-content of a box:



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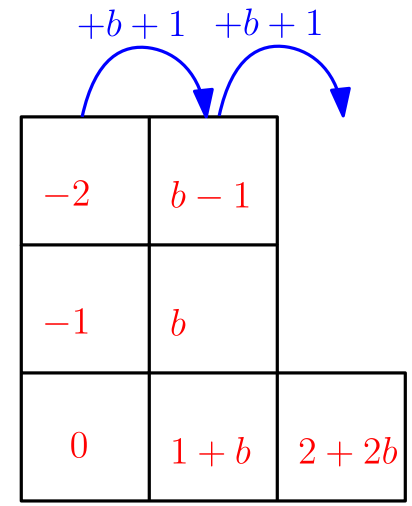

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Theorem [Chapuy, D. '20]

$$(1+b)\frac{td}{dt}\log \tau_k^{(b)} = \sum_{\mathbf{M}_\infty} \kappa_G(\mathbf{M}_\infty) t^{|\mathbf{M}_\infty|} b^{\mathrm{MON}(\mathbf{M}_\infty)},$$

rooted infinite-constellations, orientable or not with some regular condition

similar weight as before (degrees of faces + degrees of vertices labeled 0) which additionally depends on G

II. Integrable hierarchies and matrix models

KP hierarchy

Kadomtsev–Petriashvili hierarchy of partial differential equations (PDE):

- $F_{2,2} - F_{3,1} + \frac{1}{12}F_{1,1,1,1} + \frac{1}{2}F_{1,1}^2 = 0,$
- $F_{3,2} - F_{4,1} + \frac{1}{6}F_{2,1,1,1} + F_{2,1} \cdot F_{1,1} = 0,$
- $F_{4,2} - F_{5,1} + \frac{1}{4}F_{3,1,1,1} - \frac{1}{120}F_{1,1,1,1,1,1} + F_{3,1} \cdot F_{1,1} + \frac{1}{2}F_{2,1}^2 - \frac{1}{8}F_{1,1,1}^2 - \frac{1}{12}F_{1,1}F_{1,1,1,1} = 0,$
- $\dots$ where $F_{i_1,\dots,i_k} := \frac{\partial^k}{\partial p_{i_1} \cdots \partial p_{i_k}} F.$

The function $\exp(F)$ is called a tau function of the KP hierarchy

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The function $\exp(F)$ is called a tau function of the KP hierarchy

physics: studying colliding ocean waves

mathematical physics: Witten–Kontsevich theorem $\to$ the g.f. of intersection numbers is a tau-function of KdV hierarchy (KP which depends only on $p_1, p_3, p_5, \dots$)

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Theorem: [Okounkov '00, Orlov '00] + [Harnad, Guay-Paquet '15]

$\tau_{e^z}^{(0)}$ is a tau function of the KP hierarchy.

$\tau_G^{(0)}$ is a tau function of the KP hierarchy.

KP hierarchy

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Theorem: [Okounkov '00, Orlov '00] + [Harnad, Guay-Paquet '15]

$\tau^{(0)}_{e^z}$ is a tau function of the KP hierarchy.

$\tau^{(0)}_{G}$ is a tau function of the KP hierarchy.

Theorem: [Kazarian, Lando '06]

this result implies Witten–Kontsevich theorem.

KP hierarchy

Kadomtsev–Petriashvili hierarchy of partial differential equations (PDE):

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Theorem: [Okounkov '00, Orlov '00] + [Harnad, Guay-Paquet '15]

$\tau_{e^z}^{(0)}$ is a tau function of the KP hierarchy.

$\tau_G^{(0)}$ is a tau function of the KP hierarchy.

Question: Is there a similar result for $\tau_G^{(b)}$? Or at least for some special choices of G?? Or at least for some special choices of G and $b \neq 0$???

BKP hierarchy

- for general b we don't know any integrable hierarchy that will work...
- interesting case is $b = 1$:

  $\rightarrow$ for the classical case of bipartite maps $G(z) = 1 + z$ the specialization $q_i = \delta_{i=2}$ corresponds to maps and has a "matrix model":

$$\tau_G^{(b)}(q_i = \delta_{i=2}) \overset{\hbar = 1/N}{=} c_{N,b} \cdot \int_{\mathbb{R}^N} \prod_{j=1}^{N} \exp\left(\frac{-x_j^2}{2+2b} + \sum_{i \geq 1} \frac{p_i \cdot x_j^i}{i}\right) \prod_{1 \leq i < j \leq N} |x_i - x_j|^{2/(1+b)} dx$$

Gaussian β Ensemble [Okounkov '97]

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Gaussian β Ensemble [Okounkov '97]

$\rightarrow$ for $b = 1$ this corresponds to moments of Gaussian Orthogonal Ensembles:

$$\tau_G^{(1)}(q_i = \delta_{i=2}) = c_{N,b} \cdot \int_{\mathcal{H}(N)} \exp\left(\frac{-\operatorname{Tr}(H^2)}{4} + \sum_{i\geq 1} \frac{p_i \cdot \operatorname{Tr}(H^i)}{i}\right) dH$$

random symmetric matrices with i.i.d. normal entries

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Theorem: [van de Leur '01]

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$\rightarrow$ tau function of k-constellations **is not a tau function of the BKP hierarchy** for $k > 1$! (reason: no matrix model?) Matrix model $\rightarrow$ monotone Hurwitz!

HCIZ and BGW integrals via monotone Hurwitz numbers

$A, B, C \in \mathrm{GL}_N(\mathbb{C})$, $U(N)$ - unitary group

- $$\mathrm{HCIZ}_{U(N)}(A,B) := \int_{U(N)} \exp(t\,\mathrm{Tr}(AUBU^\dagger))dU$$

  Harish-Chandra–Itzykson–Zuber integral (depends only on the eigenvalues $(a_1, \ldots, a_N)$ and $(b_1, \ldots, b_N)$ of A, B

- $$\mathrm{BGW}_{U(N)}(C) := \int_{U(N)} \exp(\sqrt{t}\,\mathrm{Tr}(UC + C^\dagger U^\dagger))dU$$

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Theorem: [Harish-Chandra, Itzykson–Zuber] [Mironov–Mrozov–Semenoff]

We have the following explicit determinantal formulas:

Vandermonde determinant: $\prod_{i<j}(x_i - x_j)$

$$\mathrm{HCIZ}_{U(N)}(a_1,\dots,a_N,b_1,\dots,b_N) = \frac{\det\left(\exp(ta_ib_j)\right)_{1\le i,j\le N}}{V\left(a_1\sqrt{t},\dots,a_N\sqrt{t}\right)V\left(b_1\sqrt{t},\dots,b_N\sqrt{t}\right)}\cdot\prod_{i=1}^{N-1} i!$$

$$\mathrm{BGW}_{U(N)}(c_1,\dots,c_N) = \frac{\det\left((tc_i)^{\frac{n-j}{2}}I_{n-j}(2\sqrt{tc_i})\right)_{1\le i,j\le N}}{V\left(c_1^2t,\dots,c_N^2t\right)}\cdot\prod_{i=1}^{N-1} i!$$

$n-j$-th modified Bessel function of the first kind

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Theorem: [Collins, Mironov–Mrozov–Semenoff, Novak]

Topological expansion of HCIZ and BGW integrals is given by double and single monotone Hurwitz numbers:

$$\mathrm{HCIZ}_{U(N)}(a_1,\dots,a_N,b_1,\dots,b_N) \approx$$
$$\sum_{n\geq 0} t^n \sum_{\lambda,\mu\vdash n} \sum_{\ell\geq 0} H^{\mathrm{mon}}_{\ell,\bullet}(\lambda,\mu)\frac{1}{(-N)^\ell} p_\lambda(a_1,\dots,a_N) q_\mu(b_1,\dots,b_N)$$

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HCIZ and BGW integrals via monotone Hurwitz numbers

Let $\tau_G^{(b)}$ be the tau function of b-deformed **monotone** Hurwitz numbers (i.e. $G = \frac{1}{1-z}$)

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Topological expansion of $\beta - \mathrm{HCIZ}$ and $\beta - \mathrm{BGW}$ integrals (introduced by [Brézin–Hikami '03]) is given by double and single b-monotone Hurwitz numbers with

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$b = 1 \equiv \beta = 1$: $A, B \in \mathrm{Sym}_N(\mathbb{R}), C \in M_N(\mathbb{R})$, $O(N)$ - orthogonal group

- $$\mathrm{HCIZ}_{O(N)}(A,B) := \int_{O(N)} \exp(\tfrac{t}{2}\,\mathrm{Tr}(AOBO^\dagger))dO$$
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Question: Is it true that $\mathrm{BGW}_{O(N)}(c_1,\ldots,c_N)$ is a tau-function of the **BKP hierarchy**?

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YES

BKP hierarchy

BKP hierarchy of [van de Leur, Kac '97] of partial differential equations (PDE):

- $-F_{3,1}(N)+F_{2^2}(N)+\frac{1}{2}F_{1^2}(N)^2+\frac{1}{12}F_{1^4}(N) = S_2(N)\tau(N-2)\tau(N+2)\tau(N)^{-2}$,
- $-2F_{4,1}(N)+2F_{3,2}(N)+2F_{2,1}(N)F_{1^2}(N)+\frac{1}{3}F_{2,1^3}(N) = S_2(N)\frac{\tau(N-2)\tau(N+2)}{\tau(N)^2}(F_1(N+2) - F_1(N-2))$,
- $-6F_{5,1}(N)+4F_{4,2}(N)+2F_{3^2}(N)+4F_{3,1}(N)F_{1^2}(N)+\frac{2}{3}F_{3,1^3}(N)+4F_{2,1}(N)^2+ 2F_{2^2}(N)F_{1^2}(N)+F_{2^2,1^2}(N)+\frac{1}{3}F_{1^2}(N)^3+\frac{1}{6}F_{1^4}(N)F_{1^2}(N)+\frac{1}{180}F_{1^6}(N) = S_2(N)\frac{\tau(N-2)\tau(N+2)}{\tau(N)^2}\big(F_{1^2}(N+2) + F_{1^2}(N-2) + 2F_2(N+2) - 2F_2(N-2) + (F_1(N+2) - F_1(N-2))^2\big)$,
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The function $\tau(N) := \exp(F)$ is called a tau function of the BKP hierarchy

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- $\dots$

The function $\tau(N) := \exp(F)$ is called a tau function of the BKP hierarchy

Step 1 check that you function satisfies the BKP equation (computer simulation). If yes, there is a big chance you have a tau function of the BKP hierarchy, but how to prove it?

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- $\dots$

Proposition: [Bonzom, Chapuy, D. '21] (folklore?)

Let $\tau(N) = \sum_\lambda a_\lambda(N) s_\lambda t^{|\lambda|} \in \mathbb{C}[\mathbf{p}][[t]]$ and $(A_{i,j})_{i,j\in\mathbb{Z}}$ s.t.

$$a_\lambda(N) = \begin{cases} \mathrm{Pf}(A_{\lambda_i+N-i,\lambda_j+N-j})_{1\le i,j\le N} & \text{for } N \text{ even,} \\ \mathrm{Pf}(A_{\lambda_i+N-i,\lambda_j+N-j})_{1\le i,j\le N+1} & \text{for } N \text{ odd.} \end{cases} \quad \forall \ell(\lambda)\le N$$

Then $\tau(N)$ is a tau function of the BKP hierarchy.

Pfaffian

Theorem: [Cayley '1848]

Let $A = (a_{i,j})_{1\le i,j\le 2n}$ be a skew-symmetric $2n \times 2n$ matrix (i.e. $A^T = -A$).
There exists a polynomial $\mathrm{Pf}(A) \in \mathbb{C}[a_{i,j} : 1 \le i,j \le 2n]$ s.t.

$$\det(A) = \mathrm{Pf}(A)^2.$$

$$\mathrm{Pf}(A) = \sum_{\sigma\in S_{2n}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(2i-1),\sigma(2i)}.$$

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How to prove that your function is a tau function of the BKP?

Step 1 check that you function satisfies the BKP equation (computer simulation). ✓

Step 2 expand your function into Schur (computer simulation) and try to prove that the coefficients are Pfaffians. ✓

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the dimension of the irreducible representation of the orthogonal group $O(z)$ as a rational function of z

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Step 5 prove the formula you've guessed!

almost the end of the story...

Theorem: [Bonzom, Chapuy, D. '21] (BKP hierarchy for non-oriented monotone Hurwitz numbers)

$$\prod_{i-1}^{N-1}(2i)!^{-1} \cdot \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{Z_\lambda(2\mathbf{p})}{\mathrm{hook}_{2\lambda}} \prod_{\square\in\lambda} \frac{1}{2N+c_1(\square)}$$

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Theorem: [Harish-Chandra '57]

- G: compact, connected semisimple Lie group;
- $x, y \in \mathfrak{h}$ (Cartan of $\mathrm{Lie}(G)$),
- W: the associated Weyl group.
- $\langle , \rangle$: Ad-invariant inner product.
- $\Delta_\mathfrak{g}(x) = \prod_{\alpha\in\Phi_+}\langle \alpha, x\rangle$.

$$\Delta_\mathfrak{g}(x)\Delta_\mathfrak{g}(y)\int_G \exp\left(\langle \mathrm{Ad}_\mathfrak{g}\, x, y\rangle\right)dg = \frac{const(\mathfrak{g})}{|W|}\sum_{w\in W}\mathrm{sgn}(w)\exp\left(\langle w(x), y\rangle\right)$$

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is a $\tau(N)$ function of the BKP hierarchy (with an explicit formula).

Theorem: [Harish-Chandra '57]

- G: compact, connected semisimple Lie group;
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- $\langle,\rangle$: Ad-invariant inner product.
- $\Delta_\mathfrak{g}(x) = \prod_{\alpha\in\Phi_+}\langle\alpha, x\rangle$.

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Theorem: [Bonzom, Chapuy, D. '21]

Let $C \in M_{2N}(\mathbb{R})$ has a double spectrum: $C = (c_1, c_1, c_2, c_2, \ldots, c_N, c_N)$. Then

explicit matrix involving modified Bessel functions of the first kind.

$$\mathrm{BGW}_{O(2N)}(c_1, c_1, \ldots, c_N, c_N) = \frac{\mathrm{Pf}\left(M(t, c_i^2, c_j^2)\right)_{1\leq i,j\leq N}}{V\left(c_1^2 t, \ldots, c_N^2 t\right)} \cdot \prod_{i=1}^{N-1}(2i)!$$

Something that we don't understand

- Let $C \in \mathrm{GL}_N(\mathbb{R})$. We know that

$$\int_{O(N)} \exp\left(\sqrt{t}\,\mathrm{Tr}(CO)\right) dO = \sum_{n \geq 0} t^n \sum_{\lambda \vdash n} \frac{s_\lambda\left(\mathbf{p}/2(CC^T)\right)}{\mathrm{hook}_\lambda^2\, o_\lambda(1^N)}$$

  What is a direct explanation of this identity?

- We have the BKP hierarchy for three classical models of non-oriented weighted Hurwitz numbers with very different proofs. What is a better reason for the BKP? Where else it occurs?

- What about the integrability for arbitrary b?

THANK
YOU!

References:

- arXiv:2004.07824
- arXiv:2109.01499

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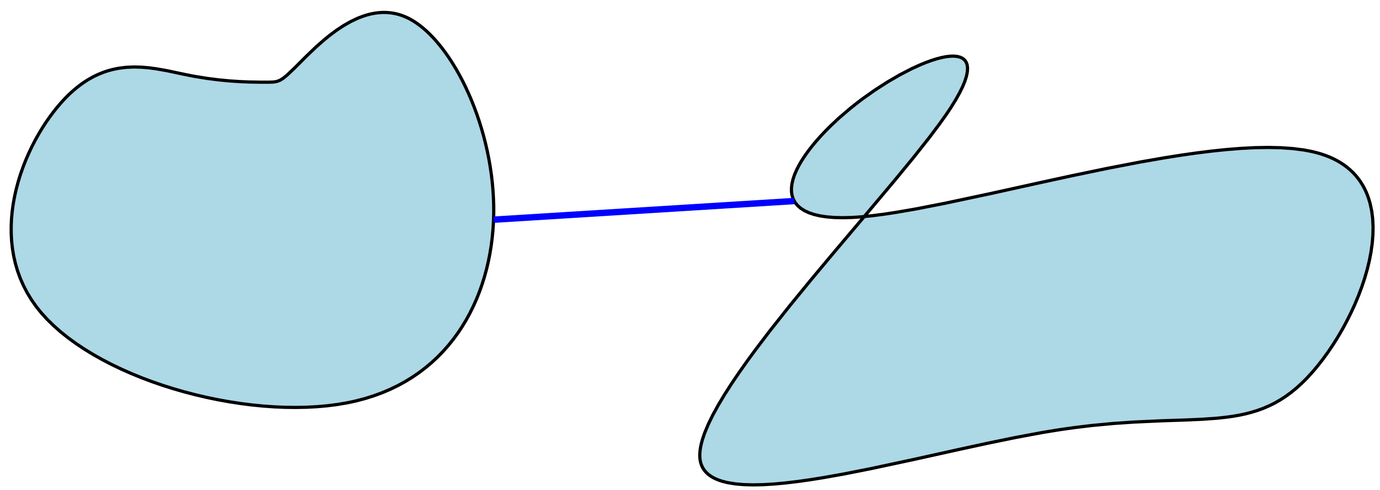

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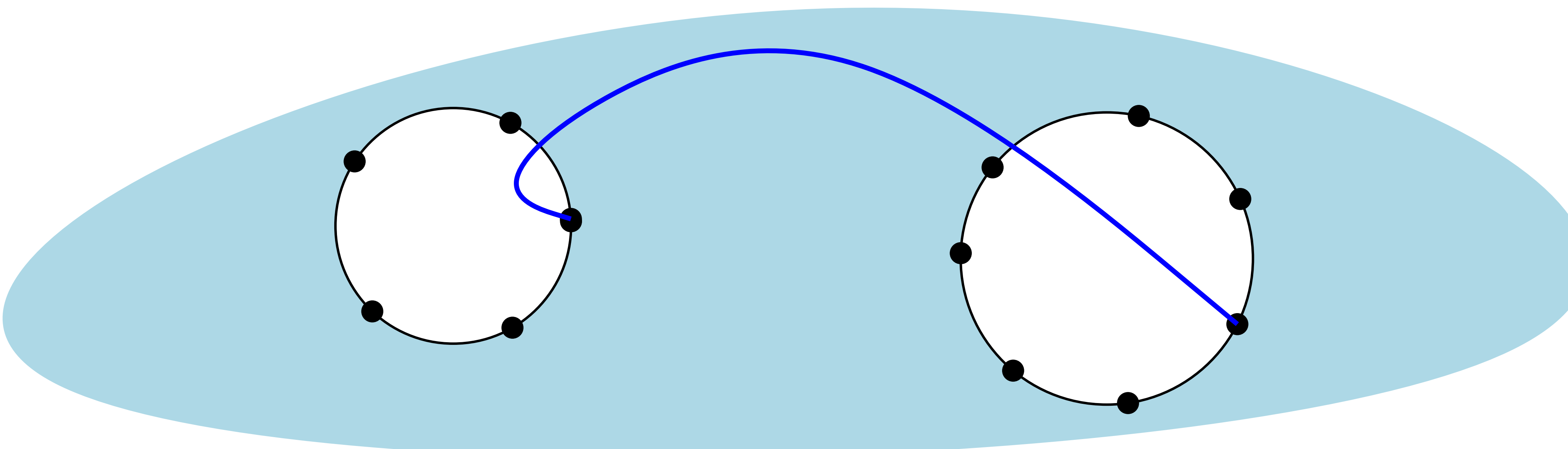

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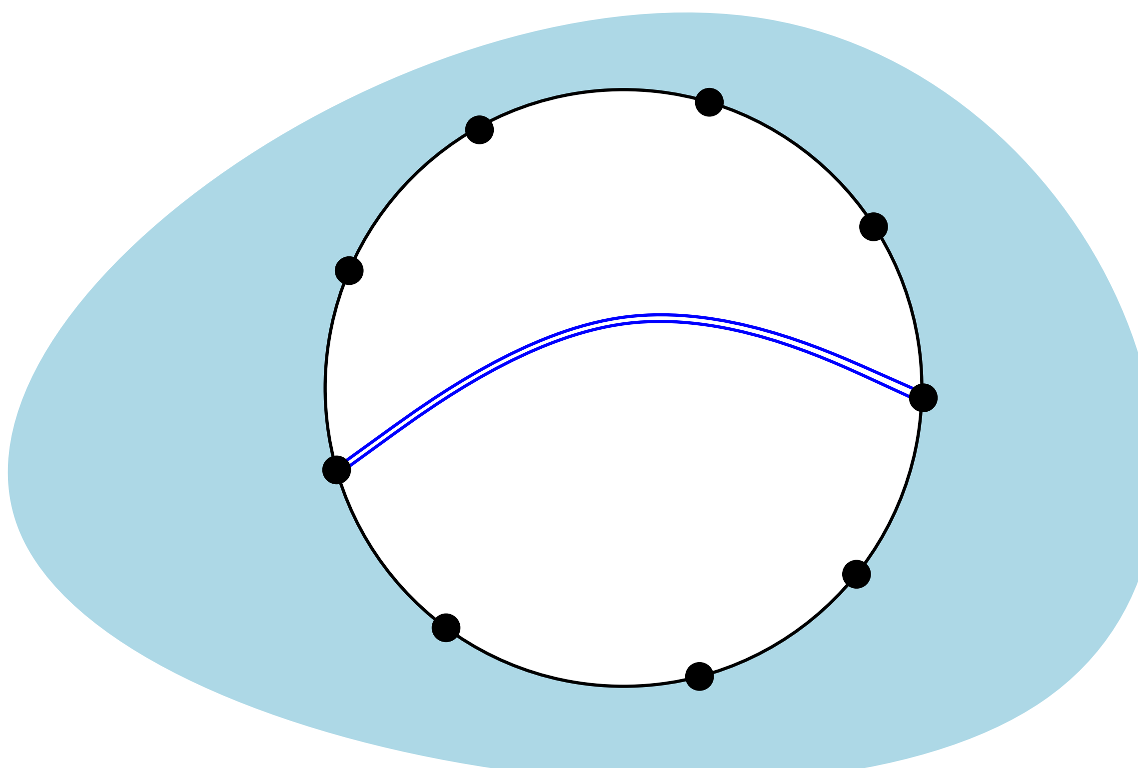

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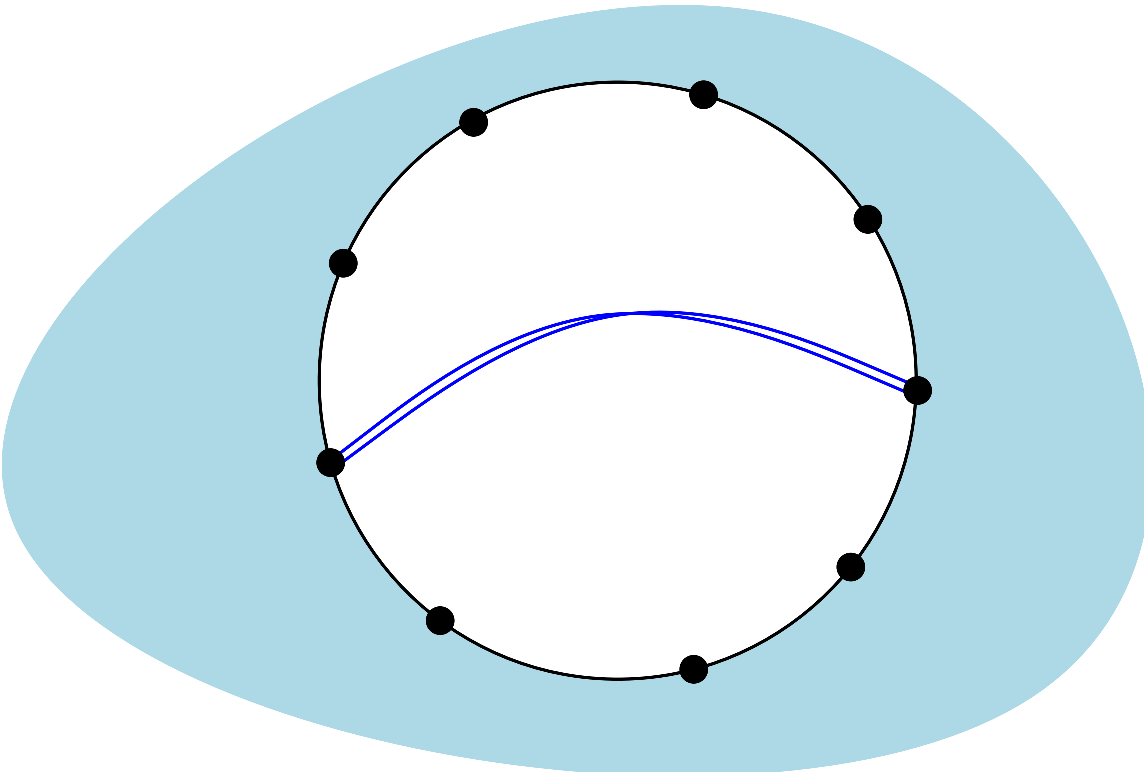

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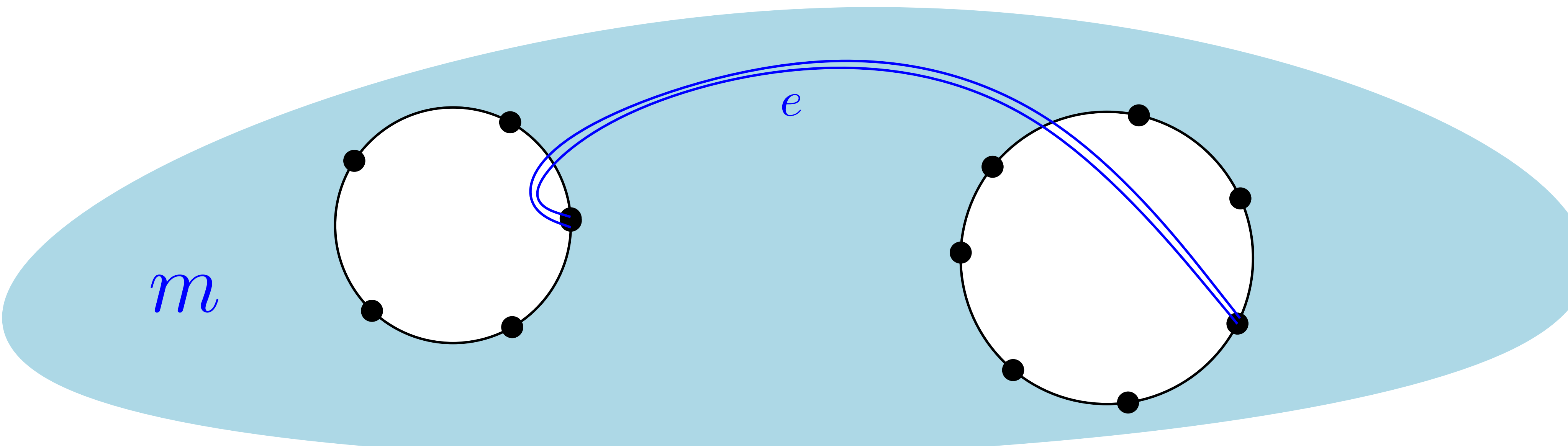

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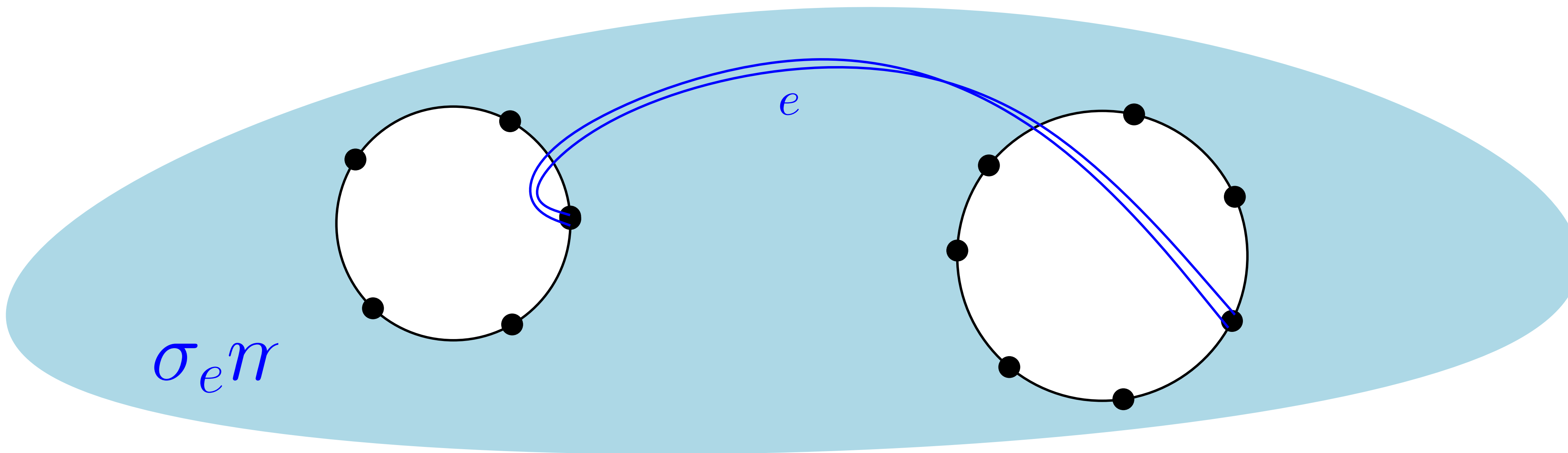

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