A bijection for rooted maps on general surfaces

Maciej Dołęga, LIAFA, Université Paris Diderot & Uniwersytet Wrocławski

joint work with

Guillaume Chapuy, CNRS & LIAFA, Université Paris Diderot

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I. Maps

Maps

= graphs embedded into a surface in a way that the complement of the image is homeomorphic to the collection of open discs called faces

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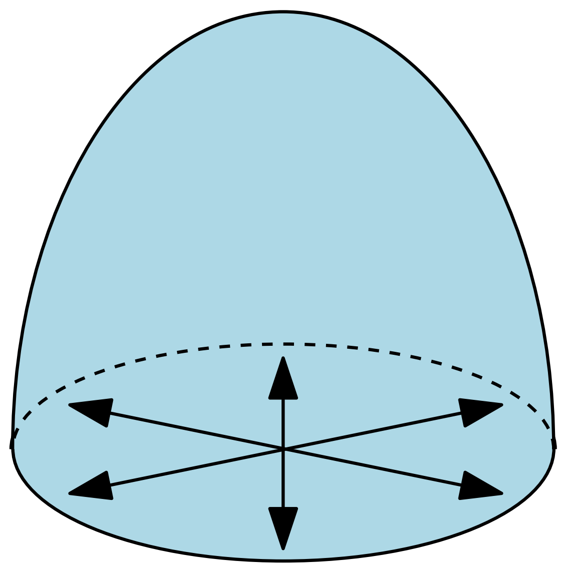
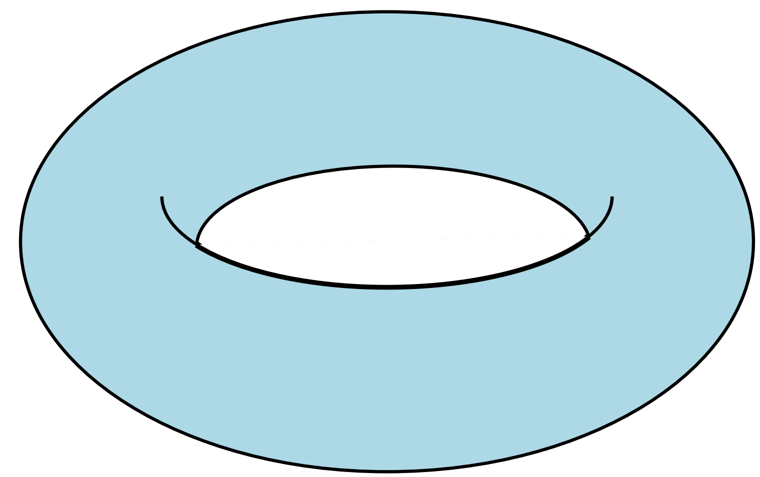


Projective plane



Torus

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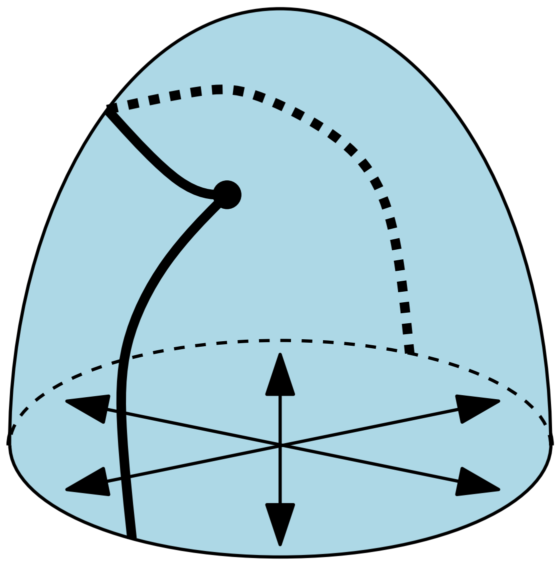
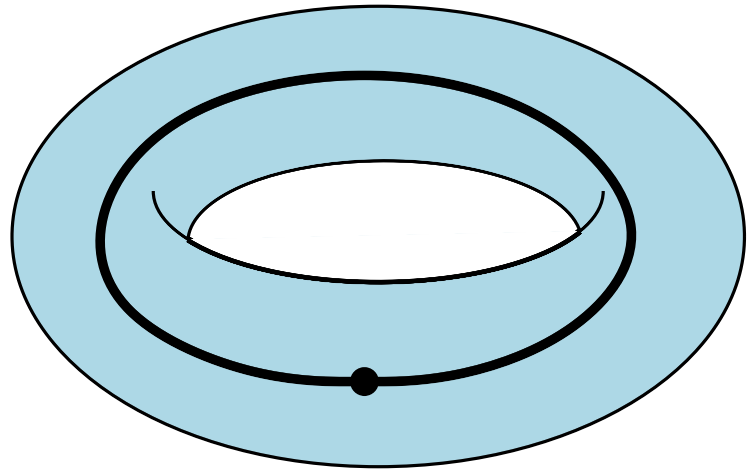


This is a map



This is not a map!

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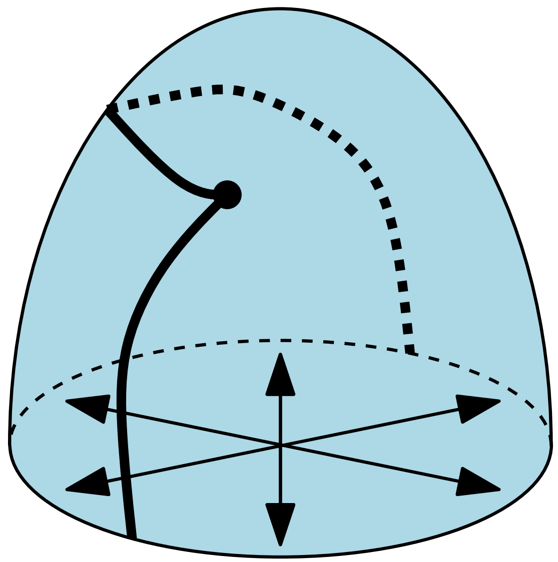

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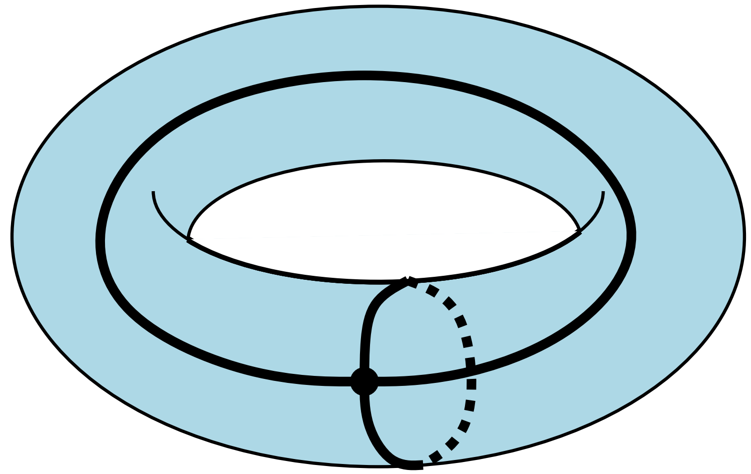

This is a map too.

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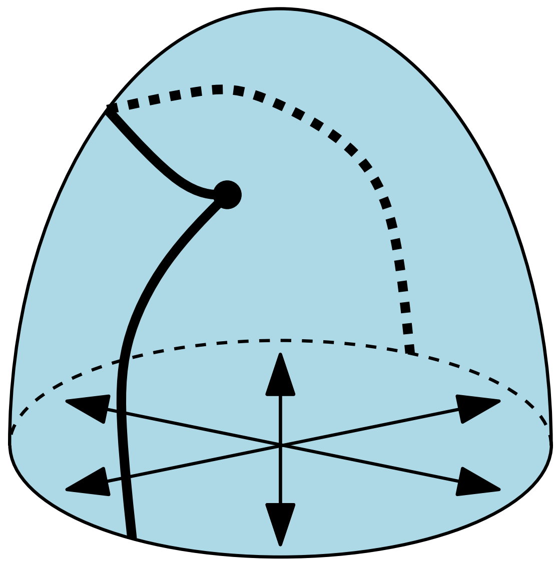

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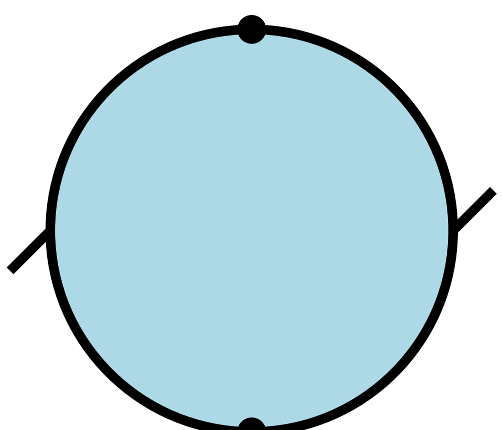

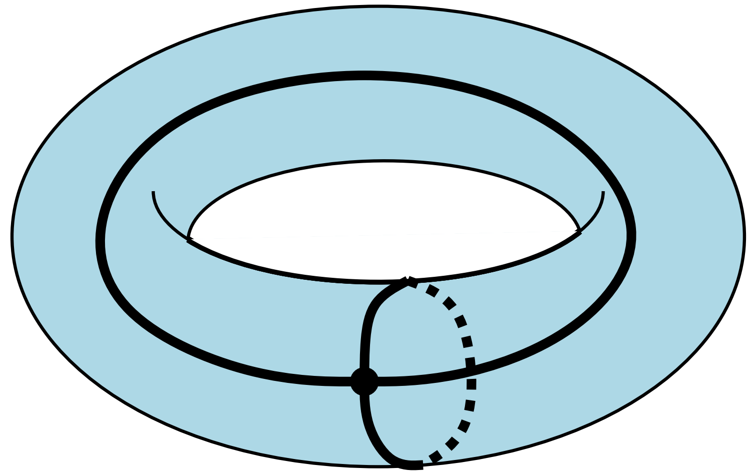

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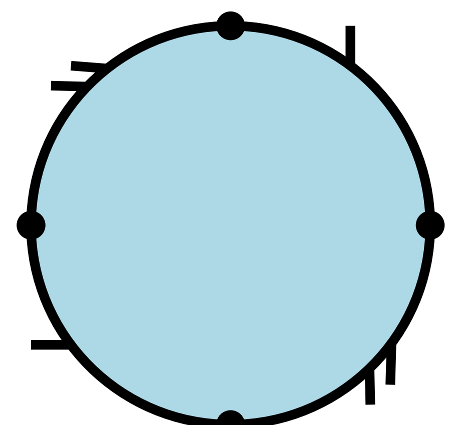

Orientable vs. non-orientable

Surfaces are classified by their Euler characterisitc: $\chi(\mathbb{S})$. The number g is the type of surface $\mathbb{S}$ if $\chi(\mathbb{S}) = 2 - 2g$. Surfaces can be:

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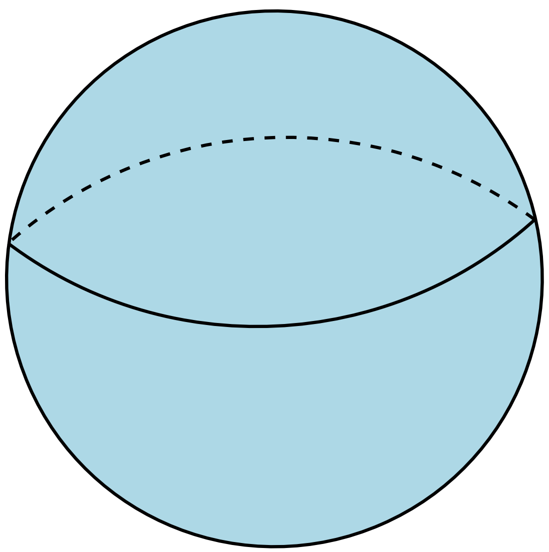
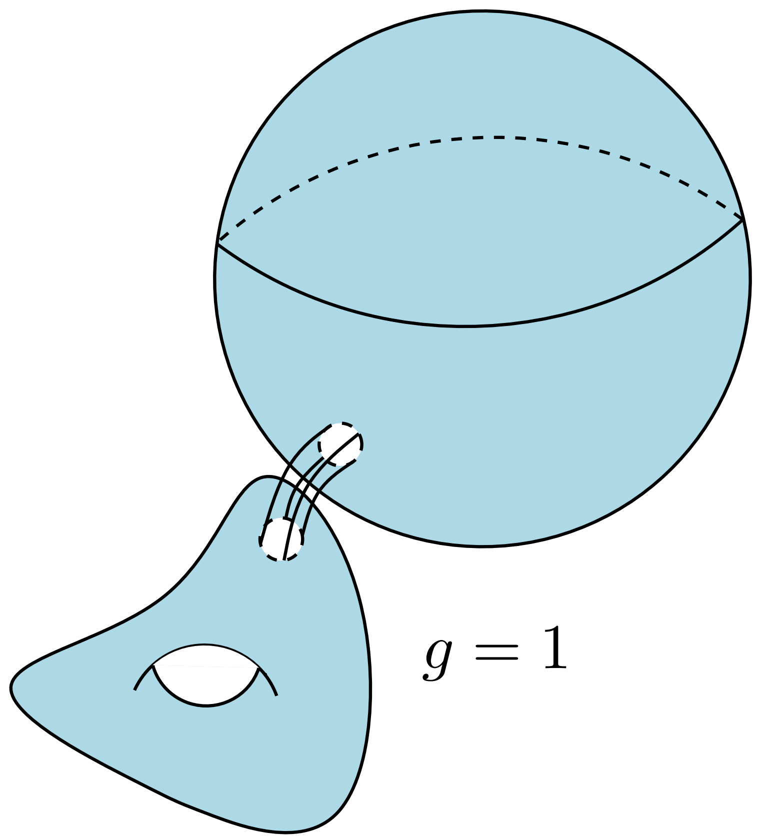
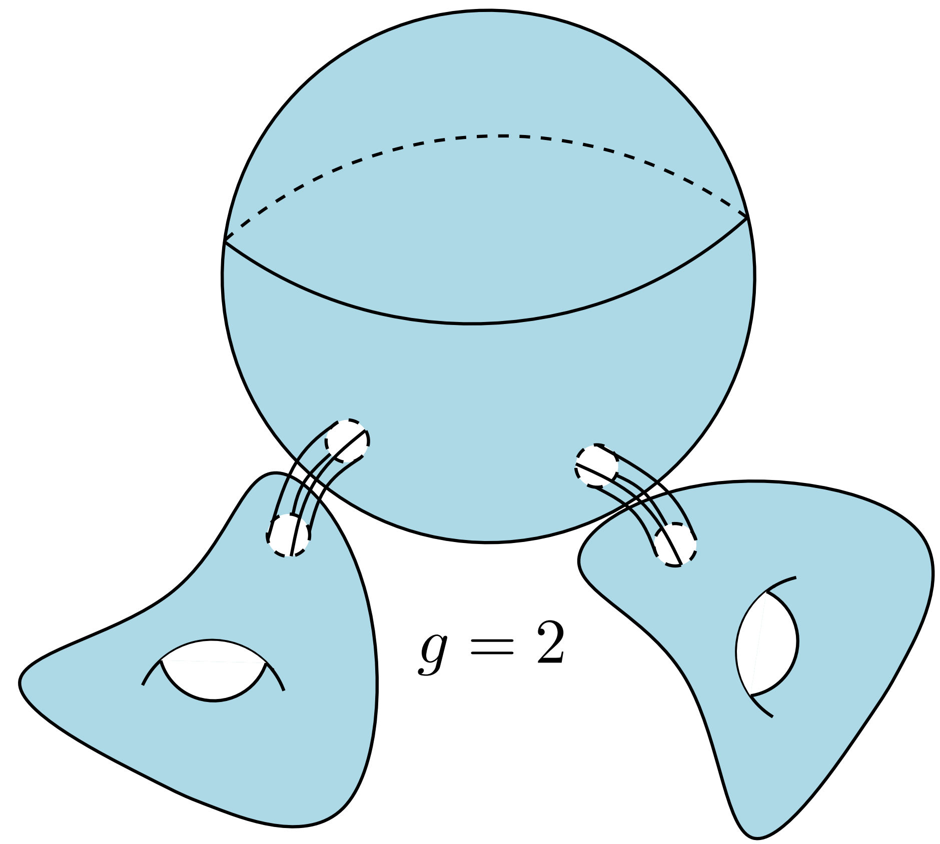


$g = 0$



$g = 1$



$g = 2$

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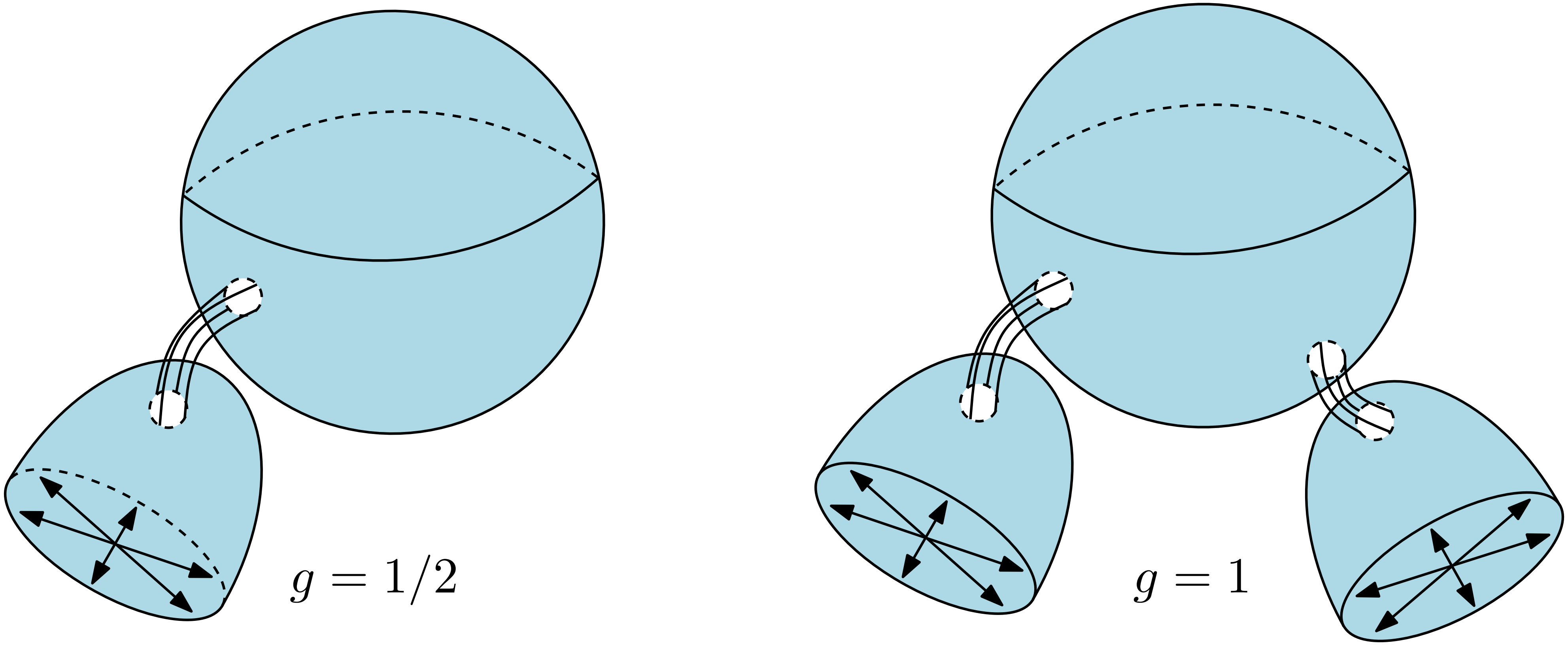

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We will say that a map M is orientable/non-orientable of type g if the underlying surface is orientable/non-orientable of type g.

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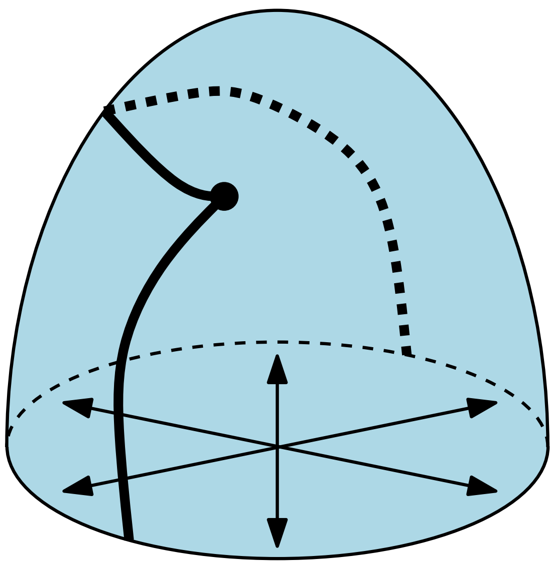
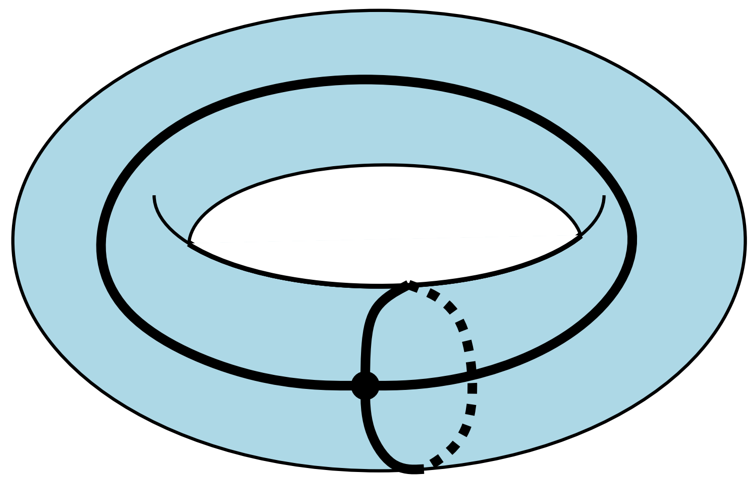


Non-orientable map of type $1/2$



Orientable map of type 1

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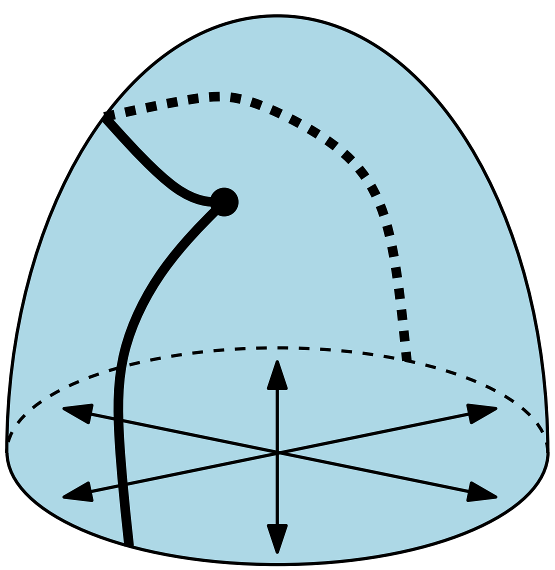
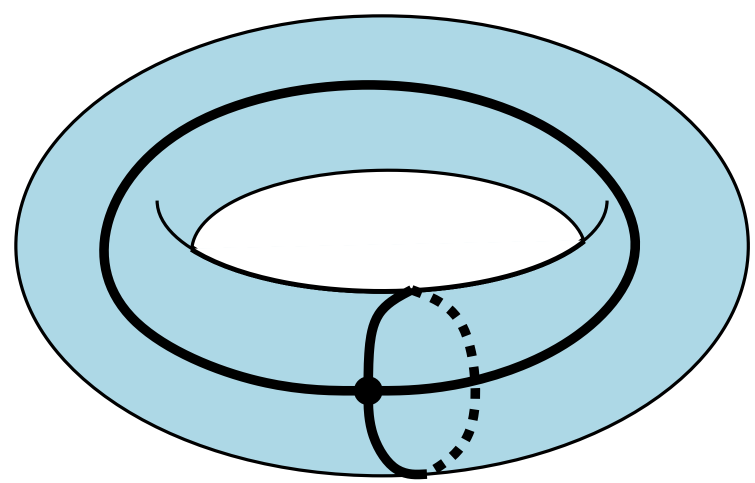
$$\chi(M) = 2 - 2g$$



$\chi(M) = ?$



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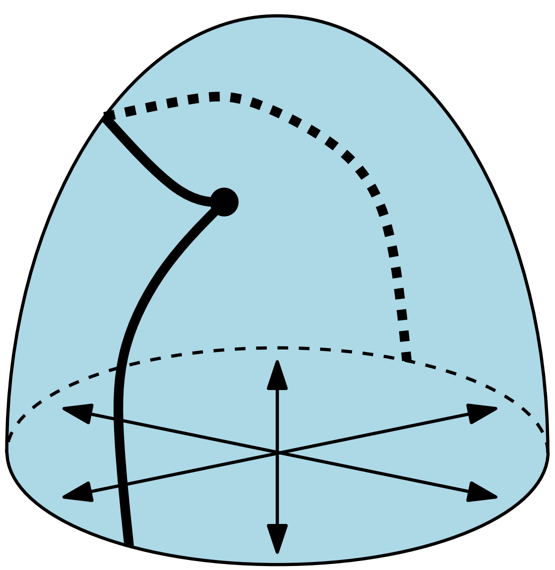


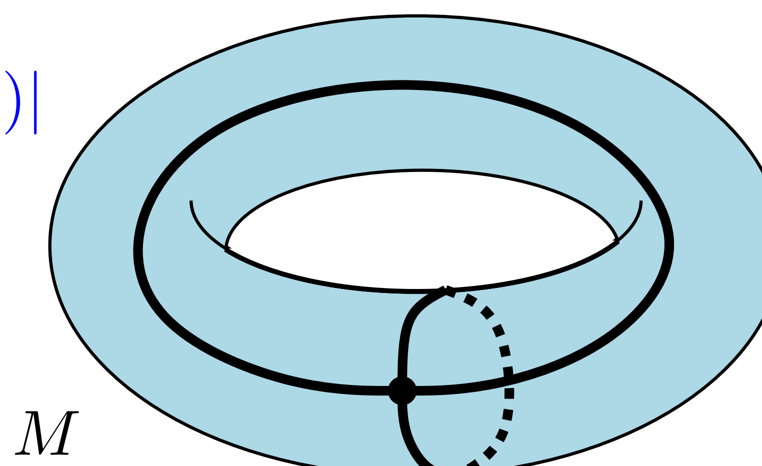
$\chi(M) = 1$

$$\chi(M) = 2 - 2g$$
$$= |F(M)| - |E(M)| + |V(M)|$$

faces of M

edges of M

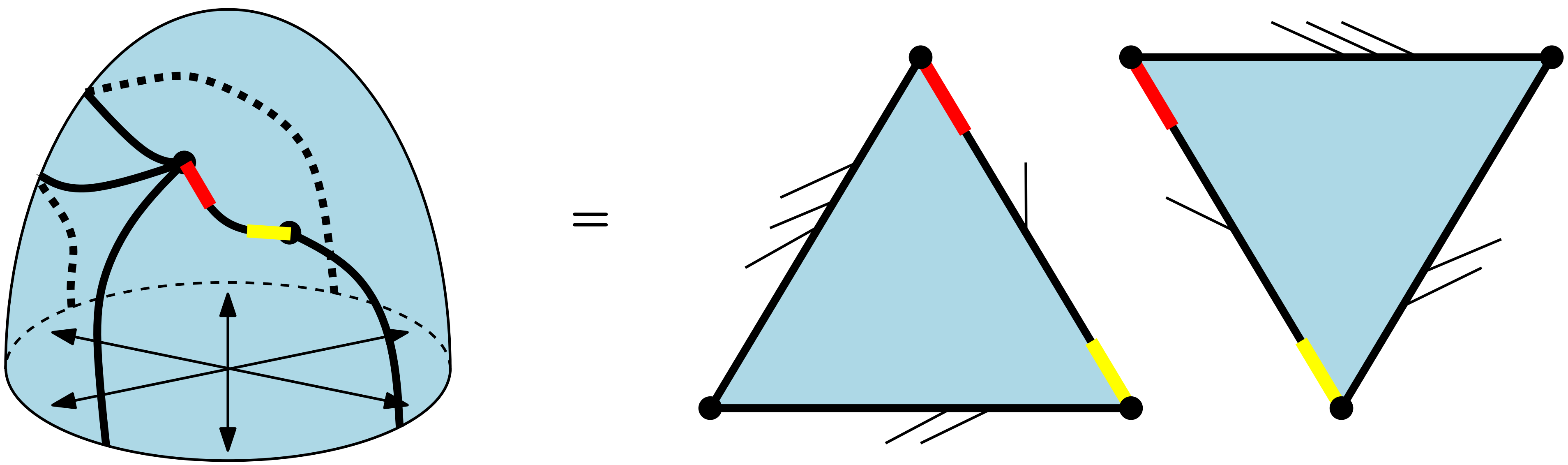
vertices of M



$\chi(M) = 0$

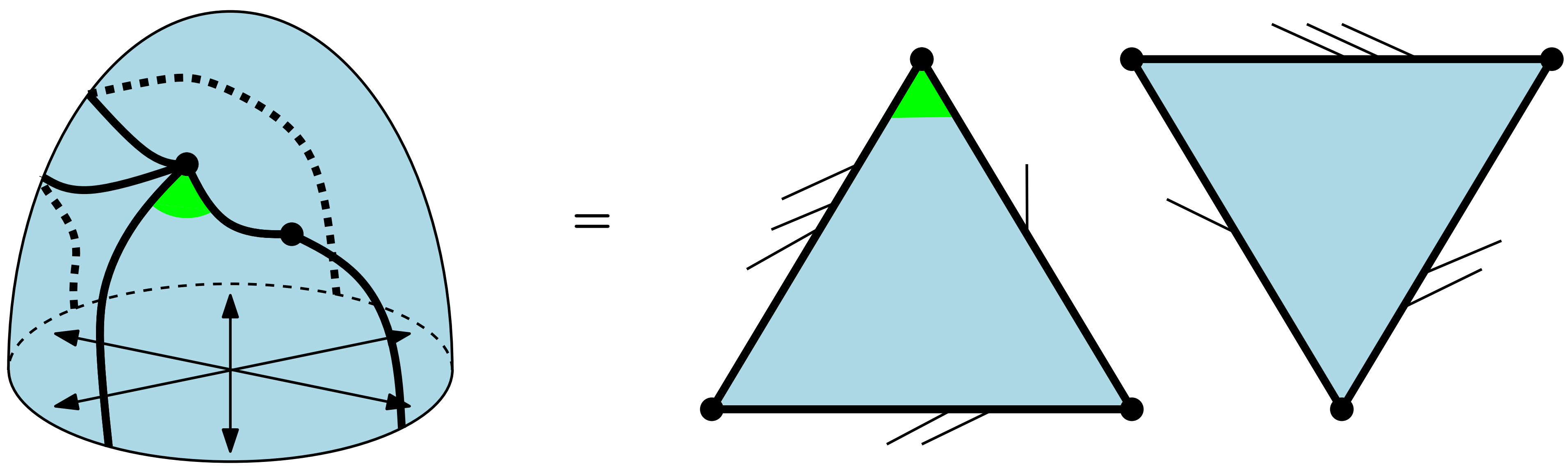
Rooted maps

Each edge consists of two half-edges.



Rooted maps

Each edge consists of two half-edges. A region between two consecutive half-edges attached to a vertex is called corner.



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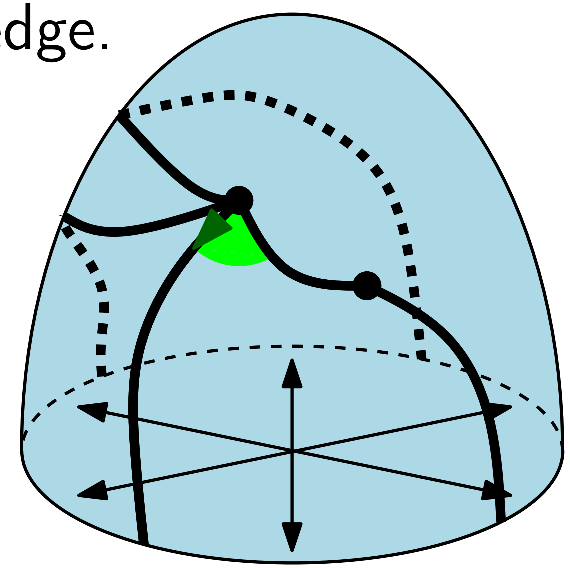
Each edge consists of two half-edges. A region between two consecutive half-edges attached to a vertex is called corner. A map is rooted if it is equipped with a distinguished half-edge (called the root), together with a distinguished side of this half-edge.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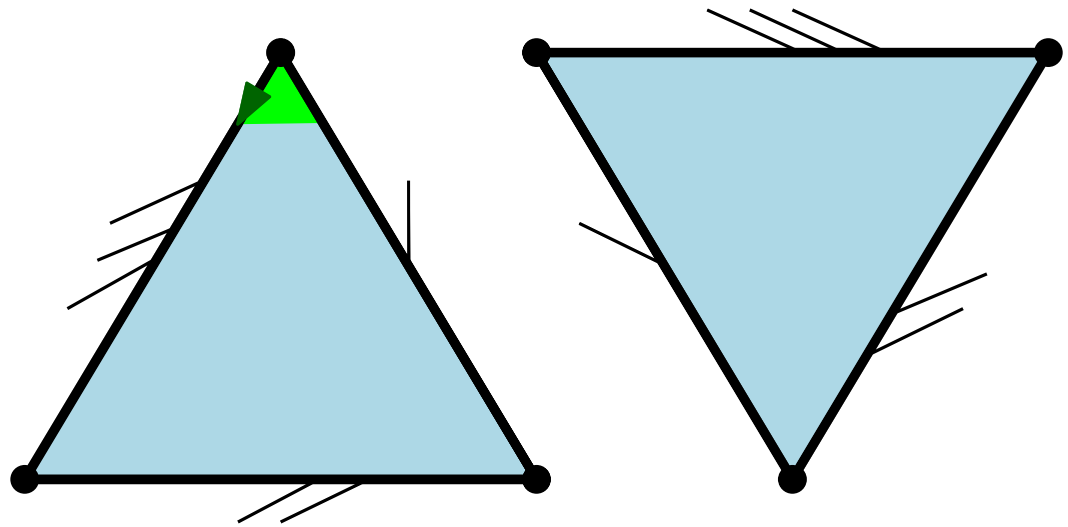

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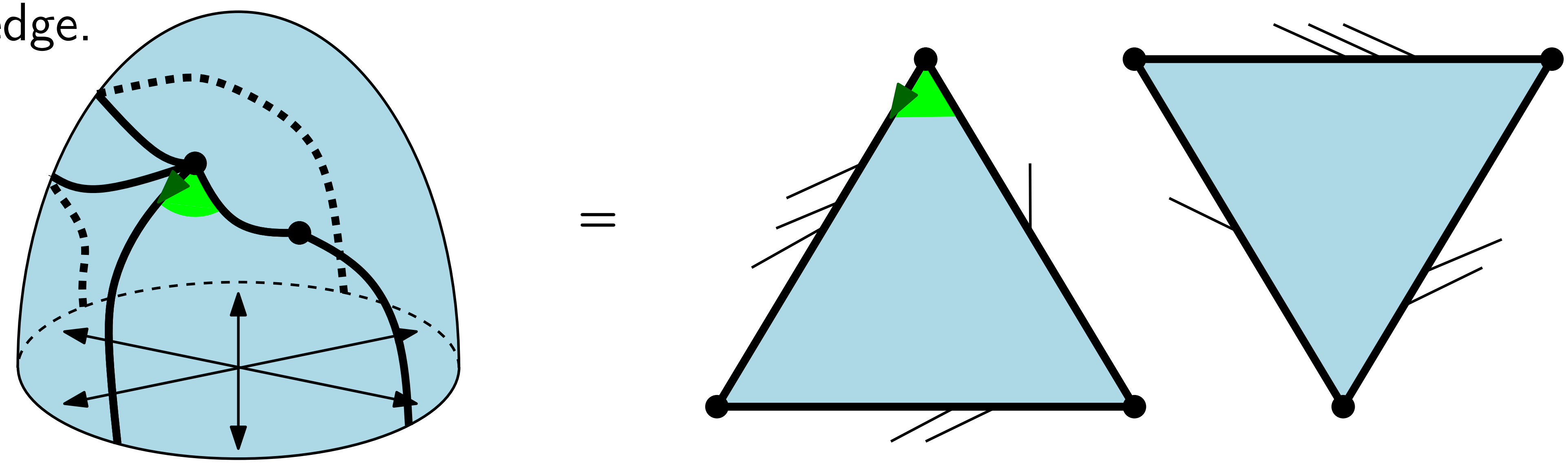

Degree of the:

• vertex is the number of half-edges incident to it,

• face is the number of edges that belong to that face (some edges can be counted twice!) = number of corners that belong to that face.

Remark:

$$\sum_{f \in F(M)} \deg(f) = \sum_{v \in V(M)} \deg(v) = 2|E(M)|.$$

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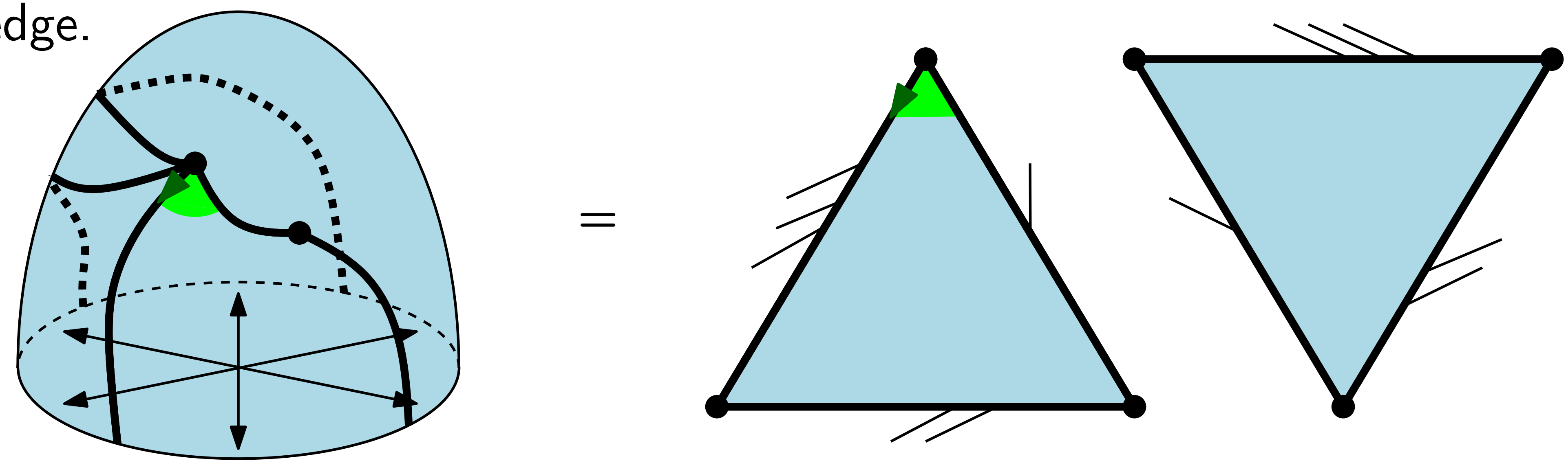

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Remark:
Tutte noticed that maps are much simpler to enumerate, when rooted, because of the lack of symmetry. From now on, all maps will be rooted!

Maps with n edges vs. bipartite quardangulations with n faces

Map M is bipartite if vertices can be colored by two different colors ($V_\bullet(M)$ - set of black vertices, $V_\circ(M)$ - set of white vertices, root vertex is black by convention) such that each edge connects two vertices of different colors.

Quadrangulation is a map with all faces of degree 4.

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Theorem [Tutte 1960]
There is a bijection between

- the set of rooted maps on $\mathbb{S}$ with n edges, l vertices and k faces of degree $\lambda_1, \dots, \lambda_k$,
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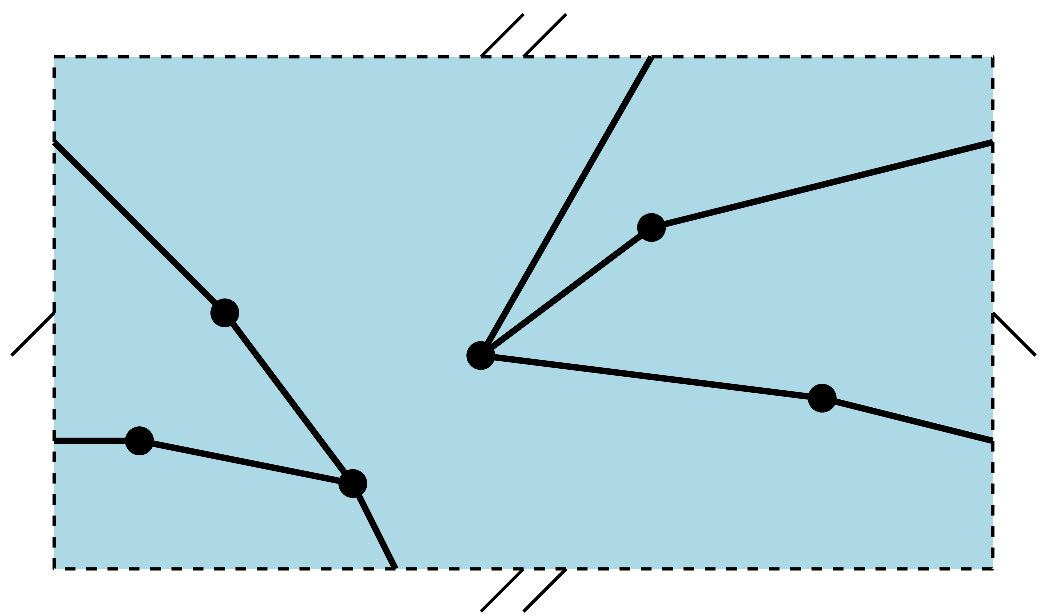

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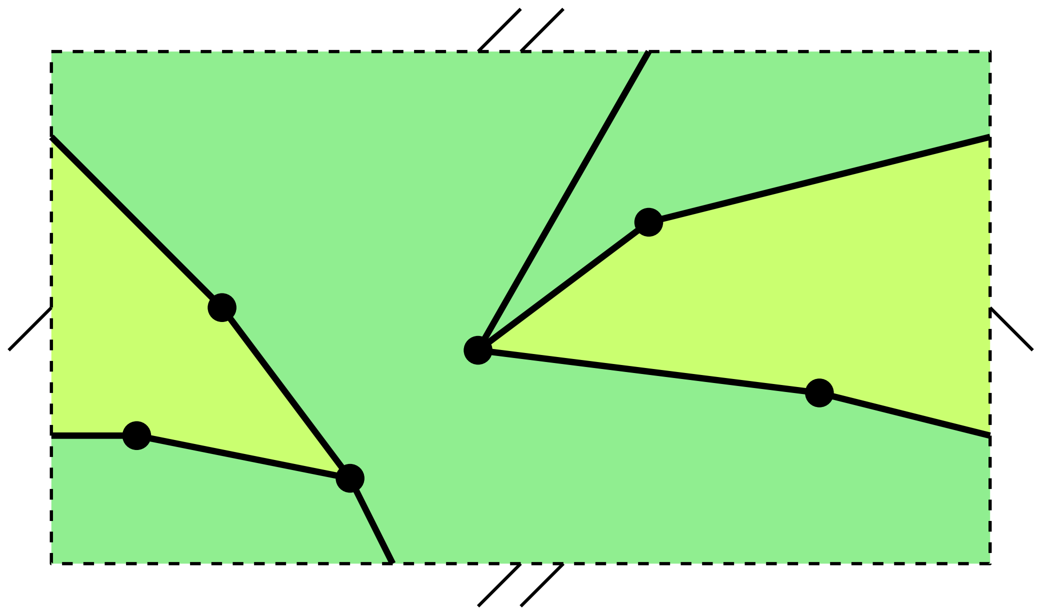

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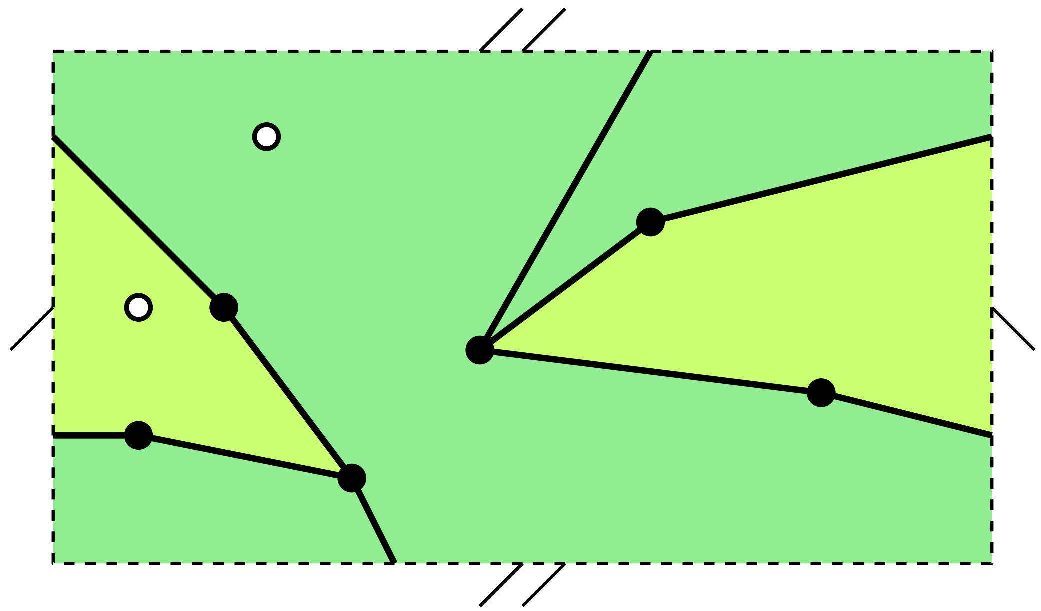

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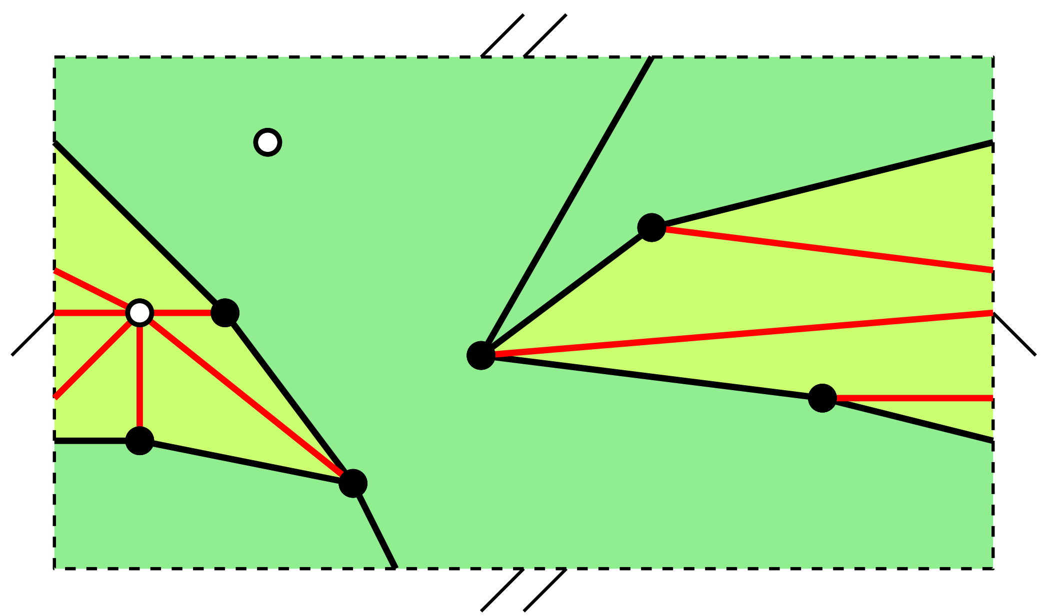

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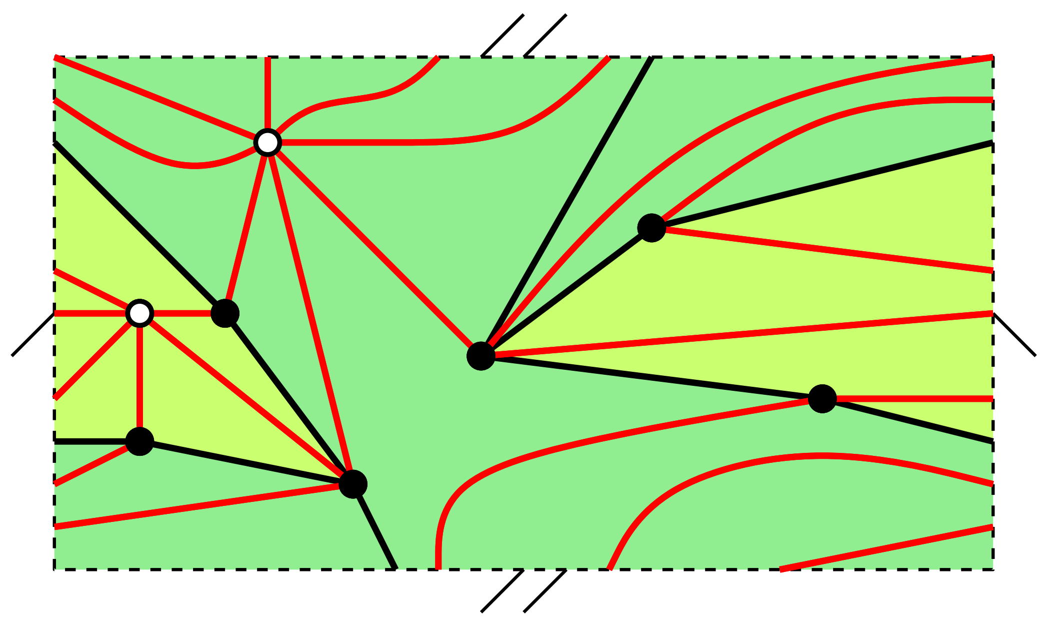

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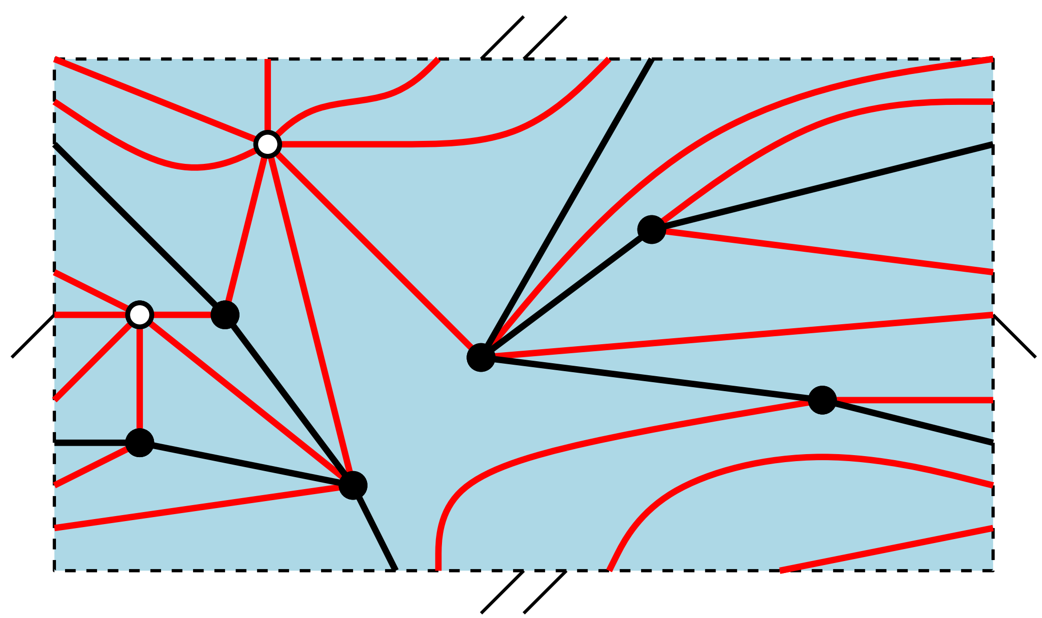

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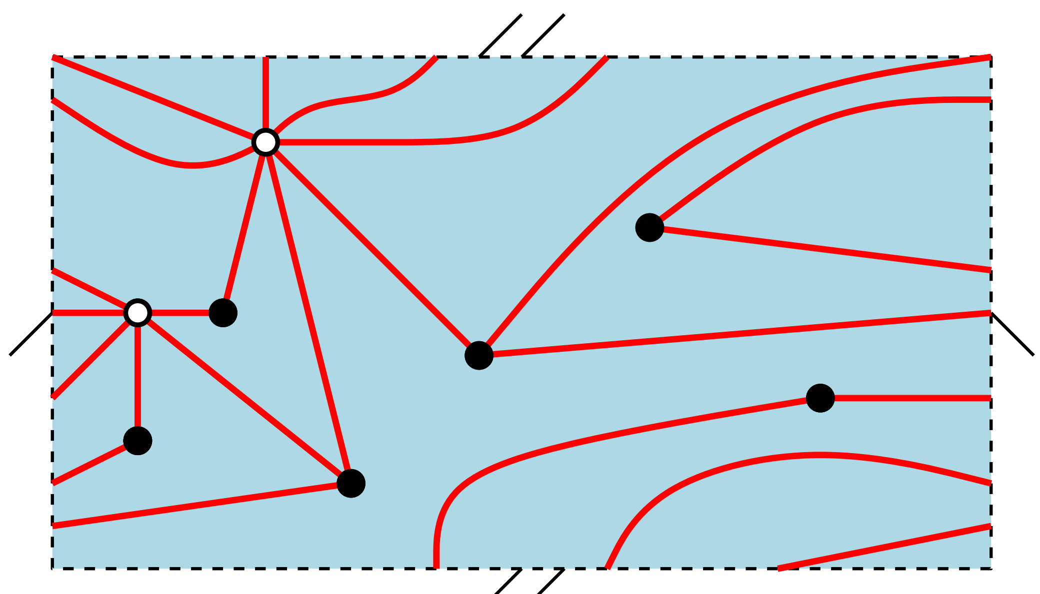

II. Bijections for bipartite quadrangulations

Labeled and well-labeled maps

A map is called labeled if its vertices are labeled by integers such that:

- the root vertex has label 1;
- if two vertices are linked by an edge, their labels differ by at most 1.

If in addition we have:

- all the vertex labels are positive,

then the unicellular map is called well-labeled.

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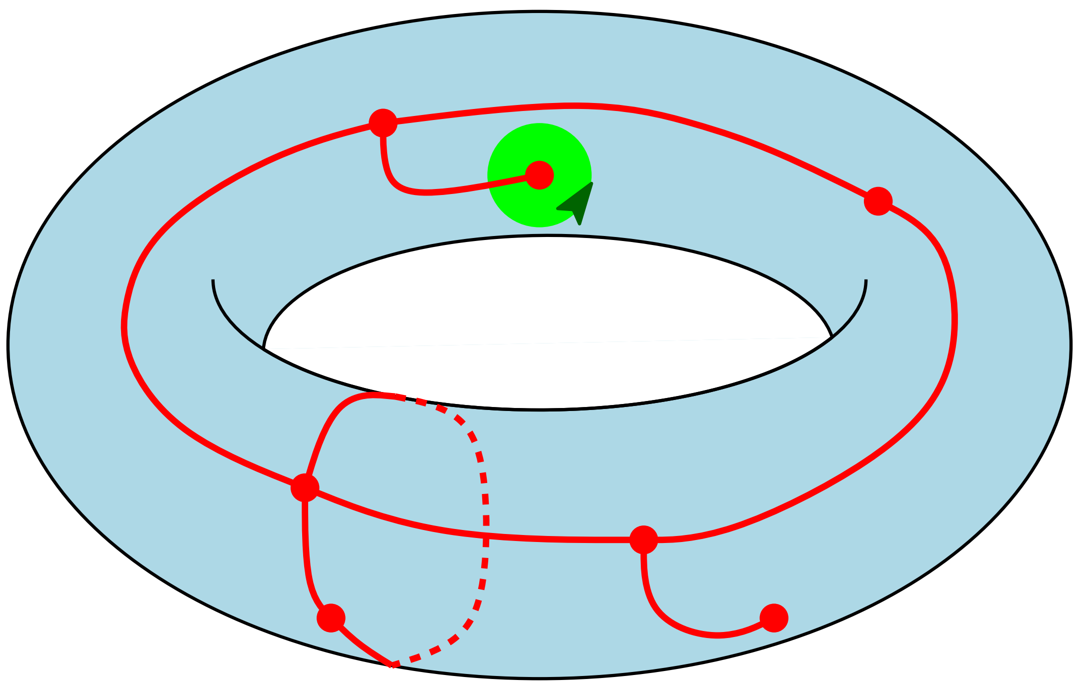

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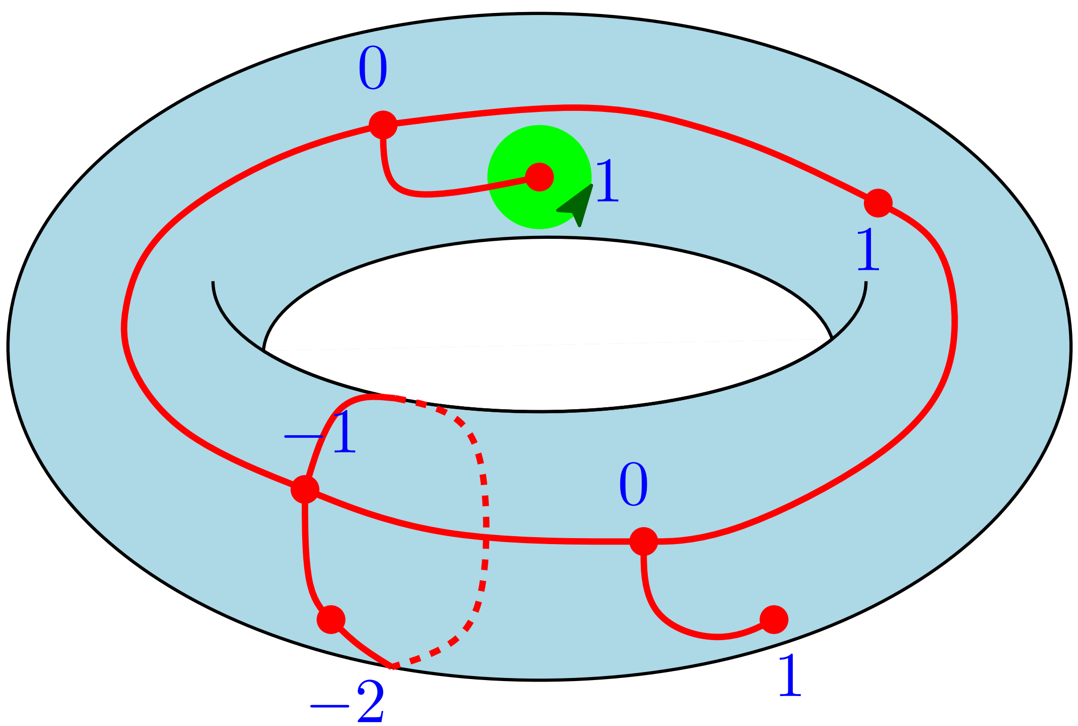

labeled map on a torus

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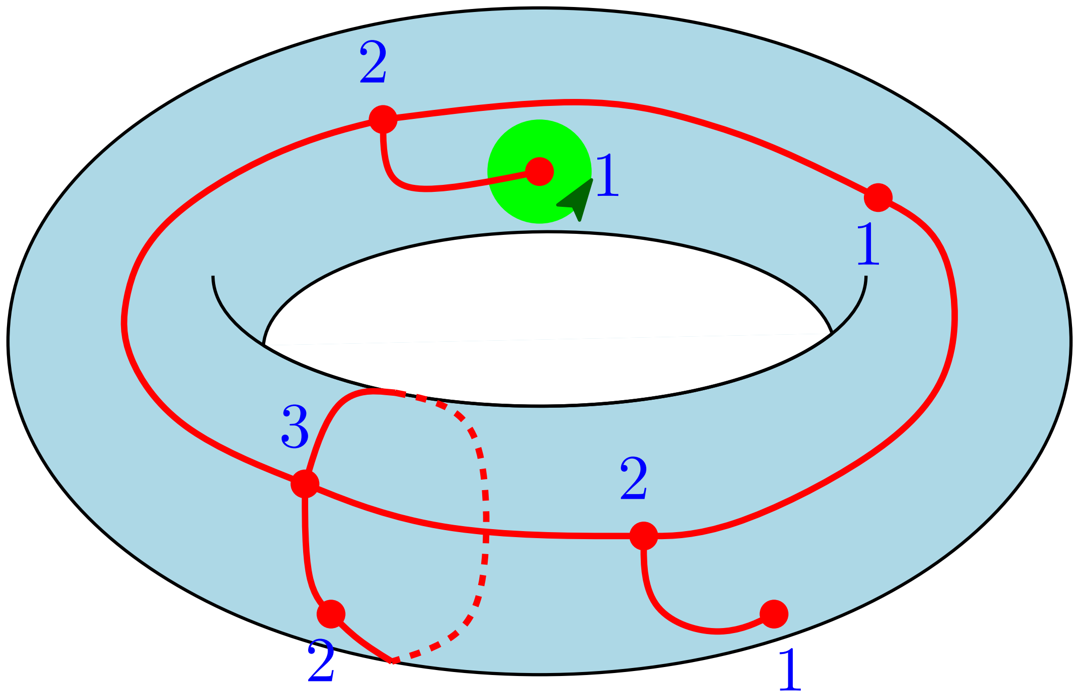

well-labeled map on a torus

Orientable case

Theorem [Marcus, Schaeffer 2009]
There is a bijection between:

- rooted, bipartite quadrangulations on ORIENTABLE surface $\mathbb{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)$;
- rooted, one-face, well-labeled maps on ORIENTABLE surface $\mathbb{S}$ with n edges and N_i vertices of label i $(i \geq 1)$;

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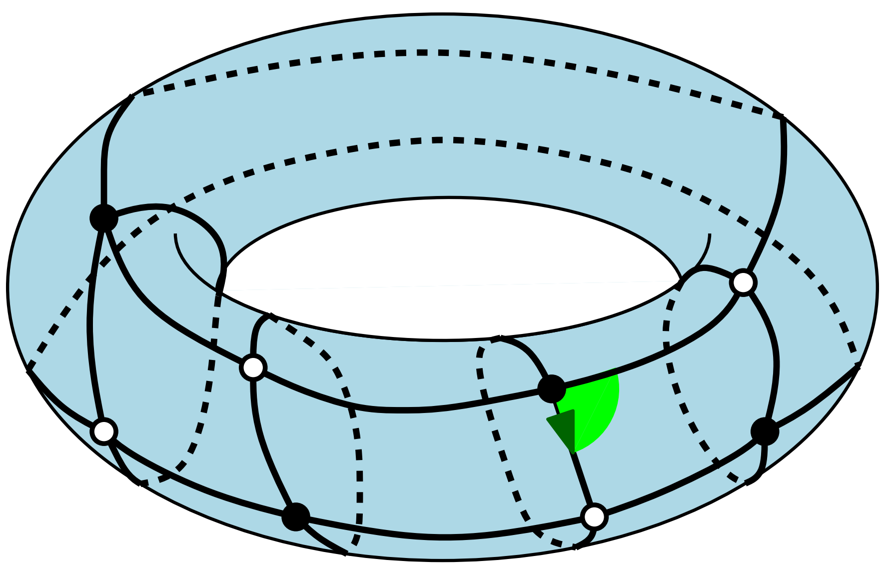

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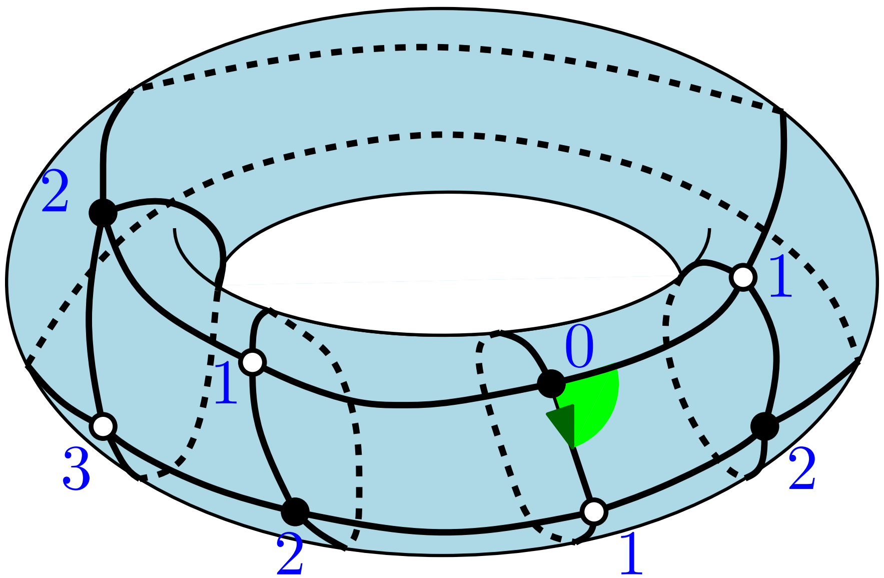

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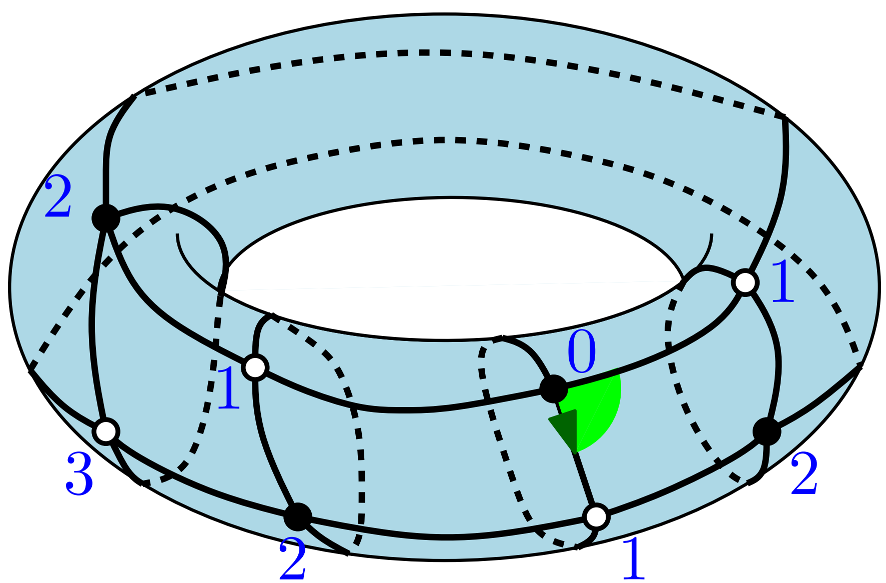

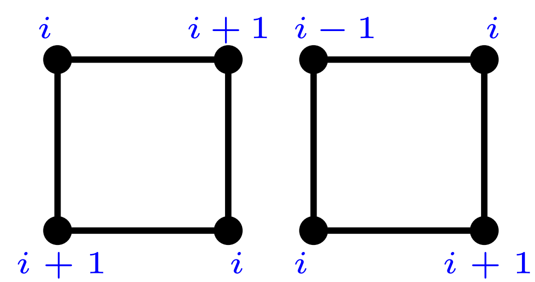

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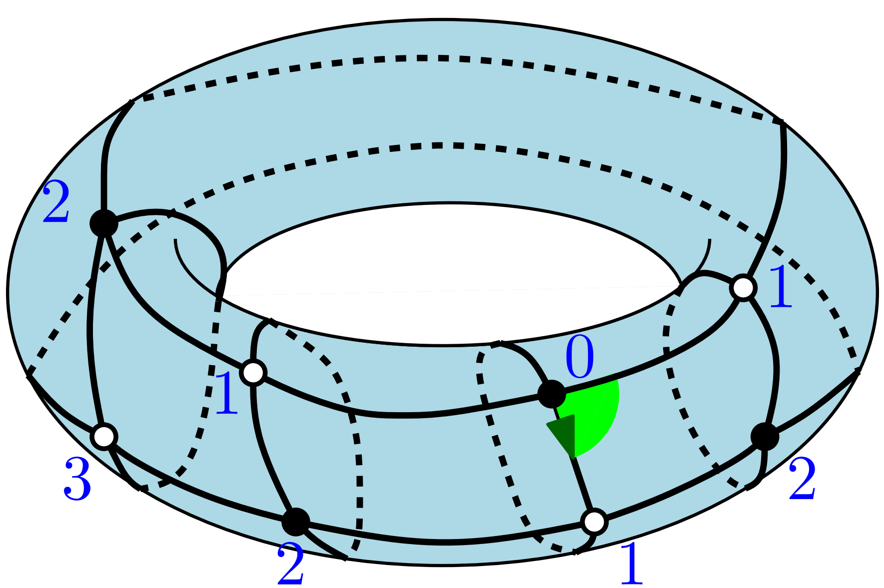

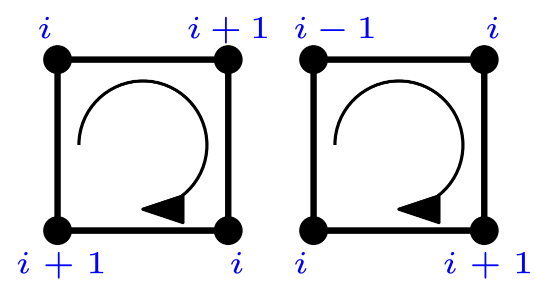

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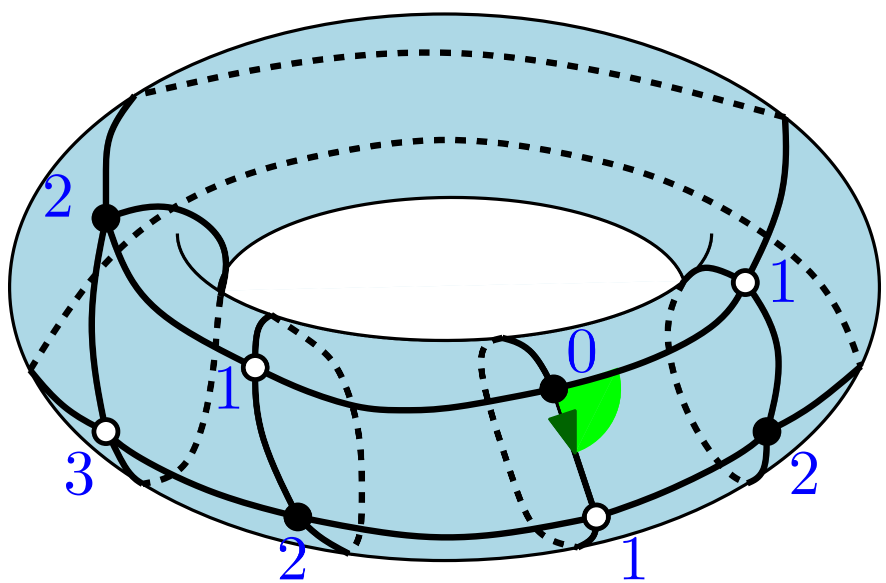

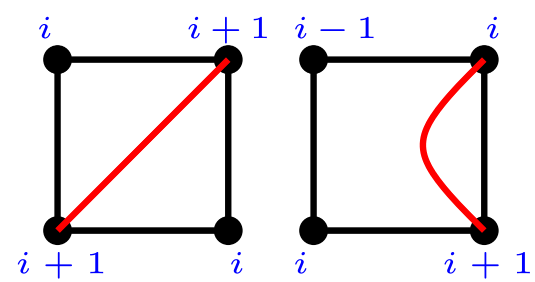

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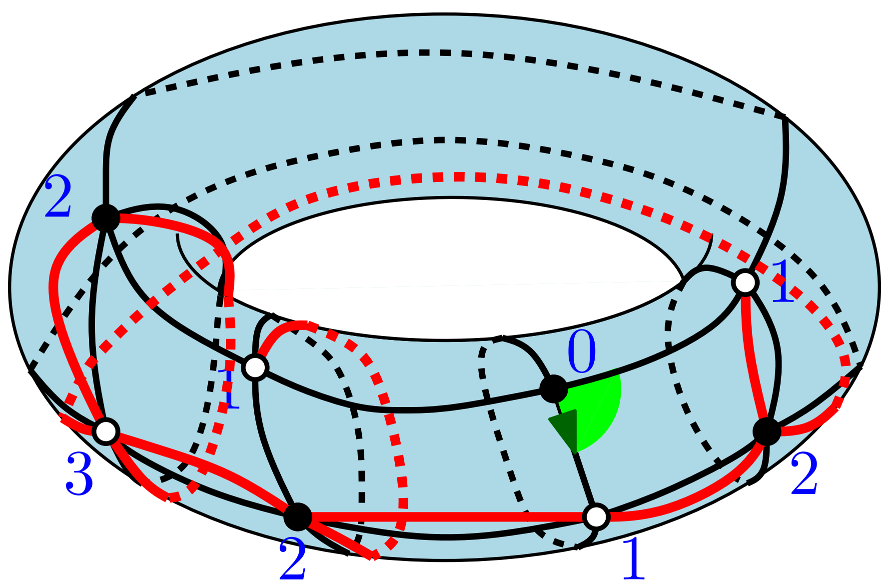

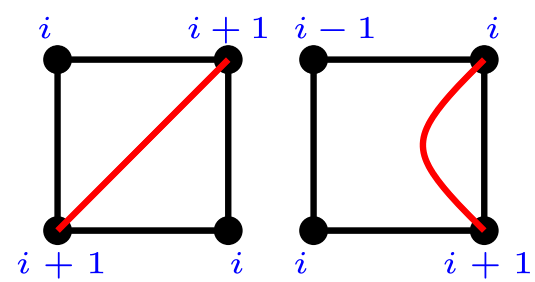

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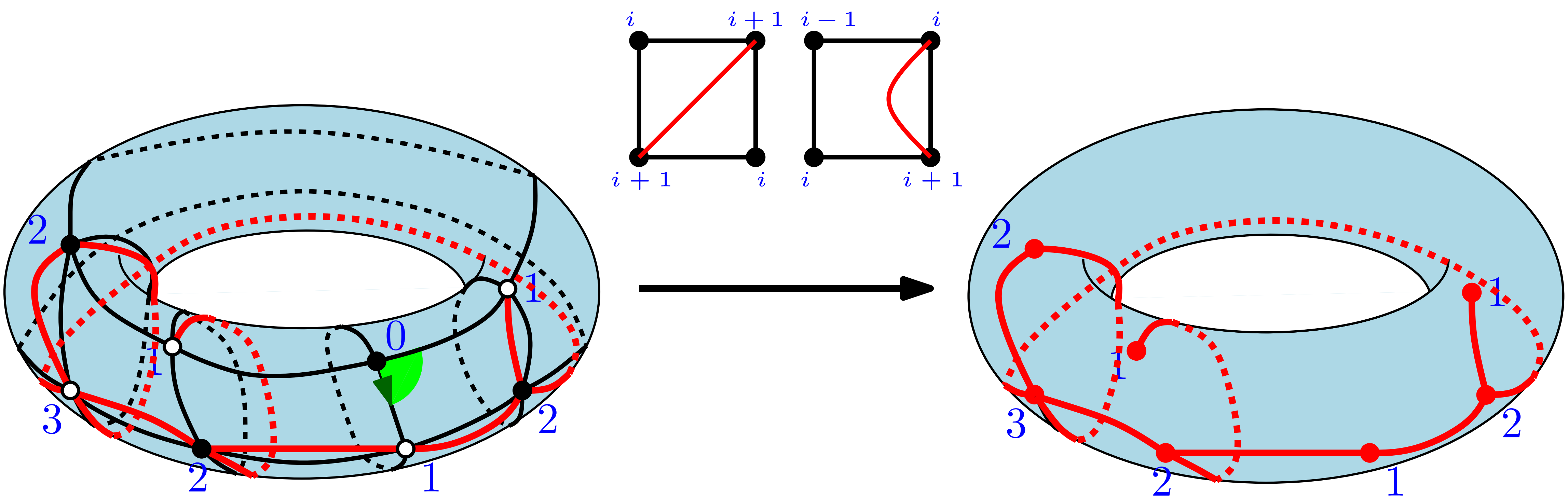

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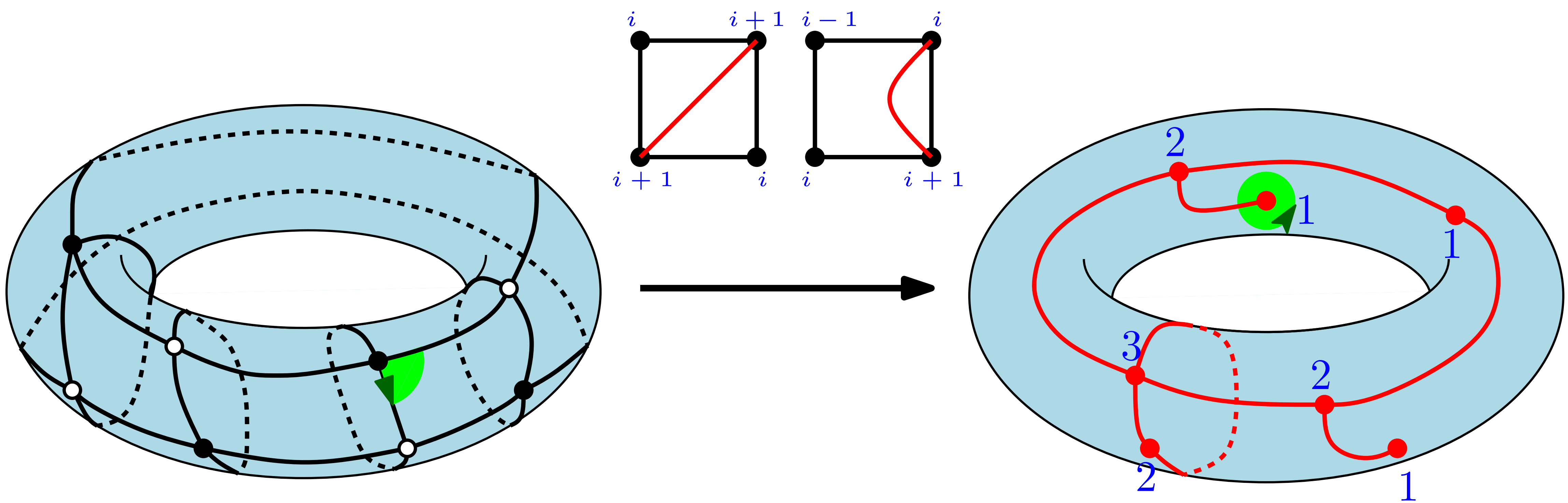

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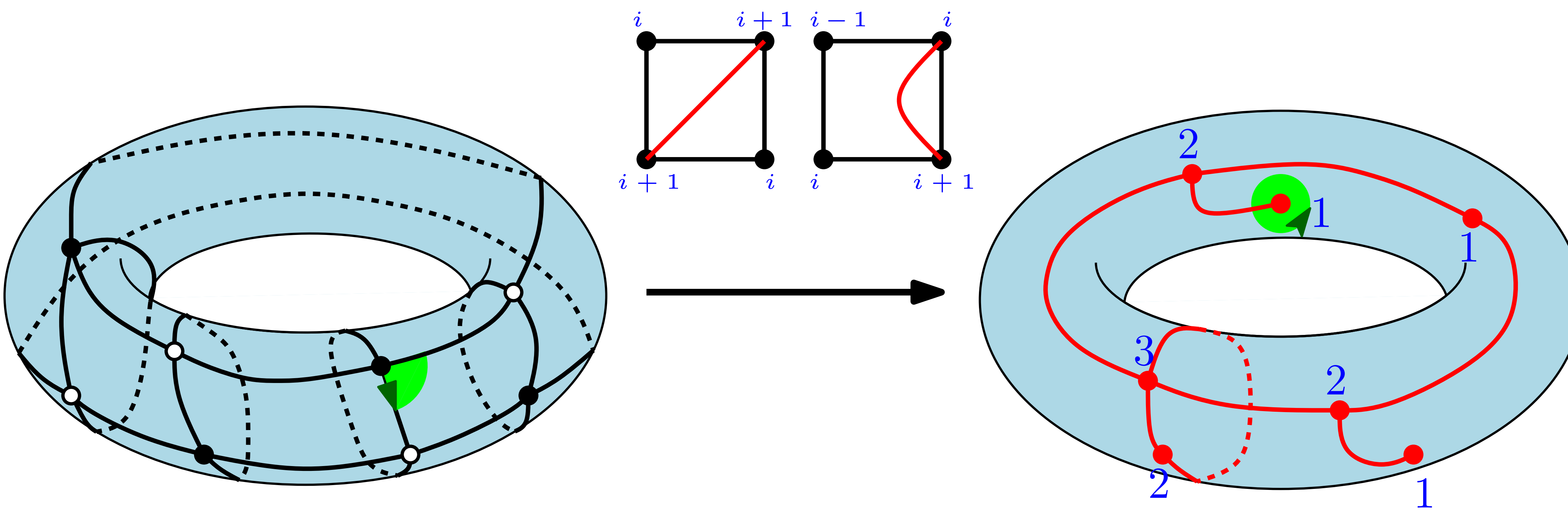

What about
non-orientable maps?

General case

Theorem [Chapuy, D. 2014]
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Idea:

To extend Marcus-Schaeffer bijection we have to use the same rules locally, but in non-orientable case we have to orient each face in some CANONICAL way.

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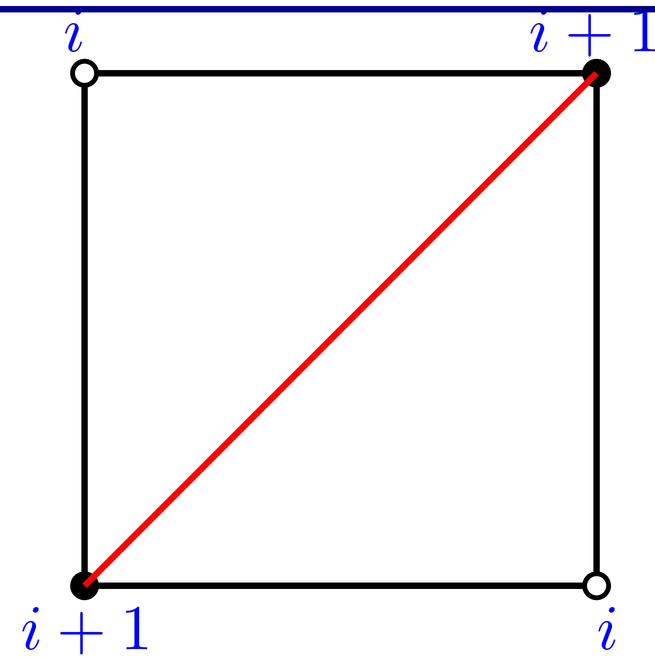

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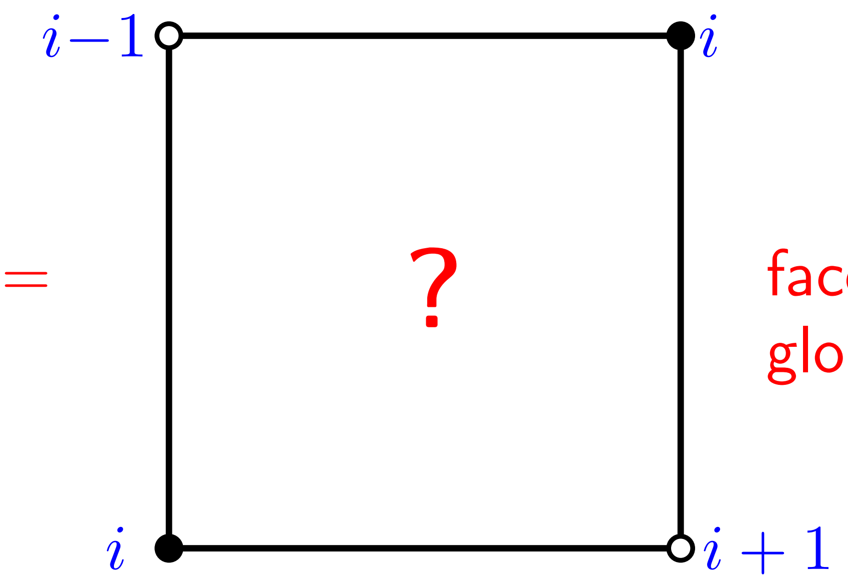
Map is non-orientable =



faces CANNOT be globally oriented!

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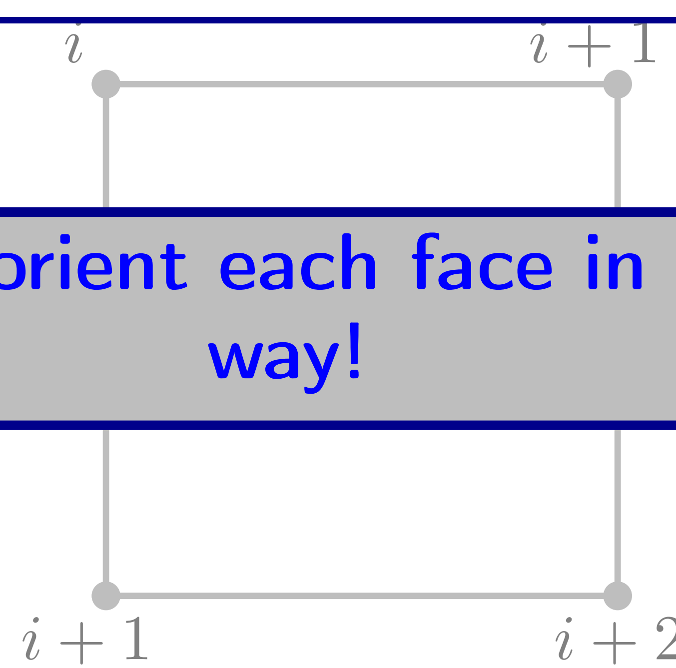
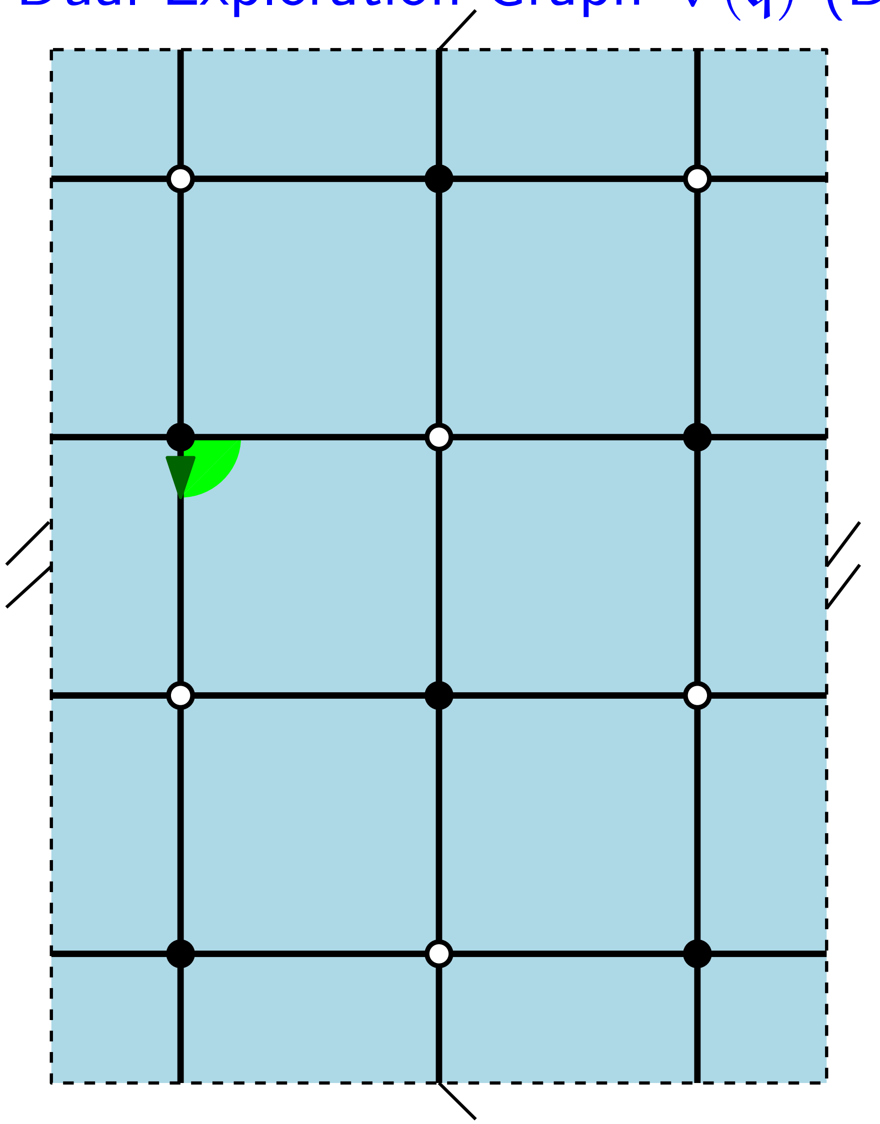


Map is iented!

We have to orient each face in some nice way!

General case (II)

We are going to orient faces of a quadrangulation $\mathfrak{q}$ by constructing recursively a Dual Exploration Graph $\nabla(\mathfrak{q})$ (DEG) on the same surface:



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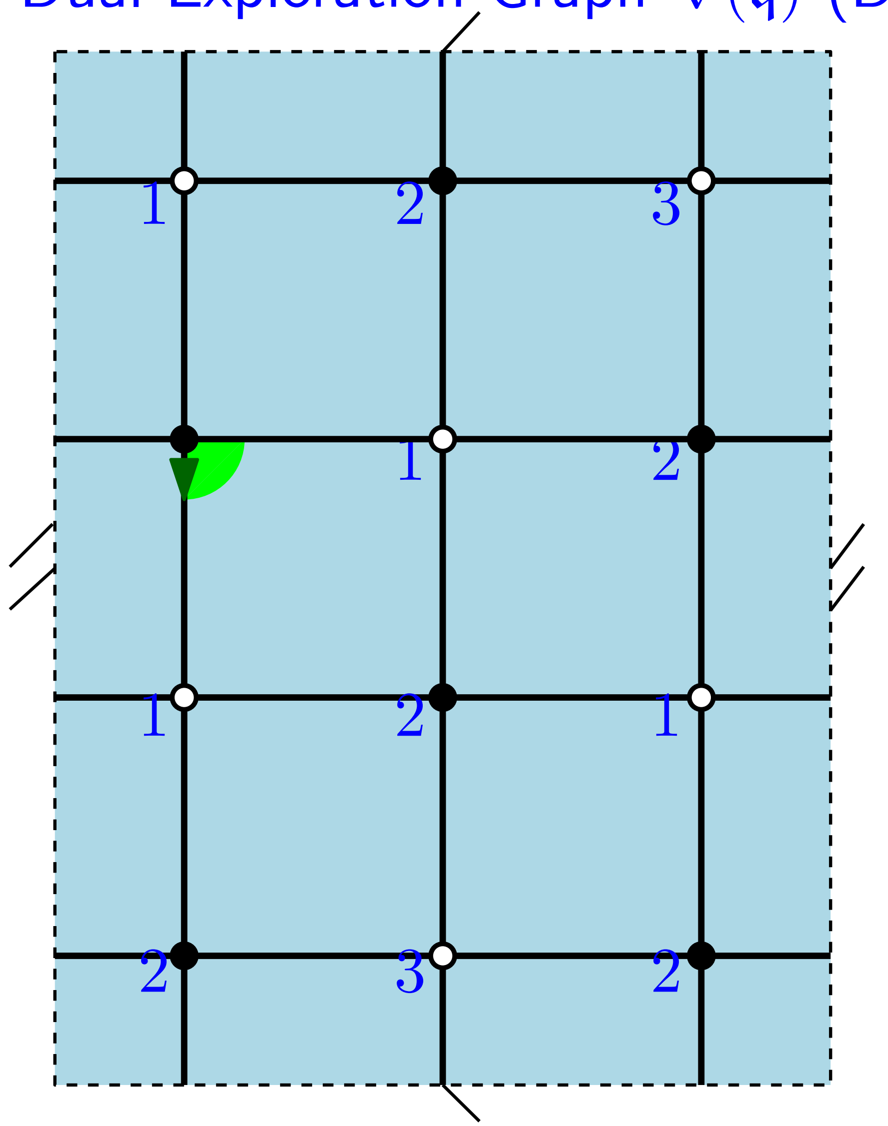

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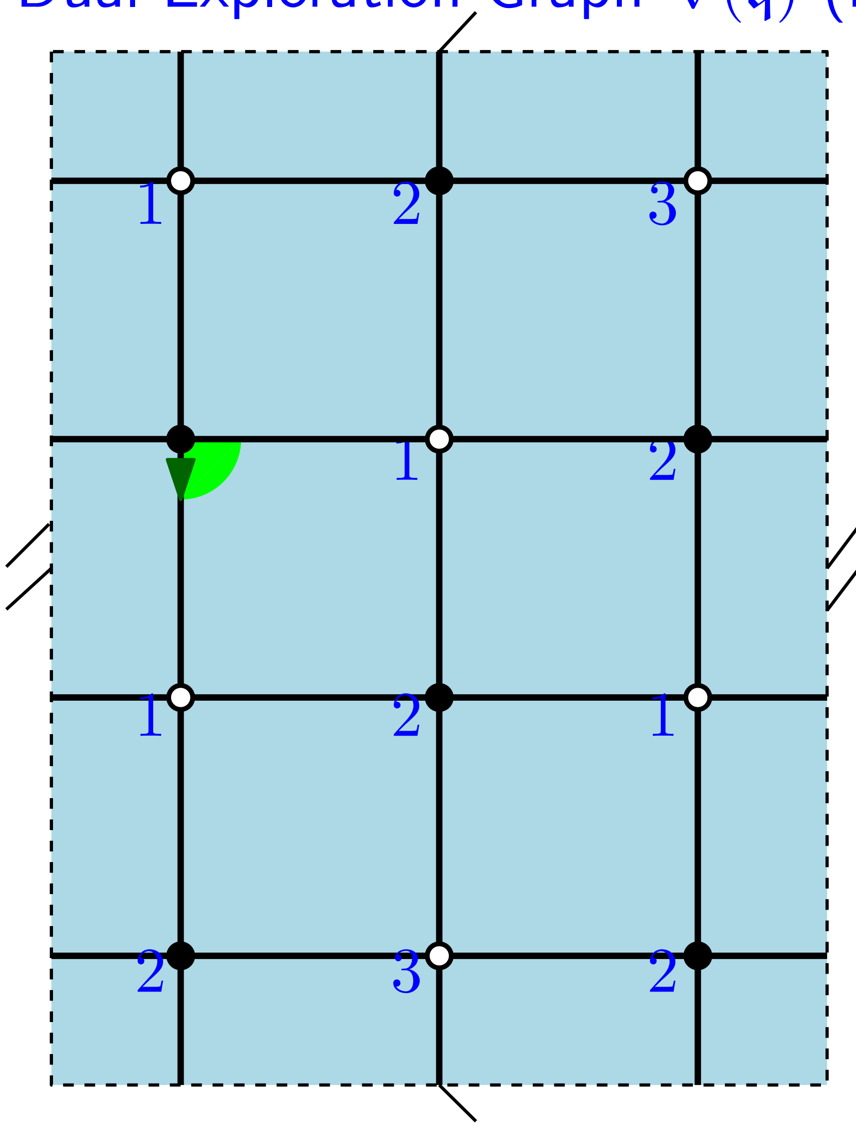
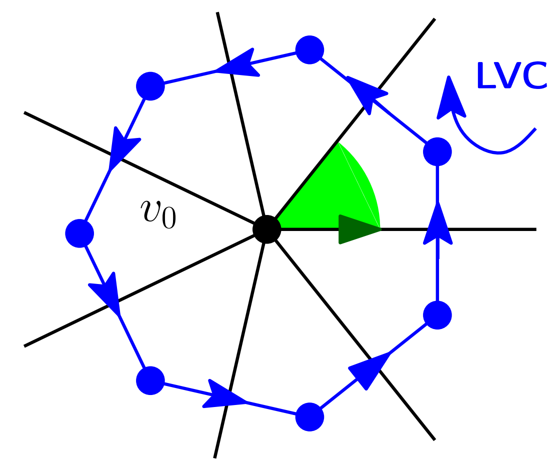


Step 0: Initialization



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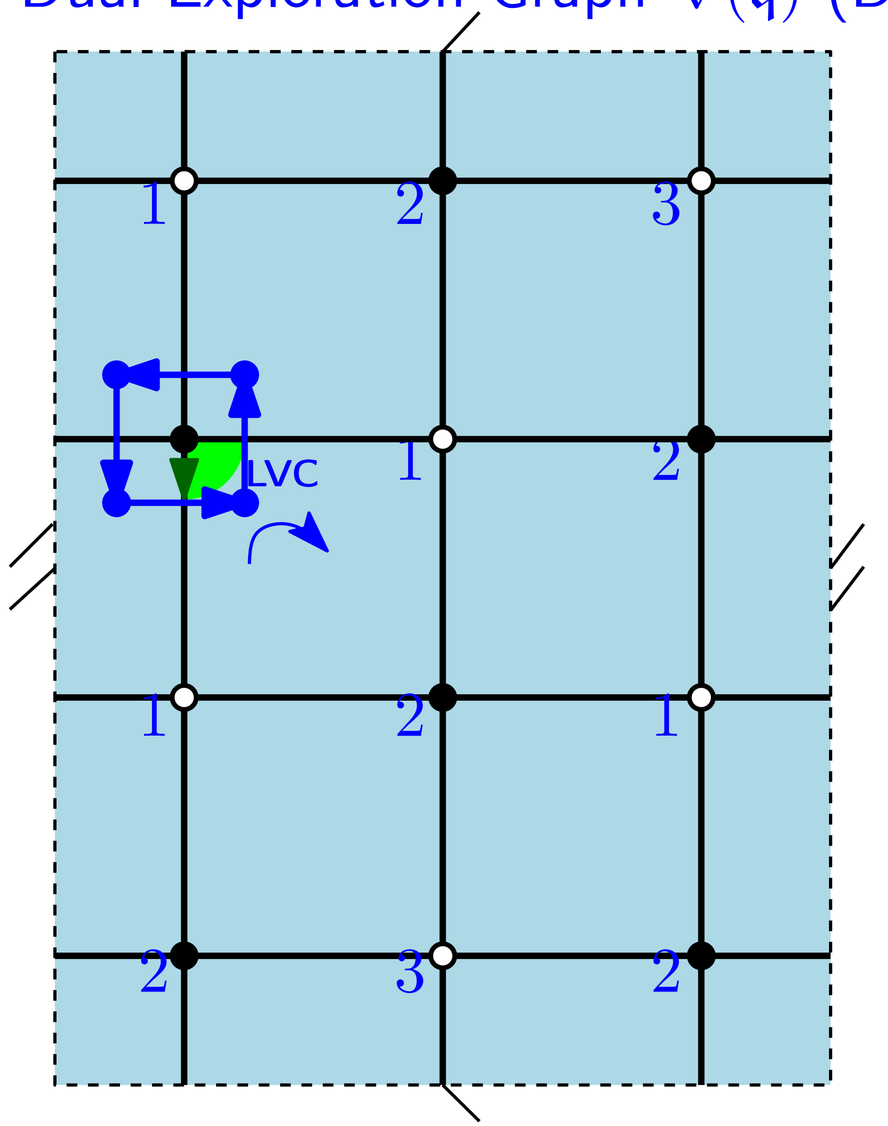

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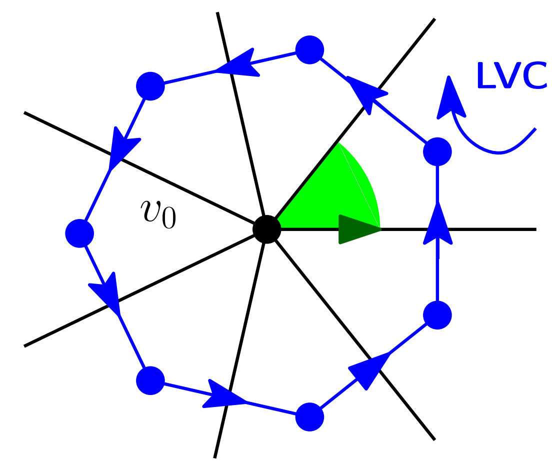

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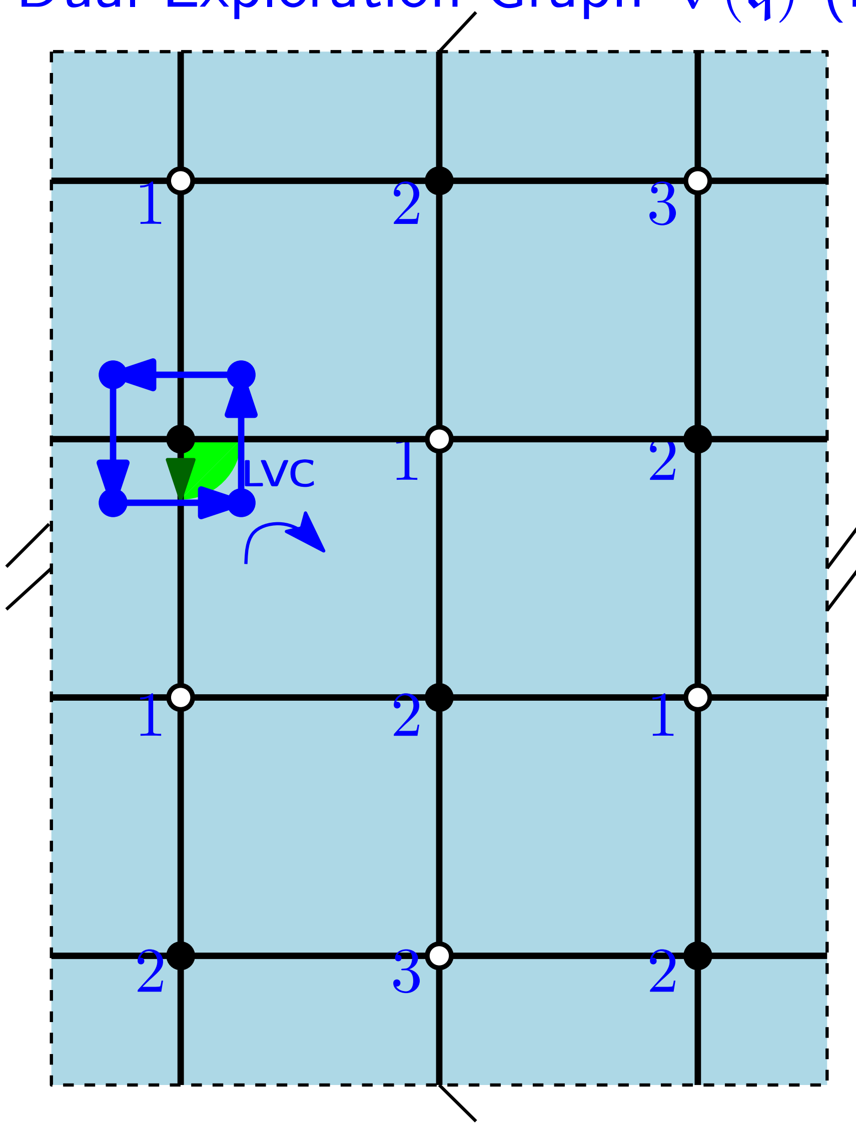

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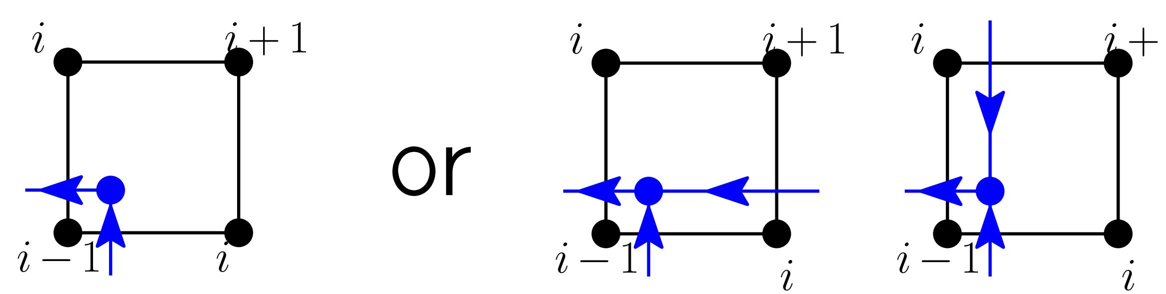

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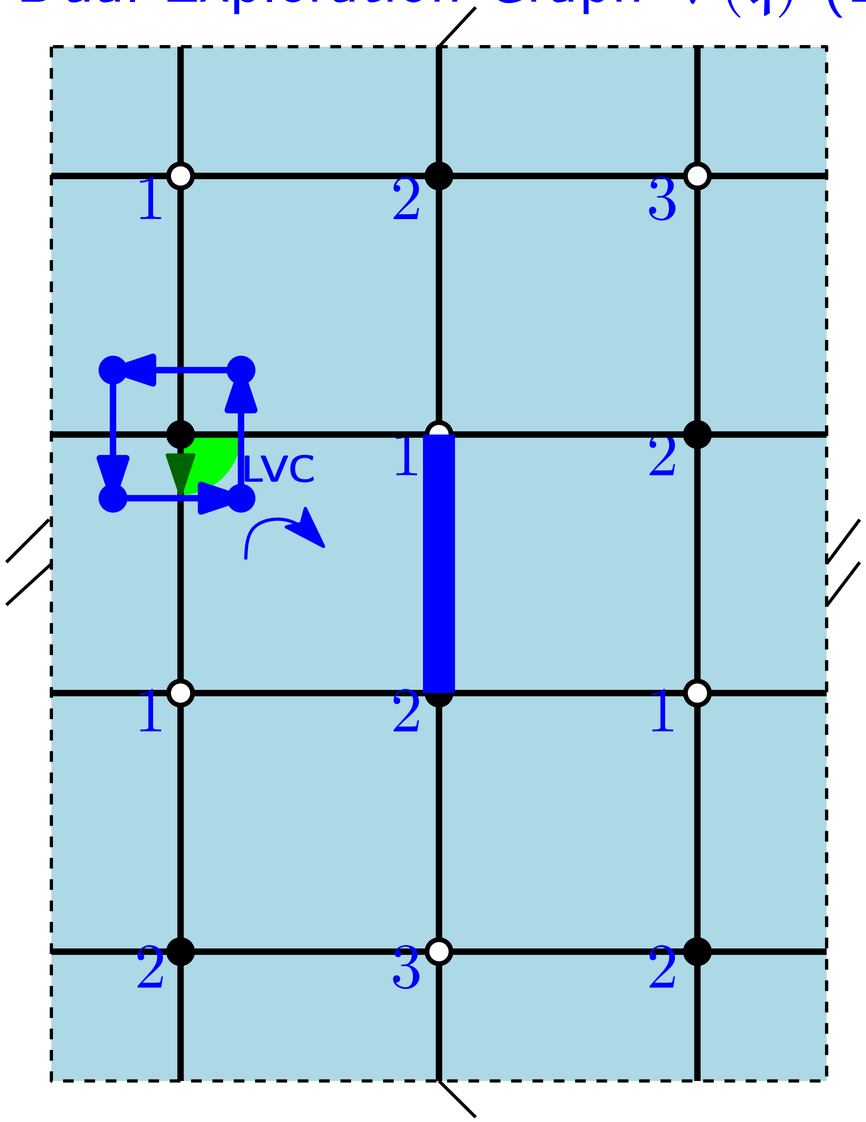

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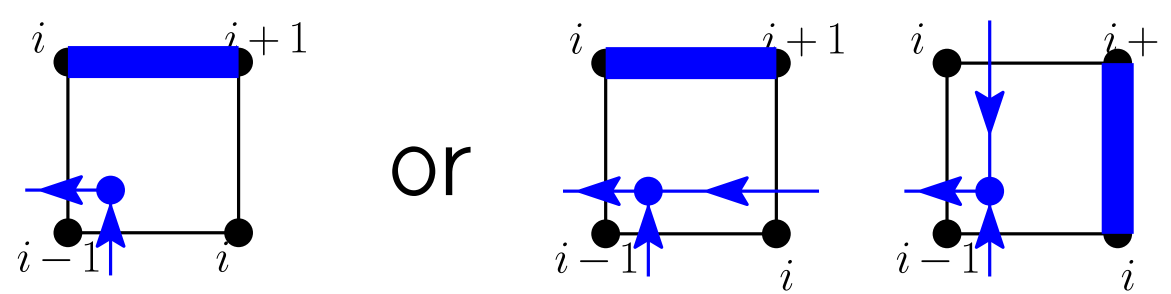

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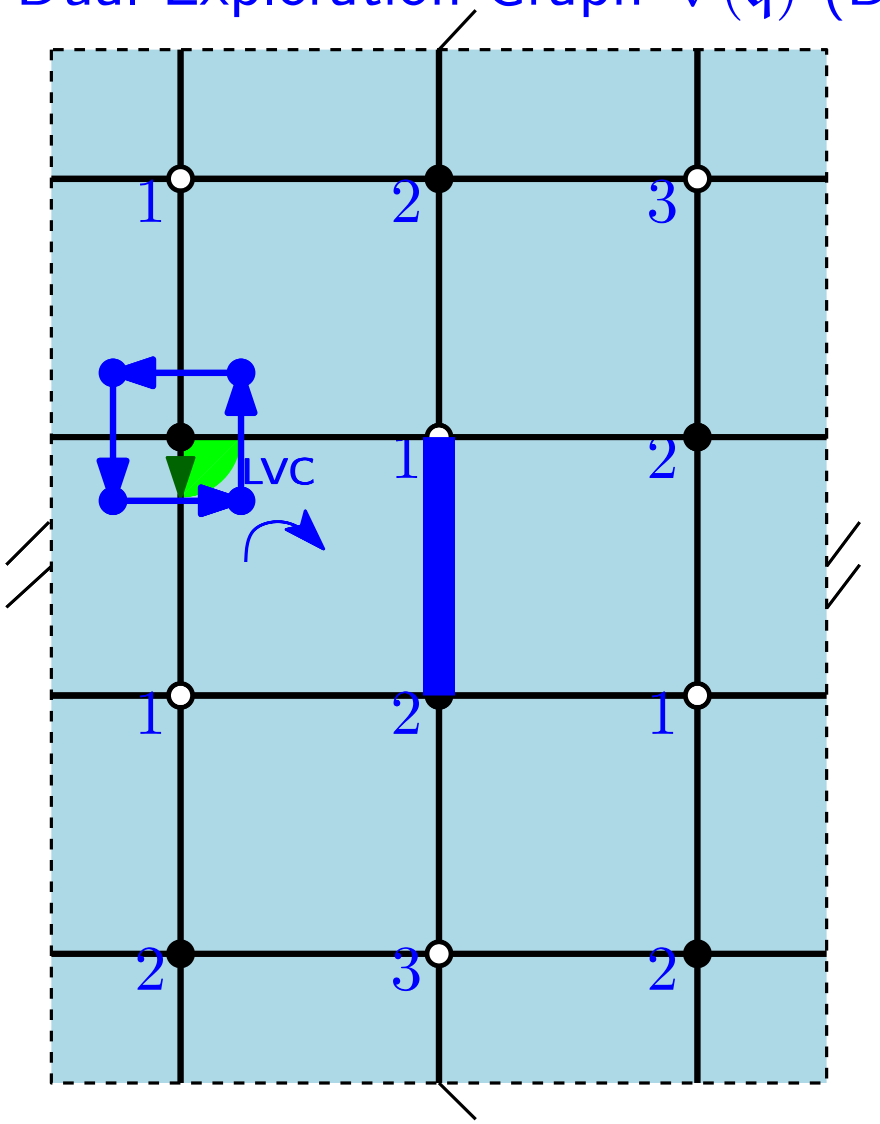

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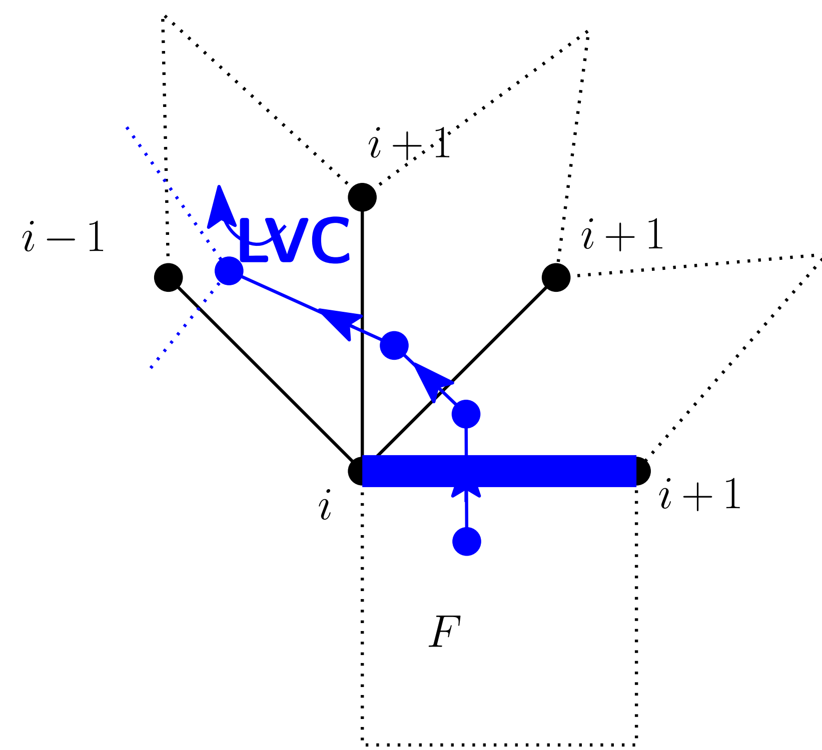

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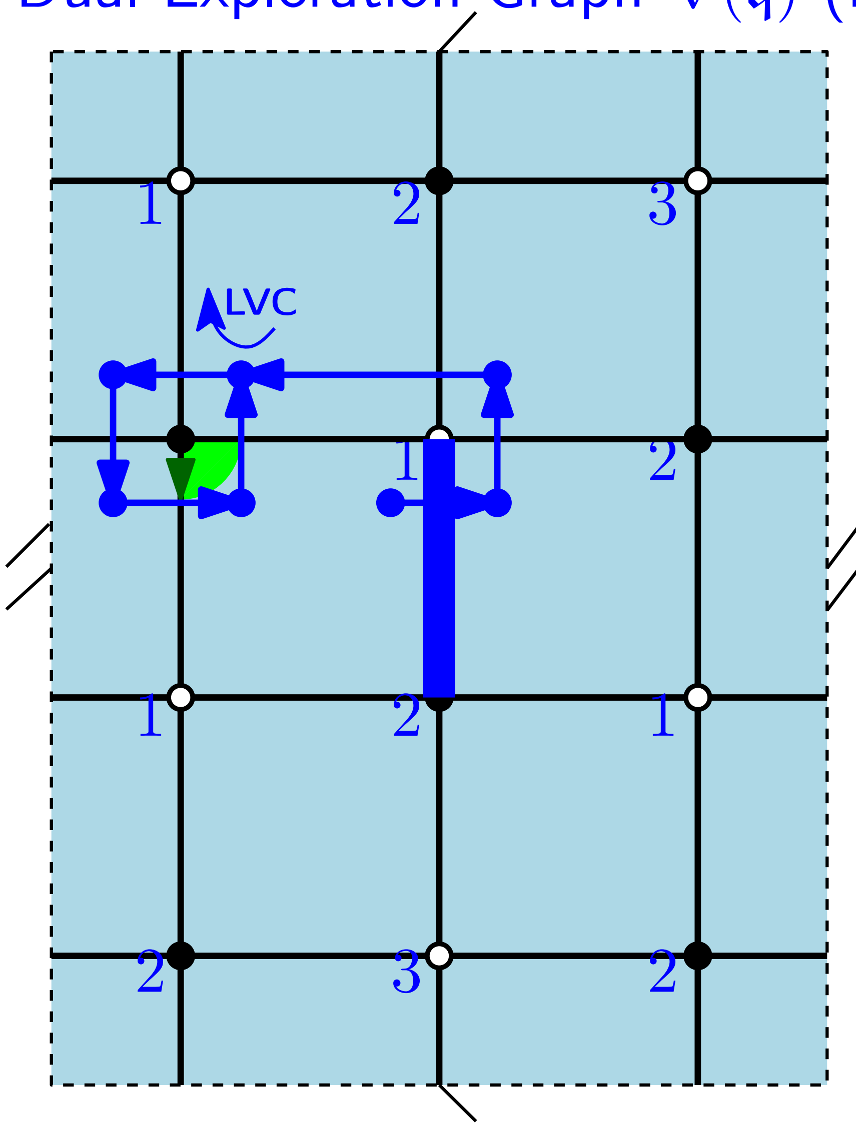

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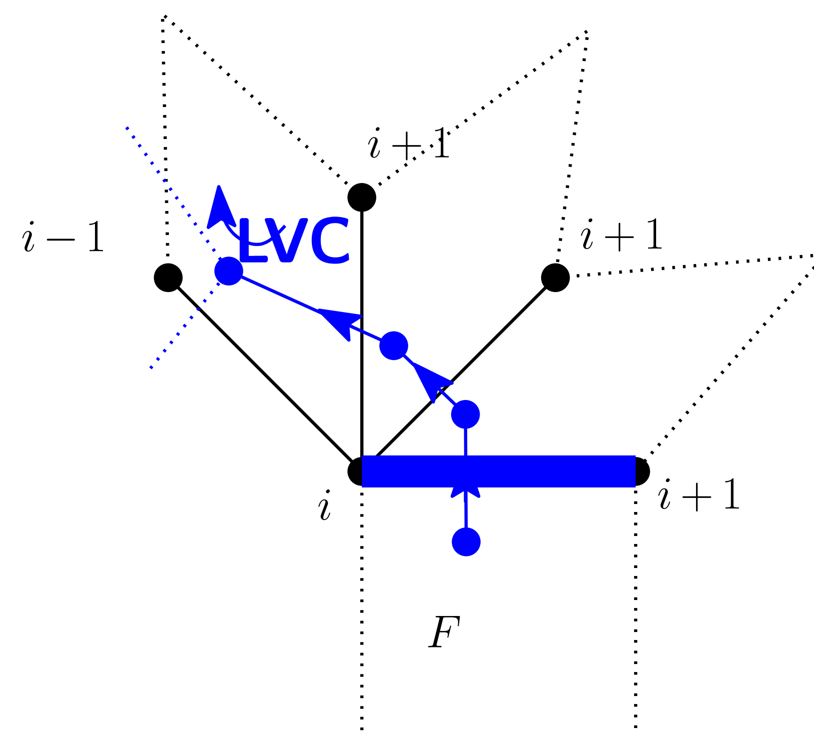

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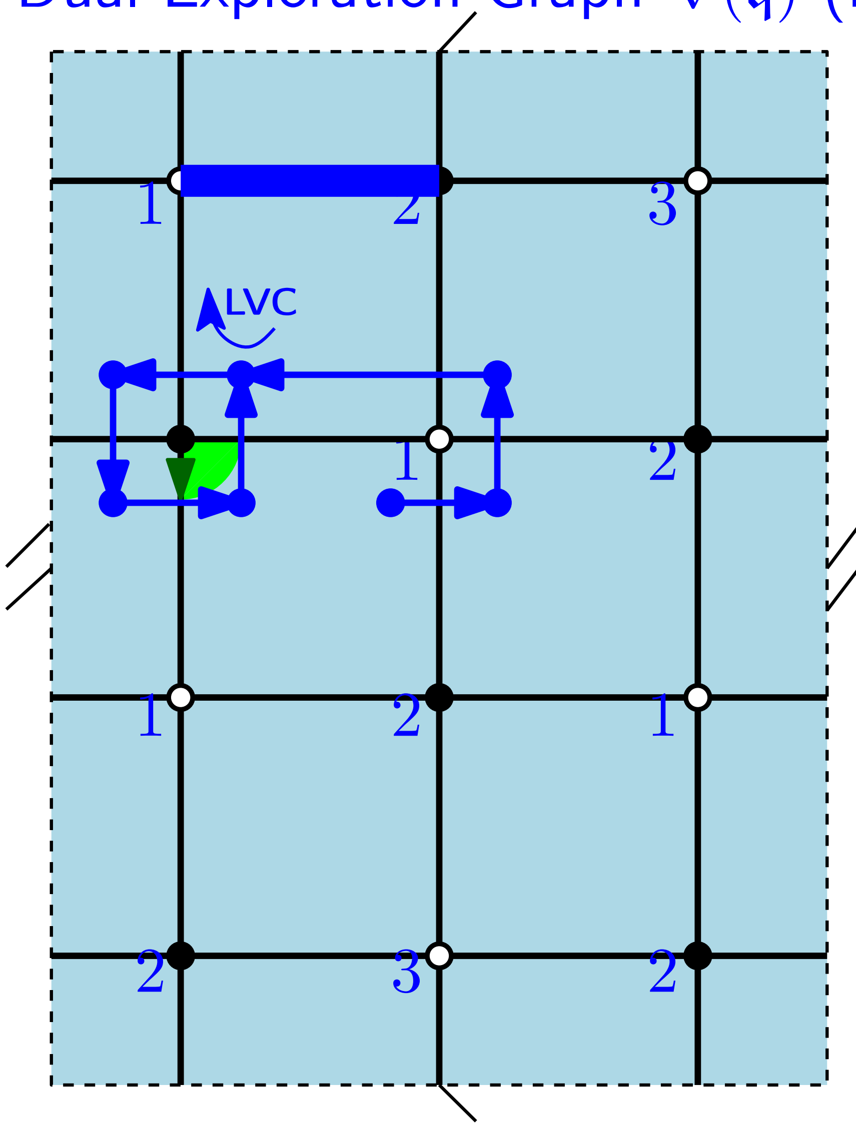

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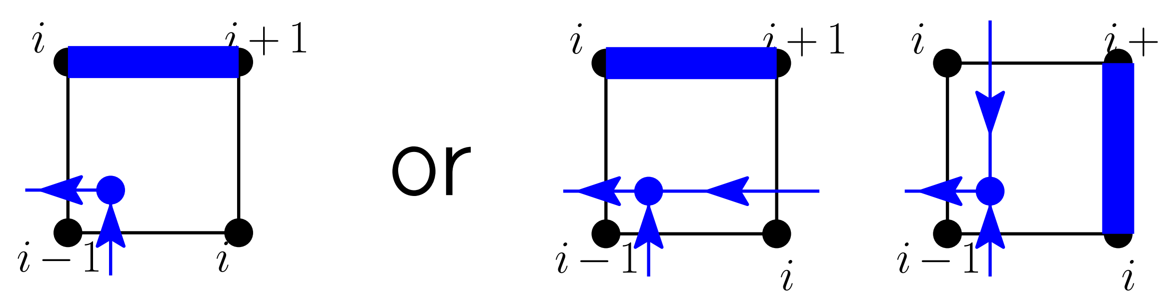

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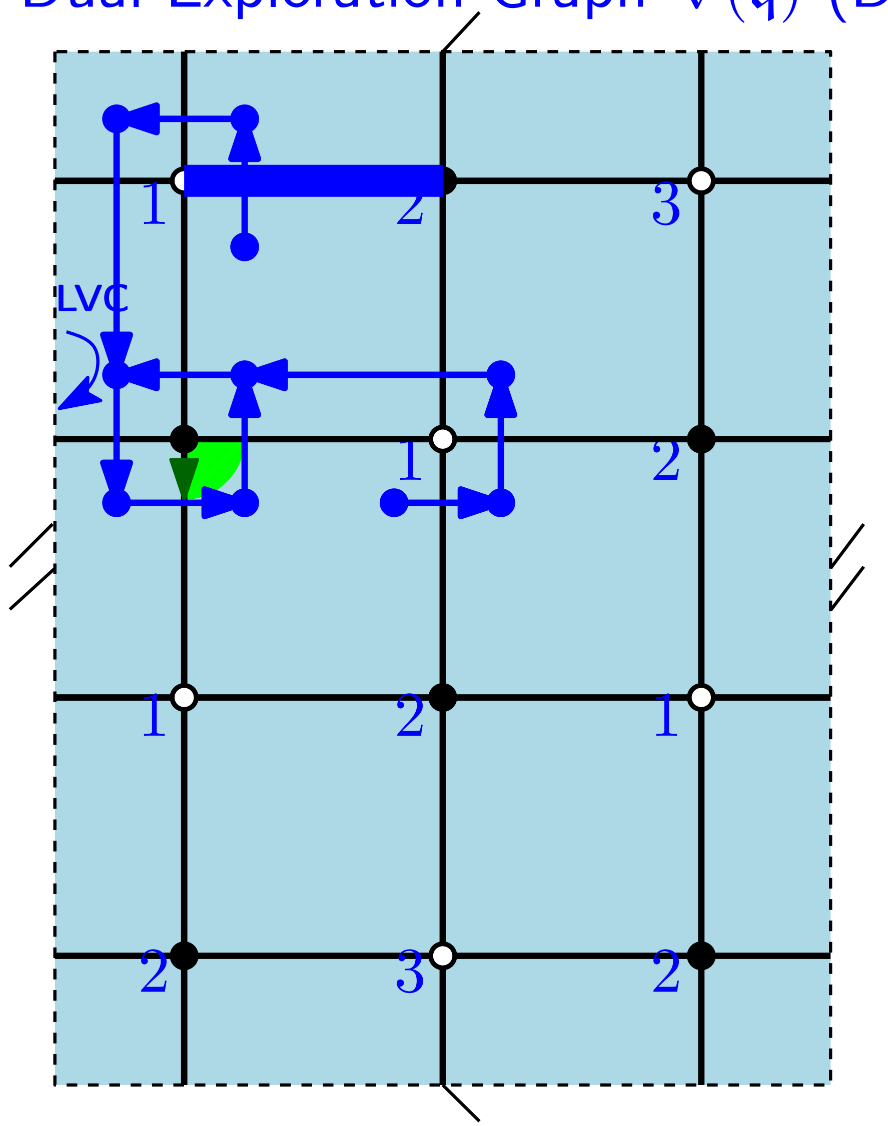

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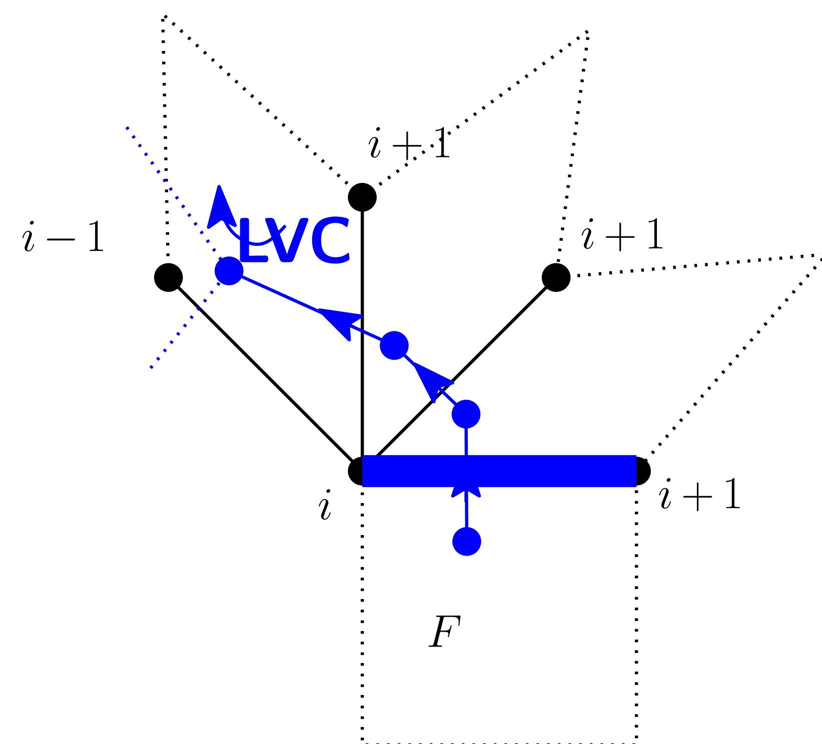

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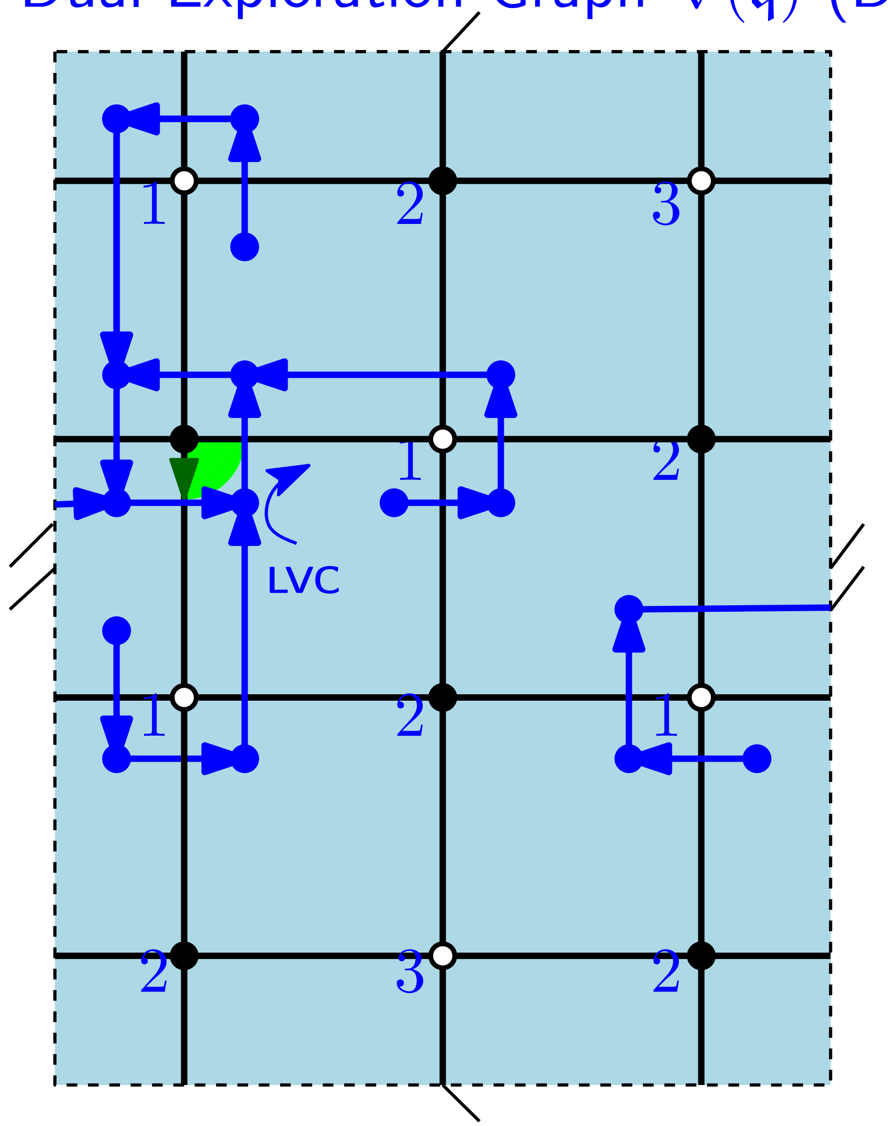

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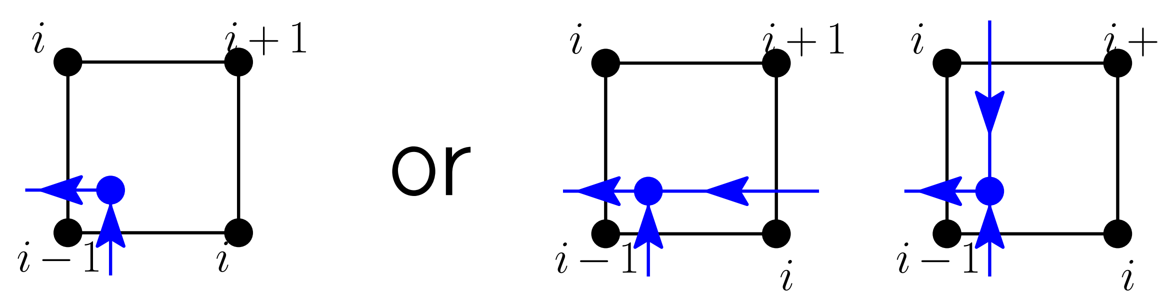

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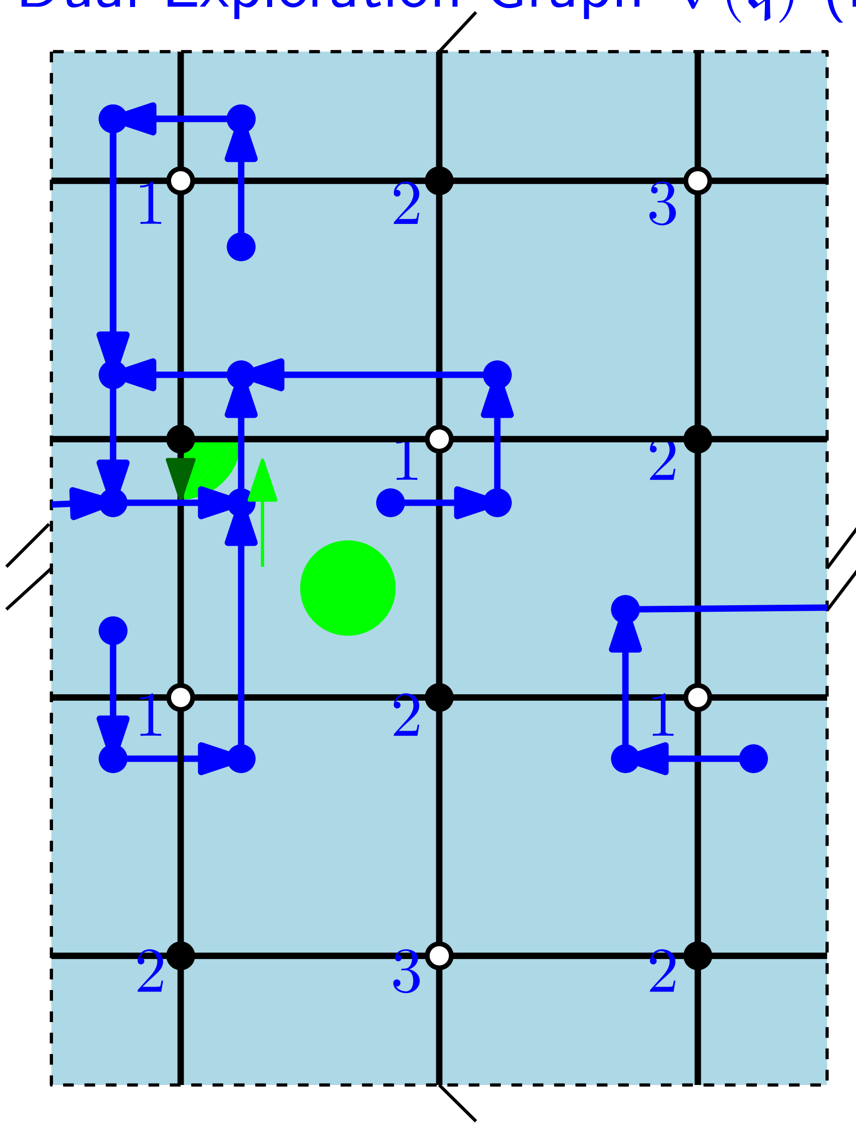

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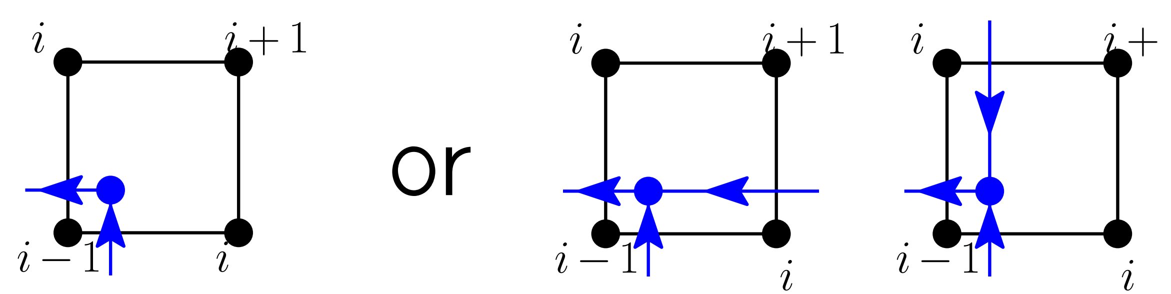

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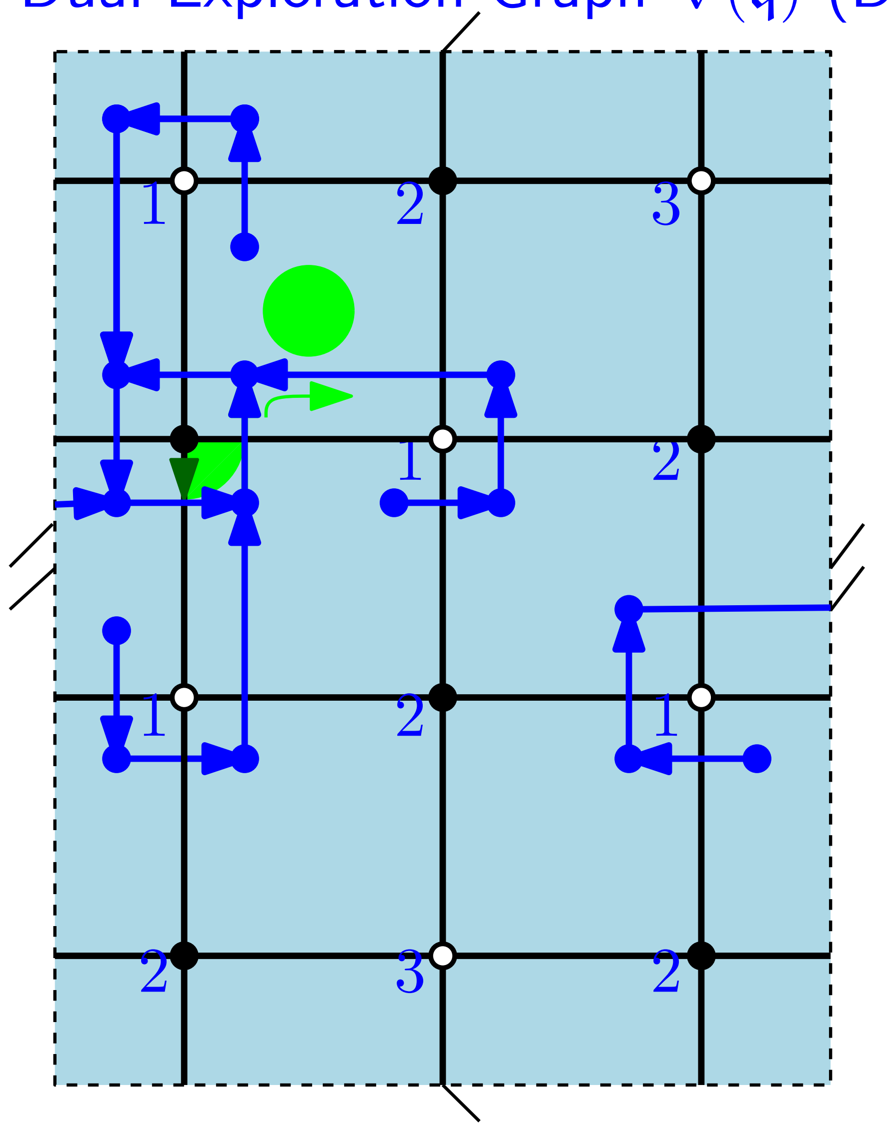

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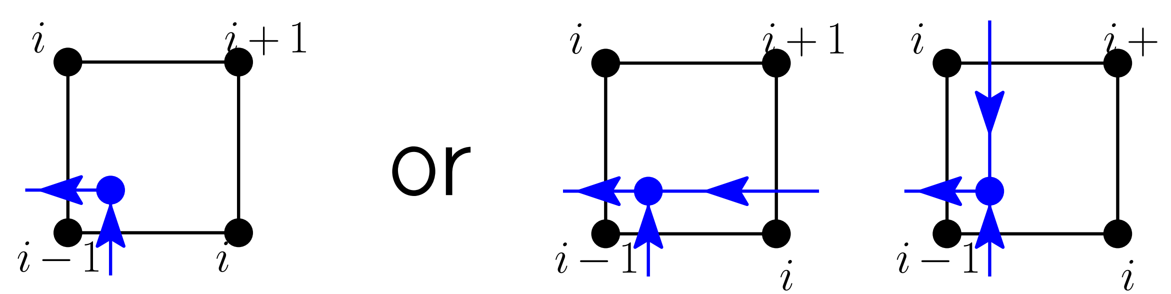

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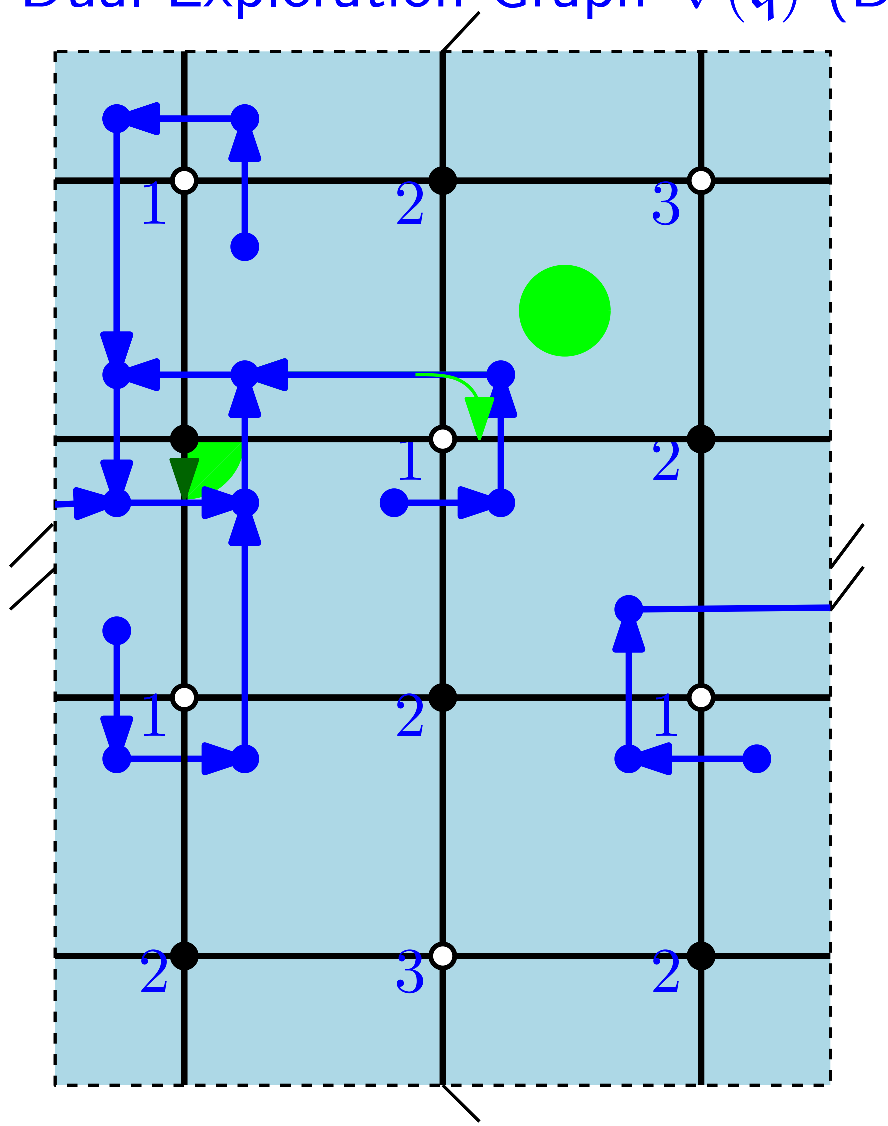

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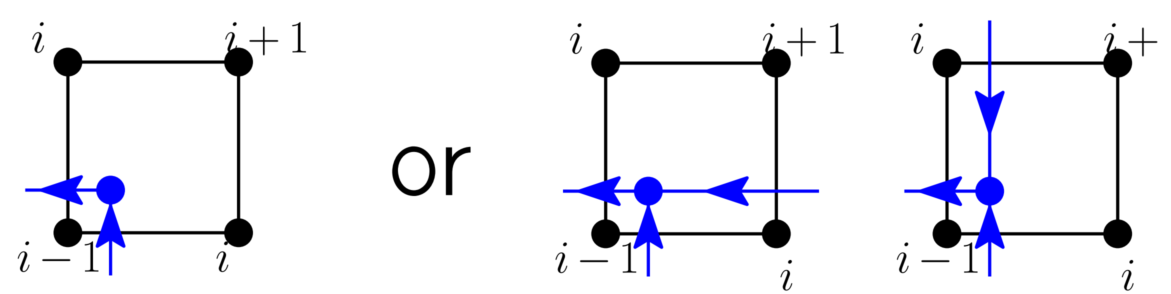

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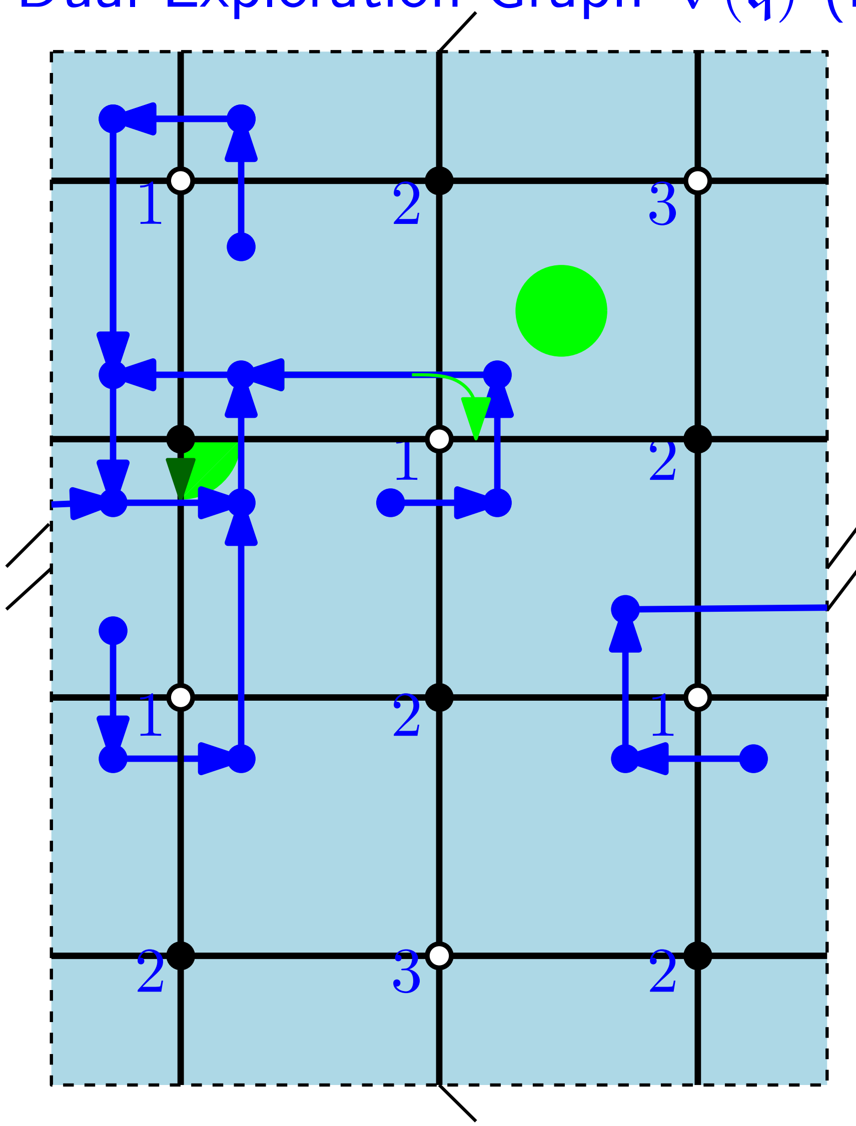

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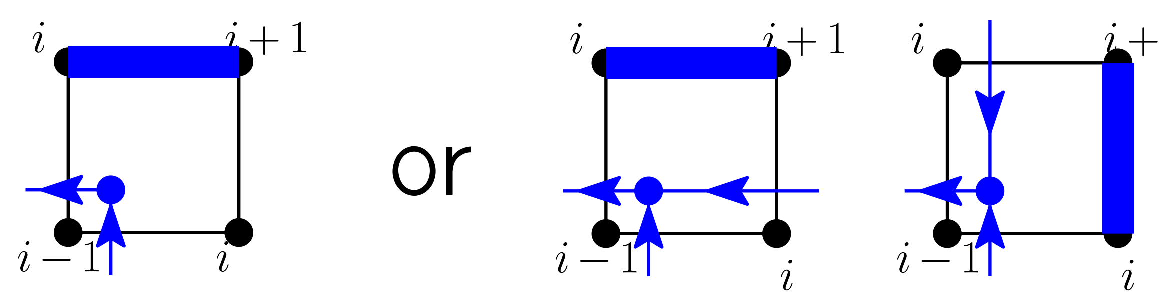

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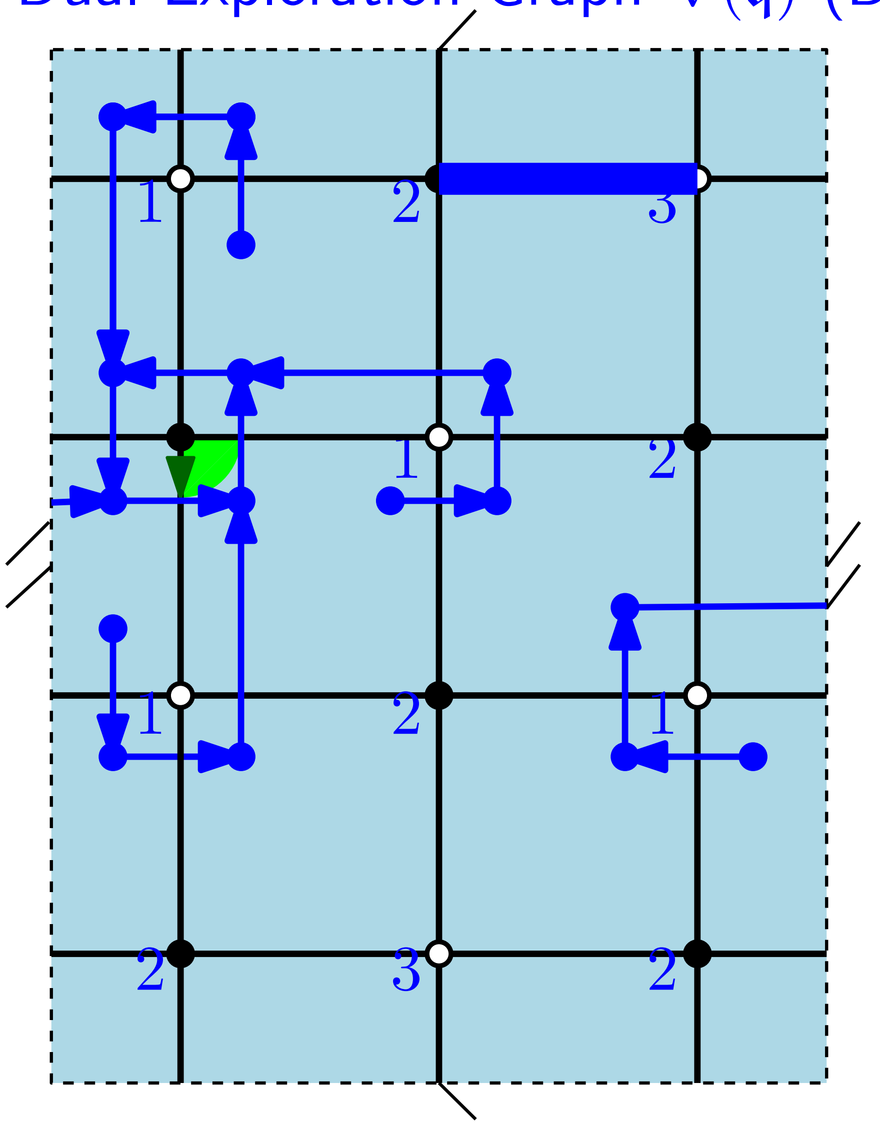

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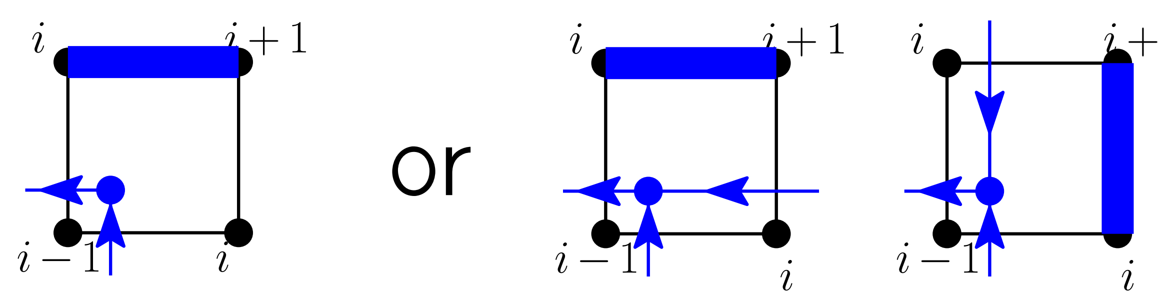

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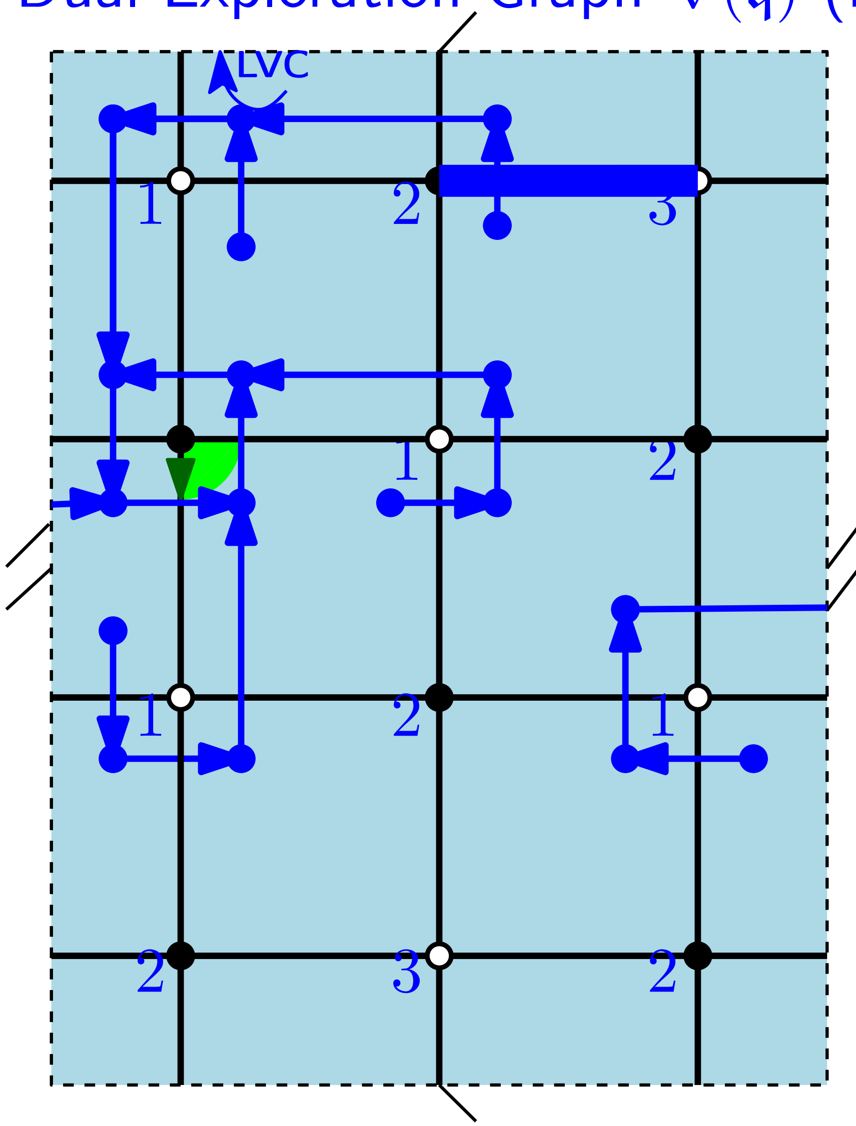

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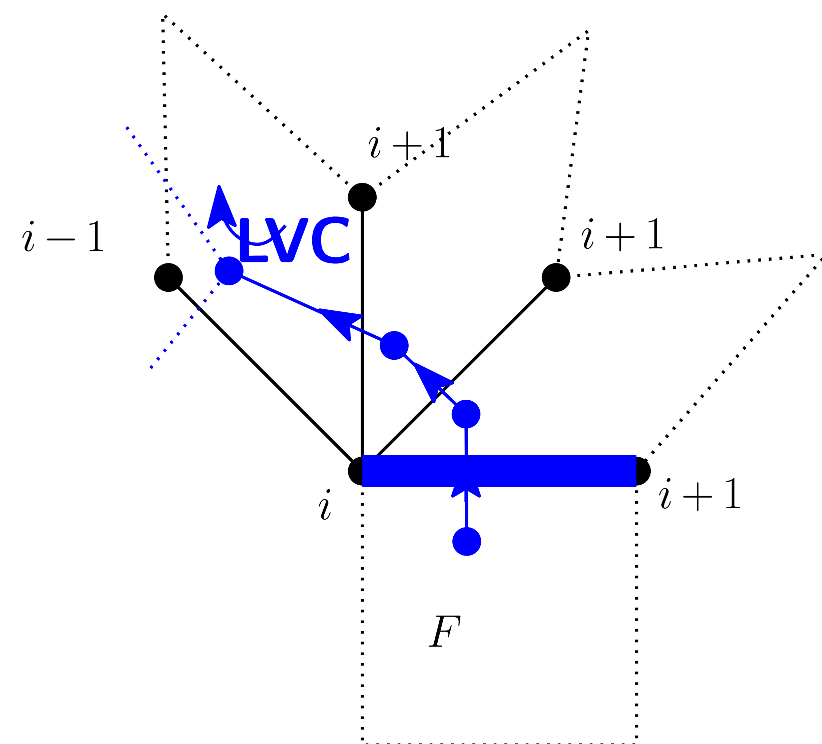

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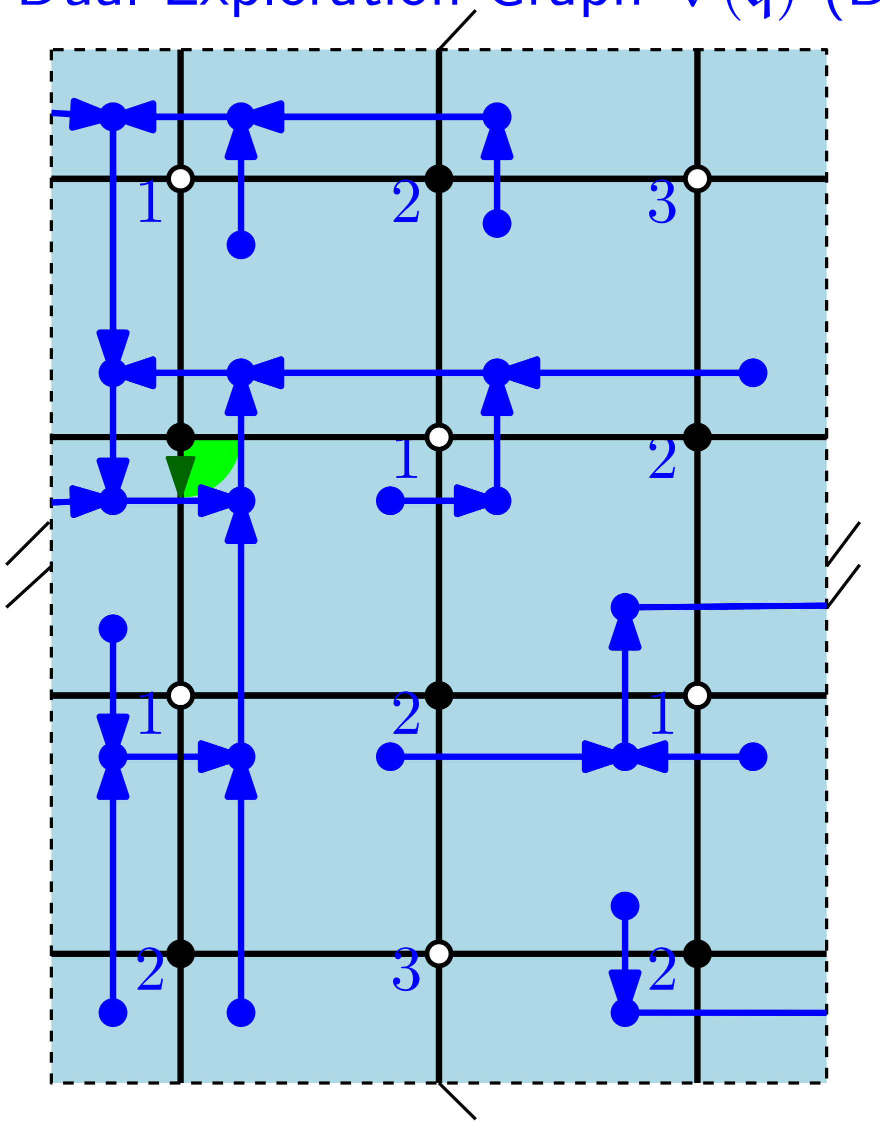

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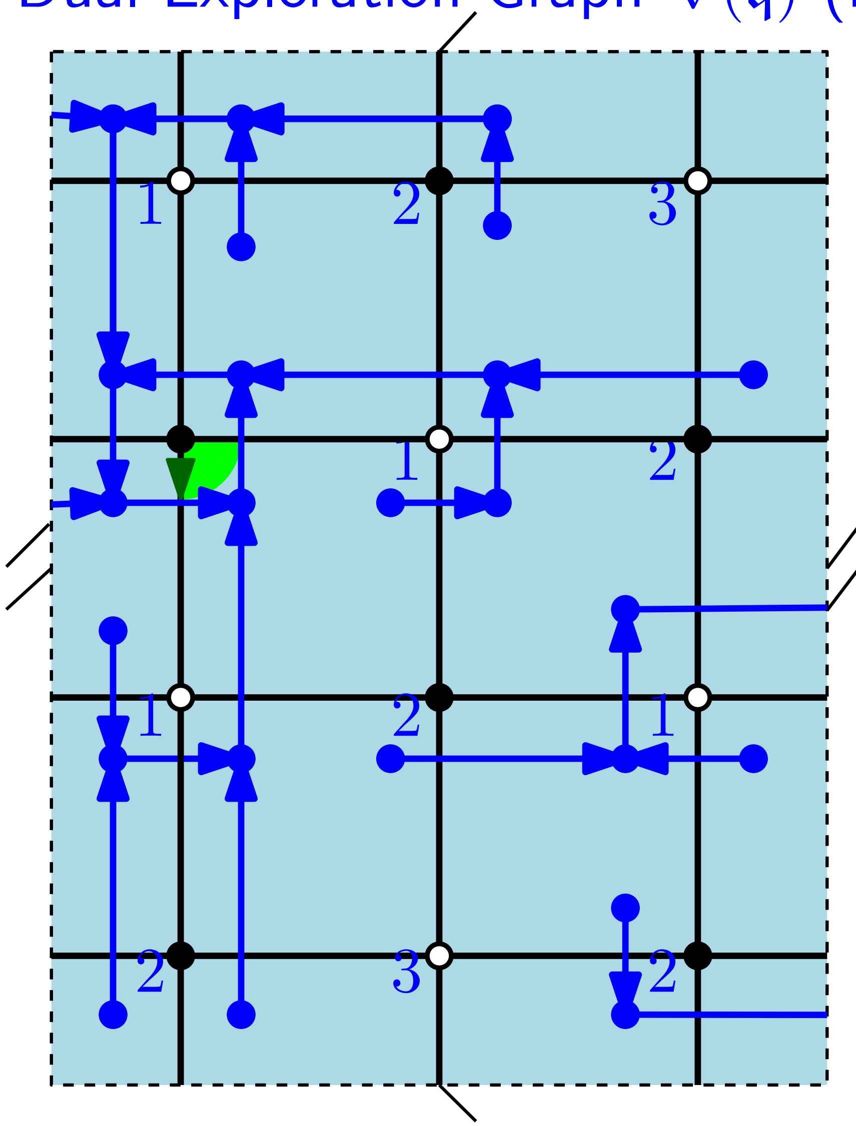
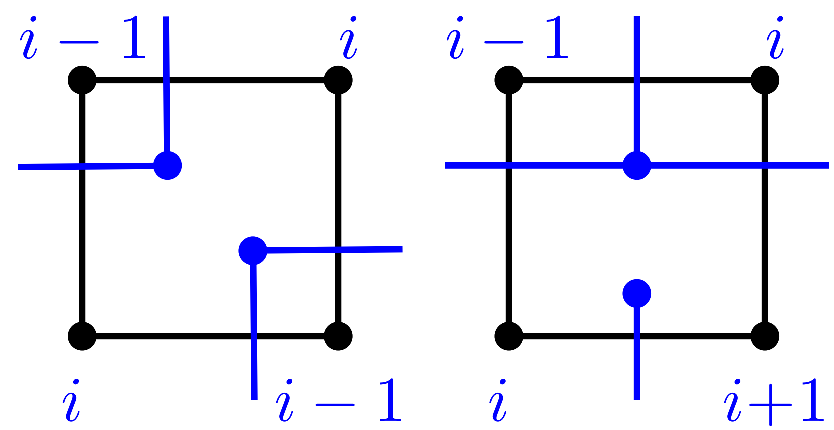


Proposition:
DEG $\nabla(\mathfrak{q})$ is formed by a unique oriented cycle encircling root vertex v_0, to which oriented trees are attached. After the construction of $\nabla(\mathfrak{q})$ is complete, each face of $\mathfrak{q}$ is of one of the two types:



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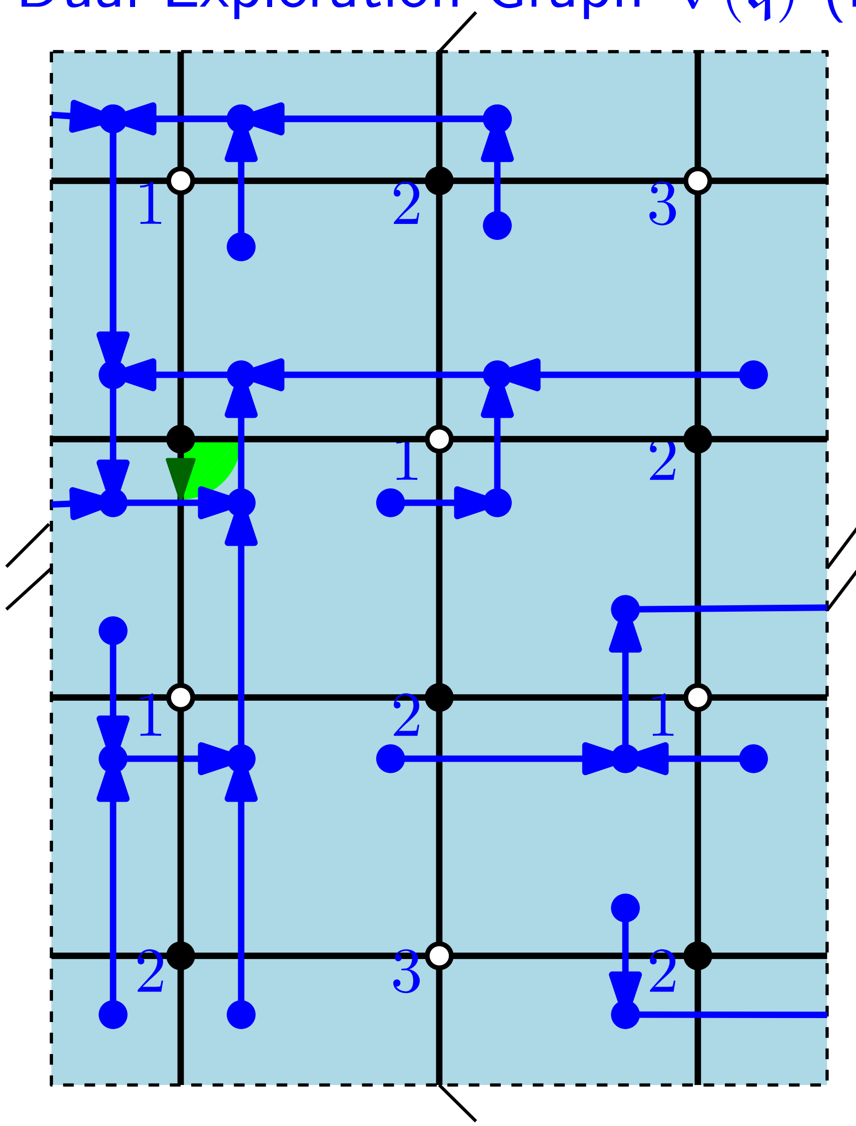
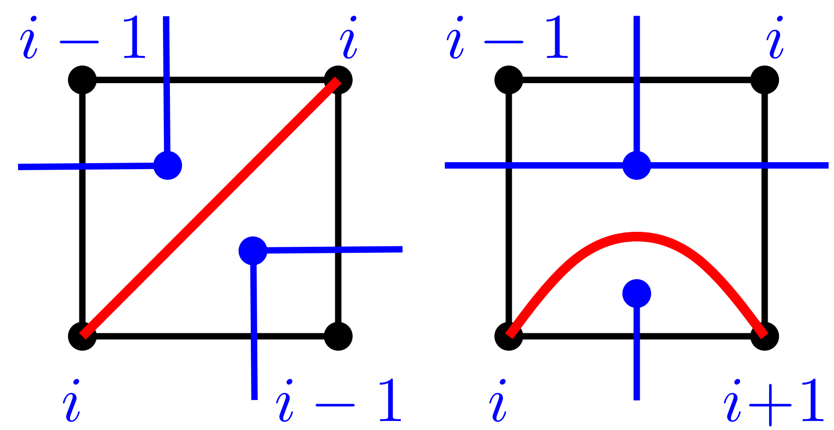


Proposition:
DEG $\nabla(\mathfrak{q})$ is formed by a unique oriented cycle encircling root vertex v_0, to which oriented trees are attached. After the construction of $\nabla(\mathfrak{q})$ is complete, each face of $\mathfrak{q}$ is of one of the two types:



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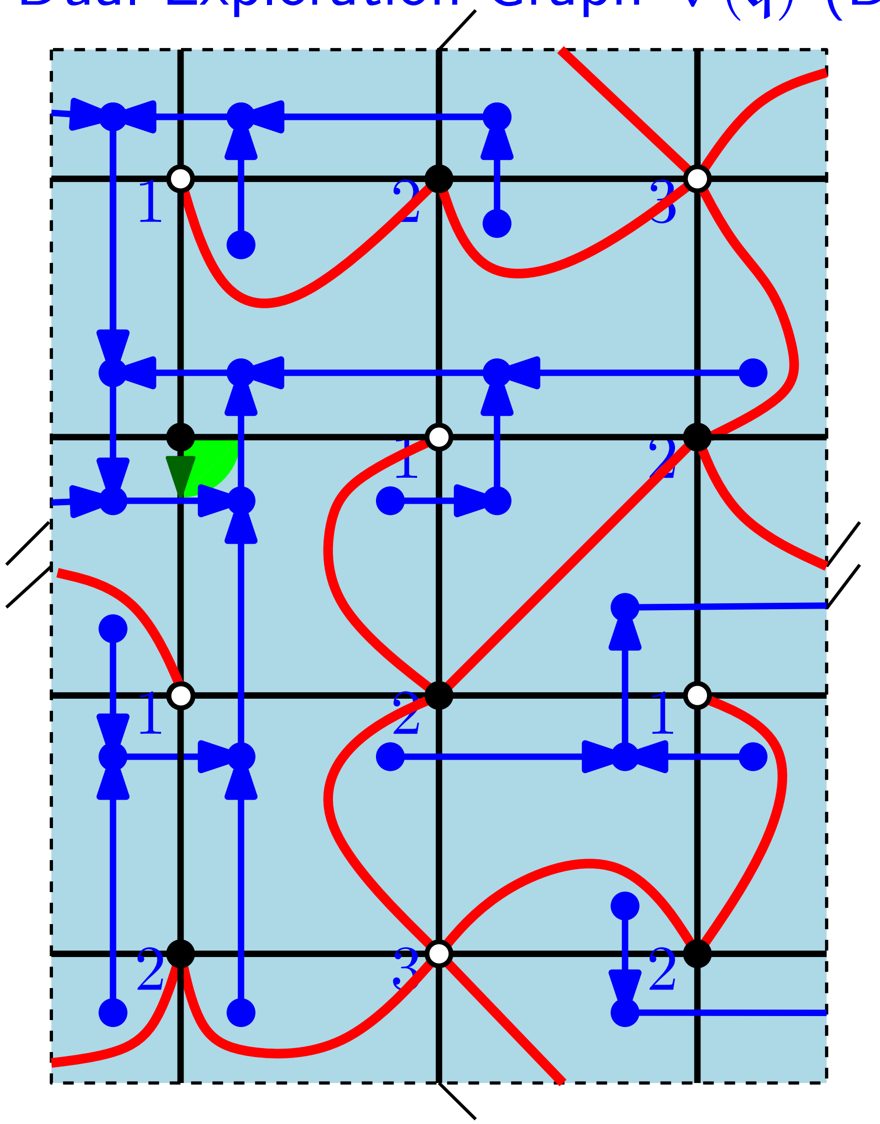

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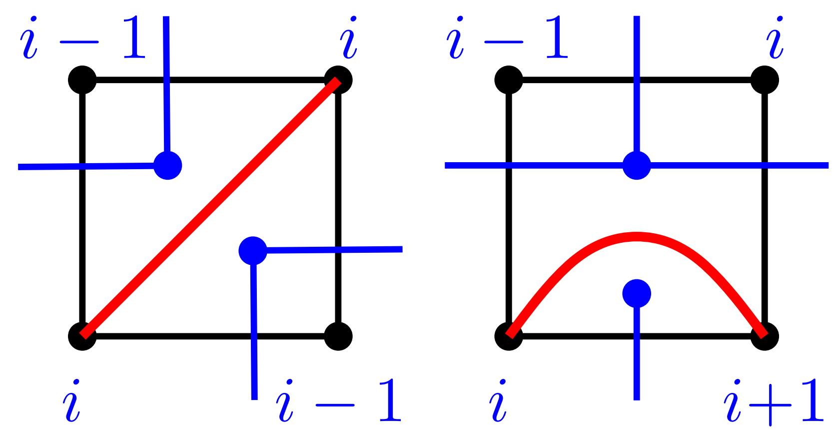

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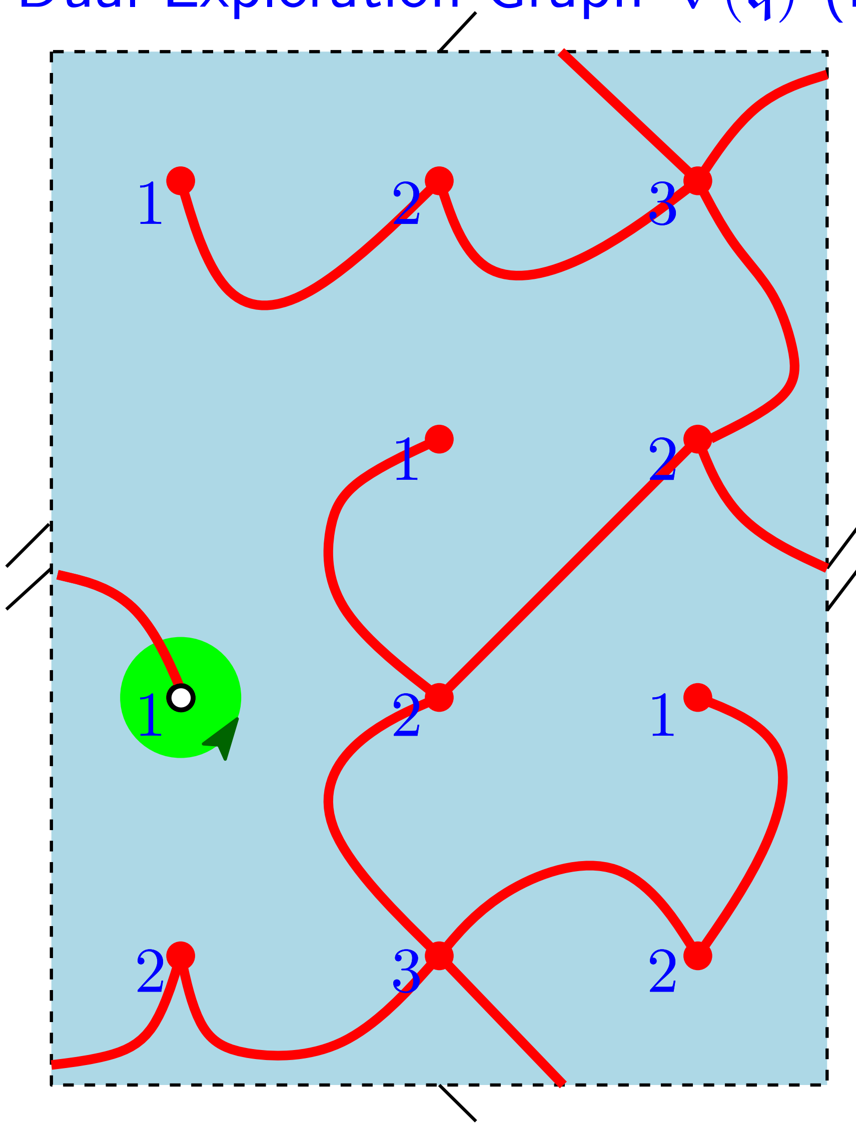

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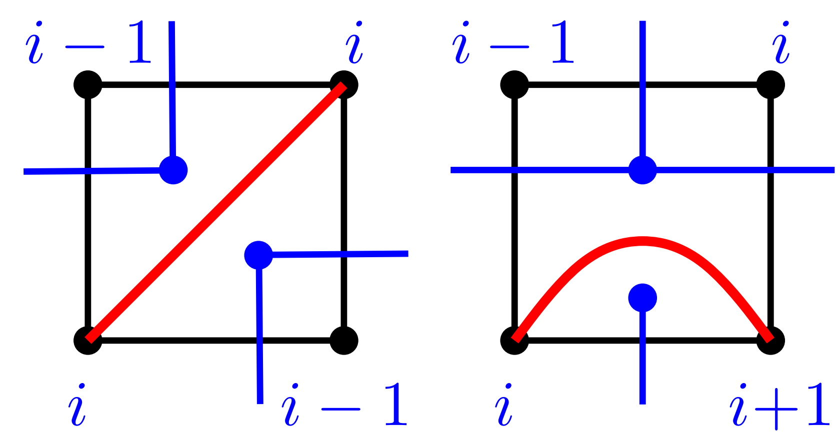

Corollary:

Red map $\phi(\mathfrak{q})$ is a one-face well-labeled rooted map with n edges, where n is the number of faces of $\mathfrak{q}$.

General case (III)

$\{$rooted, bipartite quadrangulations on $\mathbb{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)\}$

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$\{$rooted, WELL-LABELED, one-face maps on $\mathbb{S}$ with n edges and N_i vertices of label i $(i \geq 1)\}$

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Double rooting trick and Hall's marriage theorem - see next slide!

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$(\mathfrak{q}, v_0)$ - pointed, rooted quadrangulation

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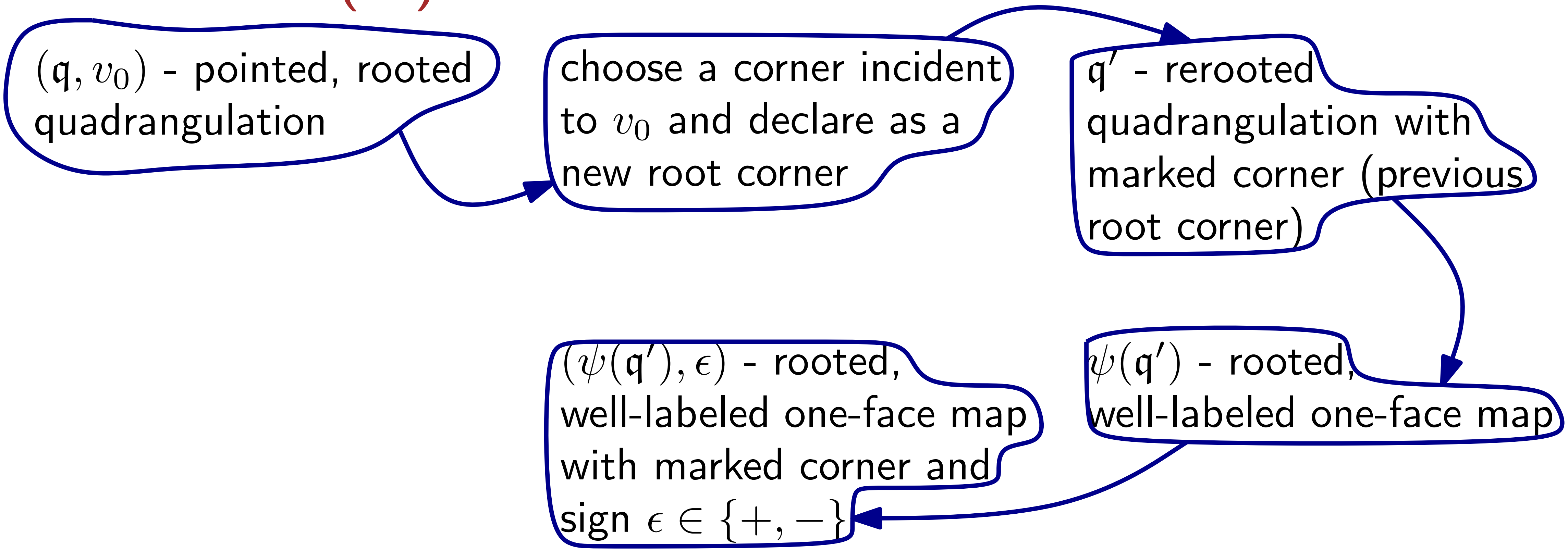

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$(\psi(\mathfrak{q}'), \epsilon)$ - rooted, well-labeled one-face map with marked corner and sign $\epsilon \in \{+, -\}$

$\mathfrak{q}'$ has $4n$ corners and map $\psi(\mathfrak{q}')$ has $2n$ corners - use some canonical way to choosa a corner in $\psi(\mathfrak{q}')$ and a sign $\epsilon \in \{+, -\}$ (for example - unique corner u incident to the root e of $\mathfrak{q}$ and sign depending on whether the label of u is smaller or greater than the other extremity of e)

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$\mathcal{Q}_{\mathbb{S},n,d}$ - set of all rooted quadrangulation on $\mathbb{S}$ with n faces and pointed vertex of degree d
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Hall's marriage theorem

Graph G has a perfect matching!

III. Applications

Enumeration - toy example

Let us try to enumerate maps with n edges on the projective plane:

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•number of rooted maps on the projective plane with n edges =

•number of rooted quadrangulations on the projective plane with n faces =

•(number of rooted, POINTED quadrangulations on the projective plane with n faces)/(number of vertices = n+1) =

•$\frac{2}{n+1}$(number of rooted, labeled, one-face maps on the projective plane with n edges)

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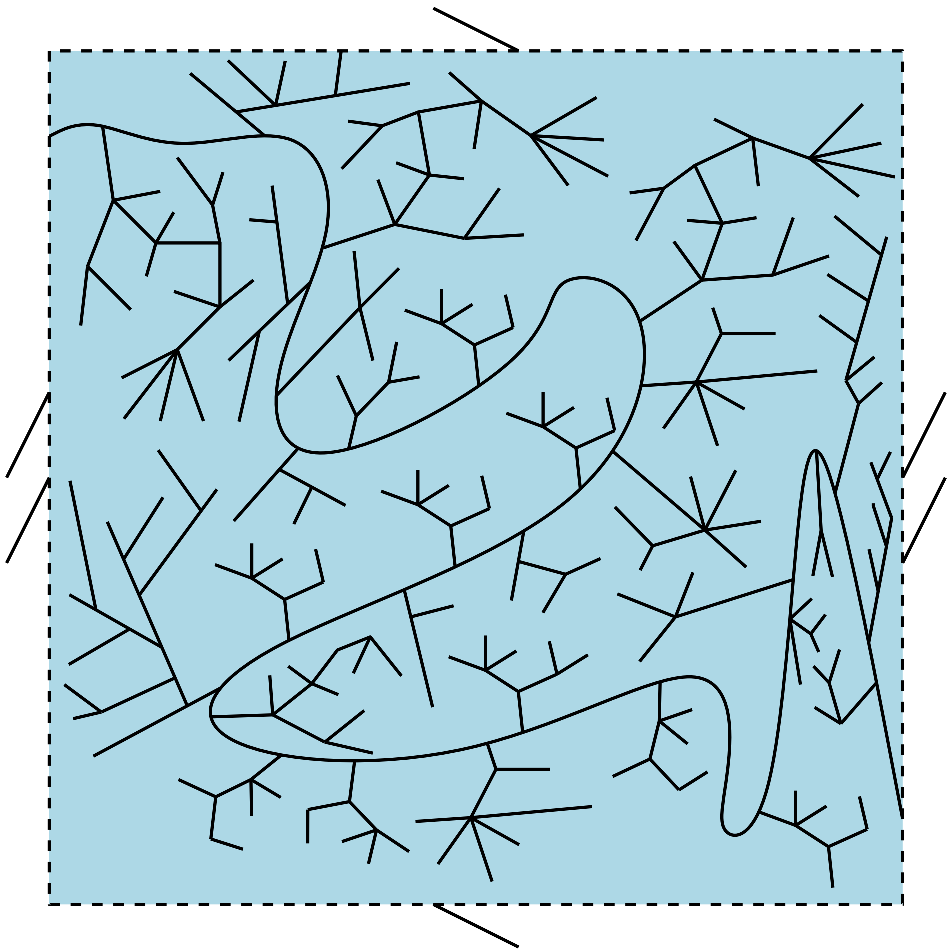

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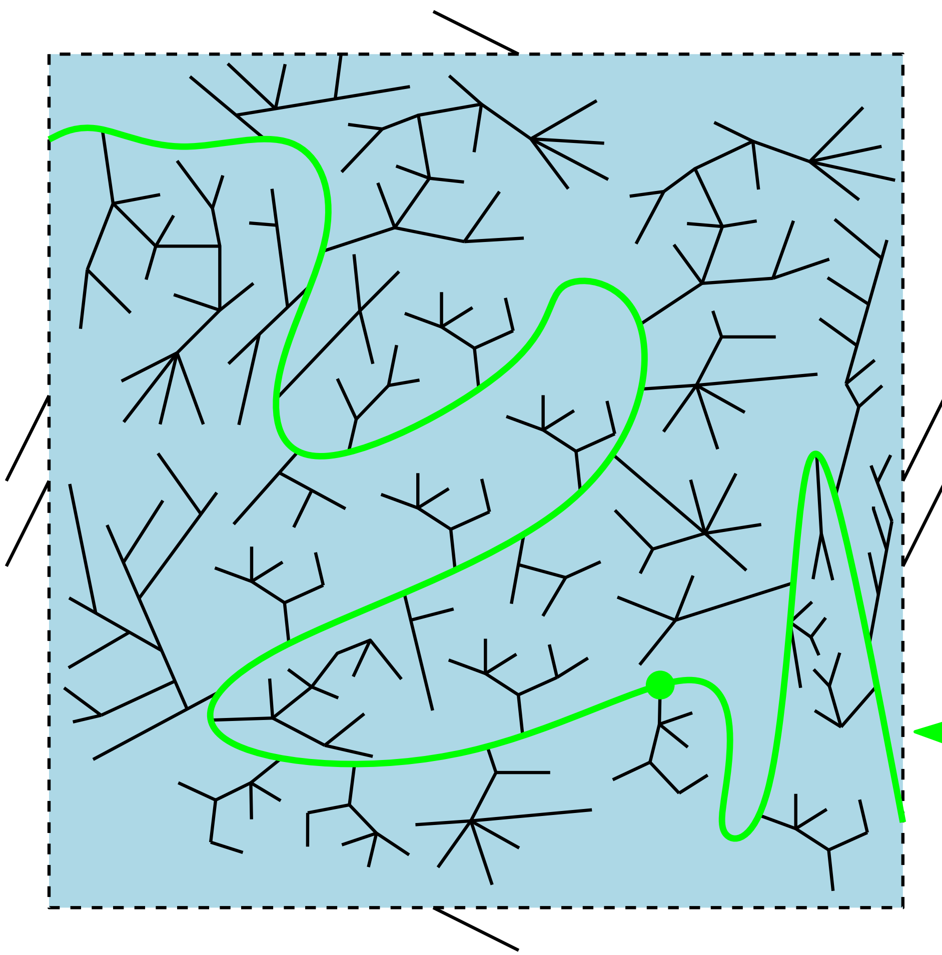

$$m_n = \frac{2}{n+1} \sum_{k \geq 1} b_k$$

Labeled cycle with k edges.

$$\boxed{b_k = \sum_{a+2b=k} \binom{k}{a,b,b}}$$

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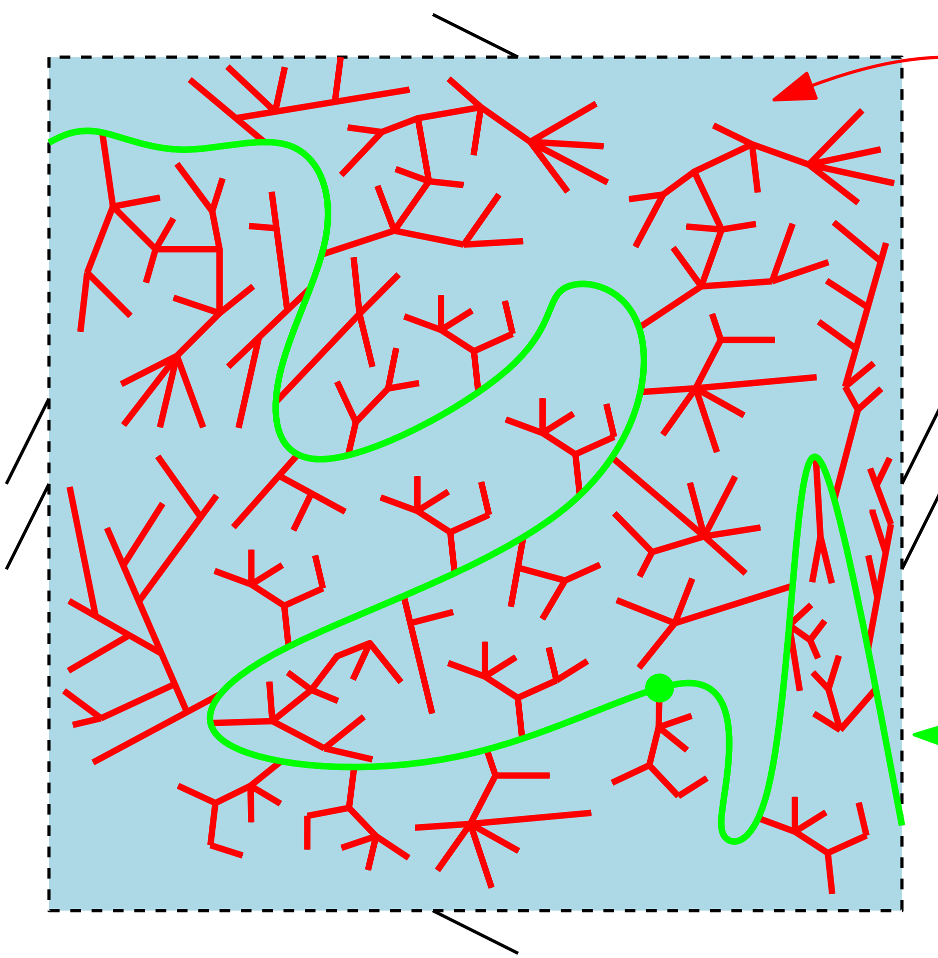

plane forests with $n-k$ edges and $2k$ components and labels.

$$m_n = \frac{2}{n+1} \sum_{k\geq 1} b_k \frac{2k}{n+k} \binom{2n-1}{n-k} \times 3^{n-k}$$

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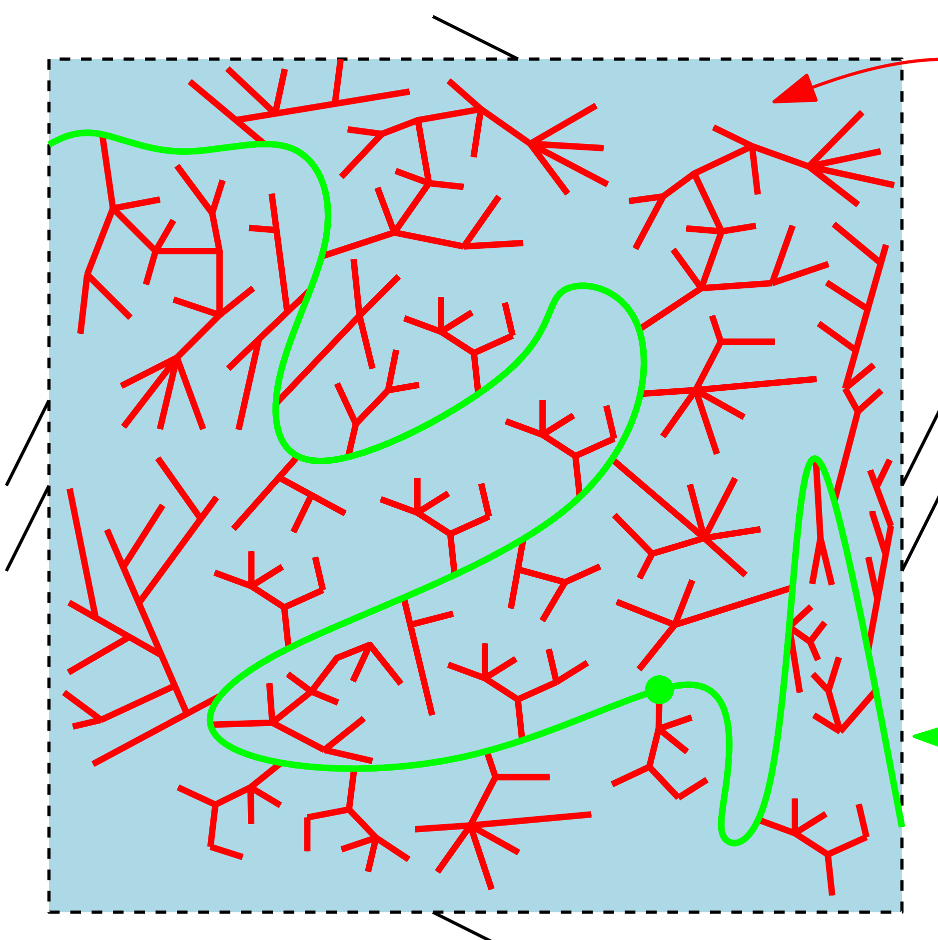

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$2n$ ways for choosing a root corner $\times$ 2 ways for choosing orientation.

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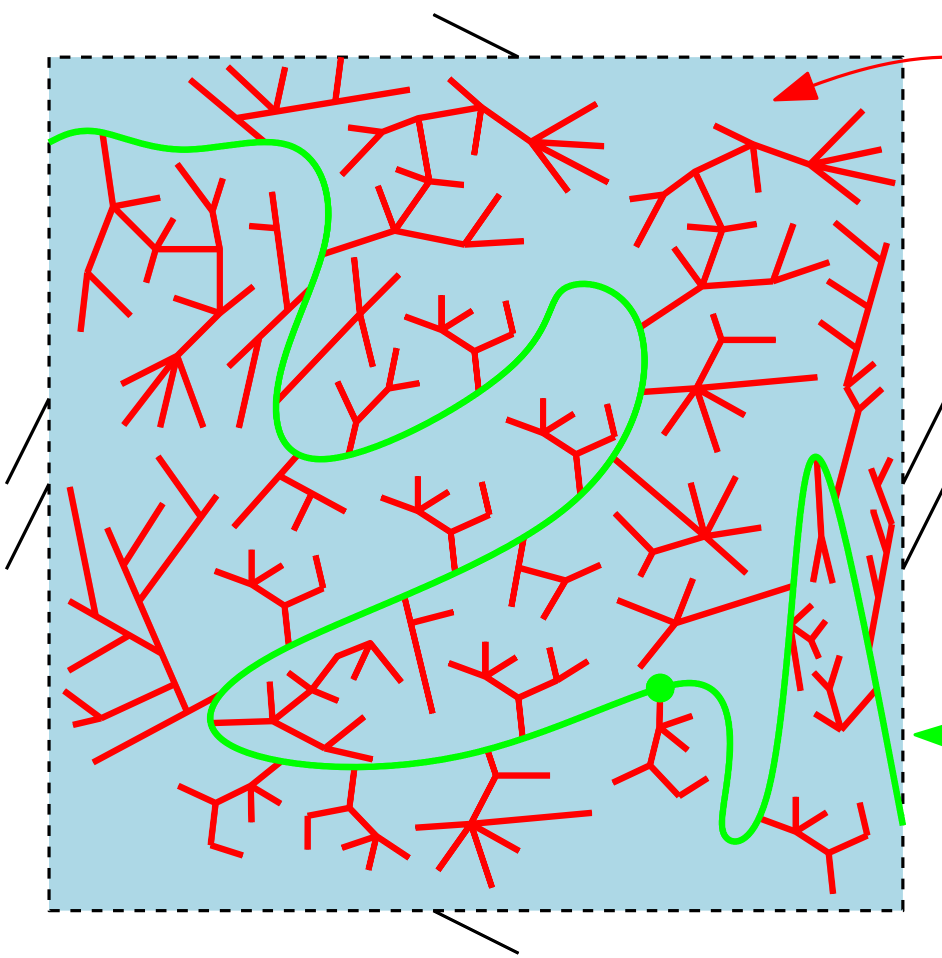

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$2n$ ways for choosing a root corner $\times$ 2 ways for choosing orientation.

$4k$ ways for deleting a second root.

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Enumeration

Theorem [Bender, Canfield 1986]

Let

$$Q_{\mathbb{S}}(t) := \sum_{n\geq 0} \vec{q}_{\mathbb{S},\bullet} t^n = \sum_{n\geq 0} (n+2-2h)\vec{q}_{\mathbb{S}}(n) t^n$$

be the generating function of rooted maps of type g pointed at a vertex or a face, by the number of edges. Moreover let $U \equiv U(t)$ and $T \equiv T(t)$ be the two formal power series defined by: $T = 1 + 3tT^2, \qquad U = tT^2(1 + U + U^2)$.
Then $Q_{\mathbb{S}}(t)$ is a rational function in U.

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Corollary [Bender, Canfield 1986]

For each $g \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$, there exists a constant p_g such that the number of rooted maps with n edges on the non-orientable surface of type g satisfies:

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Remark

Our main theorem allows us to recover Bender and Canfield results. In particular we can give some explicit (but very complicated) formula for the constant p_g.

Random maps

Let $(\mathcal{M}, v)$ be a map with distinguished vertex v. We define:

- radius of a map $\mathcal{M}$ centered at v by the quantity
$$R(\mathcal{M}, v) = \max_{u \in V(\mathcal{M})} d_{\mathcal{M}}(v, u);$$
- profile of distances from the distinguished point v (for any $r > 0$) by:
$$I_{(\mathcal{M},v)}(r) = \#\{u \in V(\mathcal{M}) : d_{\mathcal{M}}(v, u) = r\}.$$

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Theorem [Chapuy, D. 2014]

Let q_n be uniformly distributed over the set of rooted, bipartite quadrangulations with n faces on $\mathbb{S}$, and let v_0, v_* be two independent uniformly chosen vertices of q_n. Then, there exists a continuous, centered Gaussian process $L^{\mathbb{S}} = (L^{\mathbb{S}}_t, 0 \leq t \leq 1)$ such that:

- $\frac{9}{8n}^{1/4} R(q_n, v_0), \frac{9}{8n}^{1/4} R(q_n, v_*) \to \sup L^{\mathbb{S}} - \inf L^{\mathbb{S}};$
- $\frac{9}{8n}^{1/4} d_{q_n}(v_0, v_*), \frac{9}{8n}^{1/4} d_{q_n}(v_*, v_{**}) \to \sup L^{\mathbb{S}};$
- $\frac{I_{(q_n,v_0)}\left((8n/9)^{1/4}\cdot\right)}{n+2-2h}, \frac{I_{(q_n,v_*)}\left((8n/9)^{1/4}\cdot\right)}{n+2-2h} \to \mathcal{I}^{\mathbb{S}},$

where $\mathcal{I}^{\mathbb{S}}$ is defined as follows: for every non-negative, measurable $g : \mathbb{R}_+ \to \mathbb{R}_+$,
$$\langle \mathcal{I}^{\mathbb{S}}, g \rangle = \int_0^1 dt g(L^{\mathbb{S}}_t - \inf L^{\mathbb{S}}).$$

Random maps (II)

Few words about the process $L^{\mathbb{S}}$ ($\mathbb{S}$ = sphere for simplicity).

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• Contour process $c_n : [0, 2n] \to \mathbb{R}$ of the rooted, pointed quadrangulation $\mathfrak{q}_n$ with n faces: $c_n(i) = d_{\psi(\mathfrak{q}_n)}(v_i, v_0)$, where v_0 - root vertex of $\psi(\mathfrak{q}_n)$, v_i - vertex visited in the i-th step during the walk along the boundary of $\psi(\mathfrak{q}_n)$.

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• after proper normalization, the contour of uniformly chosen random rooted tree with n edges converges in distribution to the co-called normalized Brownian excursion $c^{\mathbb{S}}$ (informally - standard Brownian motion conditioned to remain non-negative on $[0, 1]$ and to take value 0 at the time 1).

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• Label process $L_n : [0, 2n] \to \mathbb{R}$ of the rooted, pointed quadrangulation $\mathfrak{q}_n$ with n faces: $L_n(i) = \ell(c_i)$, where c_0 - root corner of $\psi(\mathfrak{q}_n)$, c_i - corner visited in the i-th step during the walk along the boundary of $\psi(\mathfrak{q}_n)$.

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• after normalization by $\frac{9}{8n}^{1/4}$, label process of uniformly chosen pointed, rooted, planar quadrangulation with n faces converges to the so-called head of the Brownian snake $L^{\mathbb{S}} = (L^{\mathbb{S}}_t, 0 \leq t \leq 1)$ which is, conditionally on $c^{\mathbb{S}}$, continuous Gaussian process with covariance:

$$Cov(L^{\mathbb{S}}_s, L^{\mathbb{S}}_t) = \inf\{c^{\mathbb{S}}_u : \min(s, t,) \leq u \leq \max(s, t)\}.$$

IV. Further directions

- Generalization of the Bouttier-Di Francesco-Guitter bijection for non-orientable maps (bijection between bipartite $2p$-angulations, or, more generally bipartite maps with n faces of prescribed degrees and some kind of non-orientable mobiles?)

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- Studying random maps on ANY surface in Gromov-Hausdorff topology (convergence of bipartite quadrangulations up to extraction of SUBSEQUENCE is proved (Bettinelli, Chapuy, D.) - what about full convergance)?).

THANK
YOU!