Non-Orientable Branched Coverings, b-Hurwitz Numbers, and Positivity for Multiparametric Jack Expansions

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joint work with

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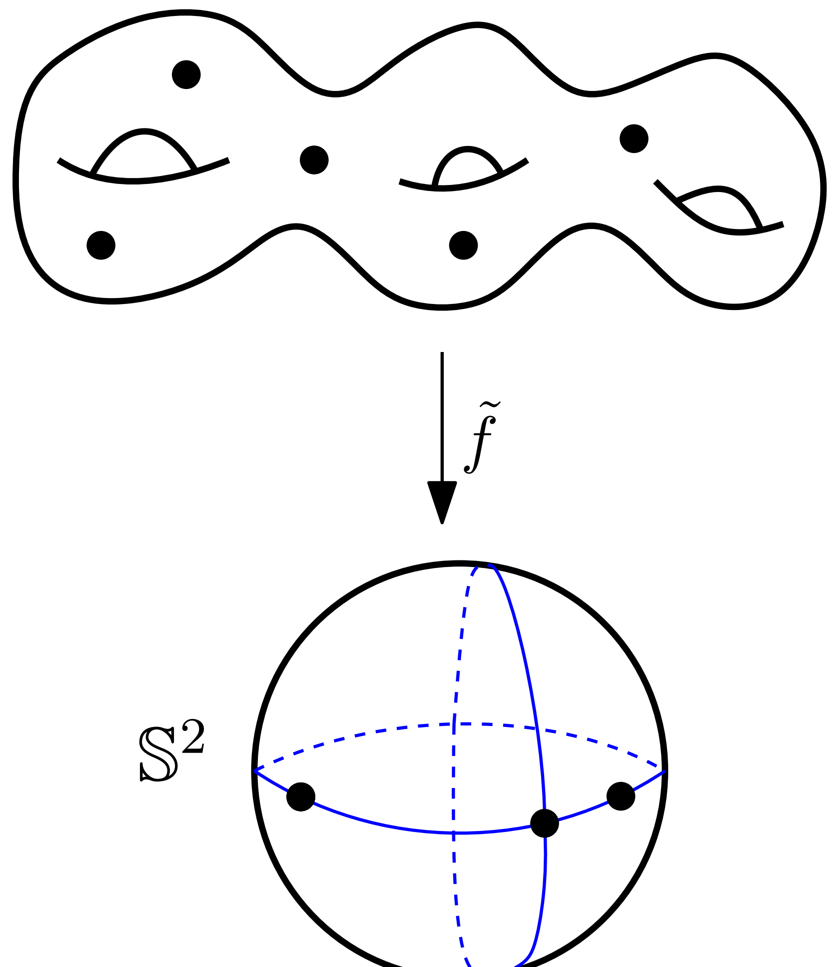
Hurwitz's problem

Problem: Classify/count all the branched coverings of the sphere $\mathbb{S}^2$



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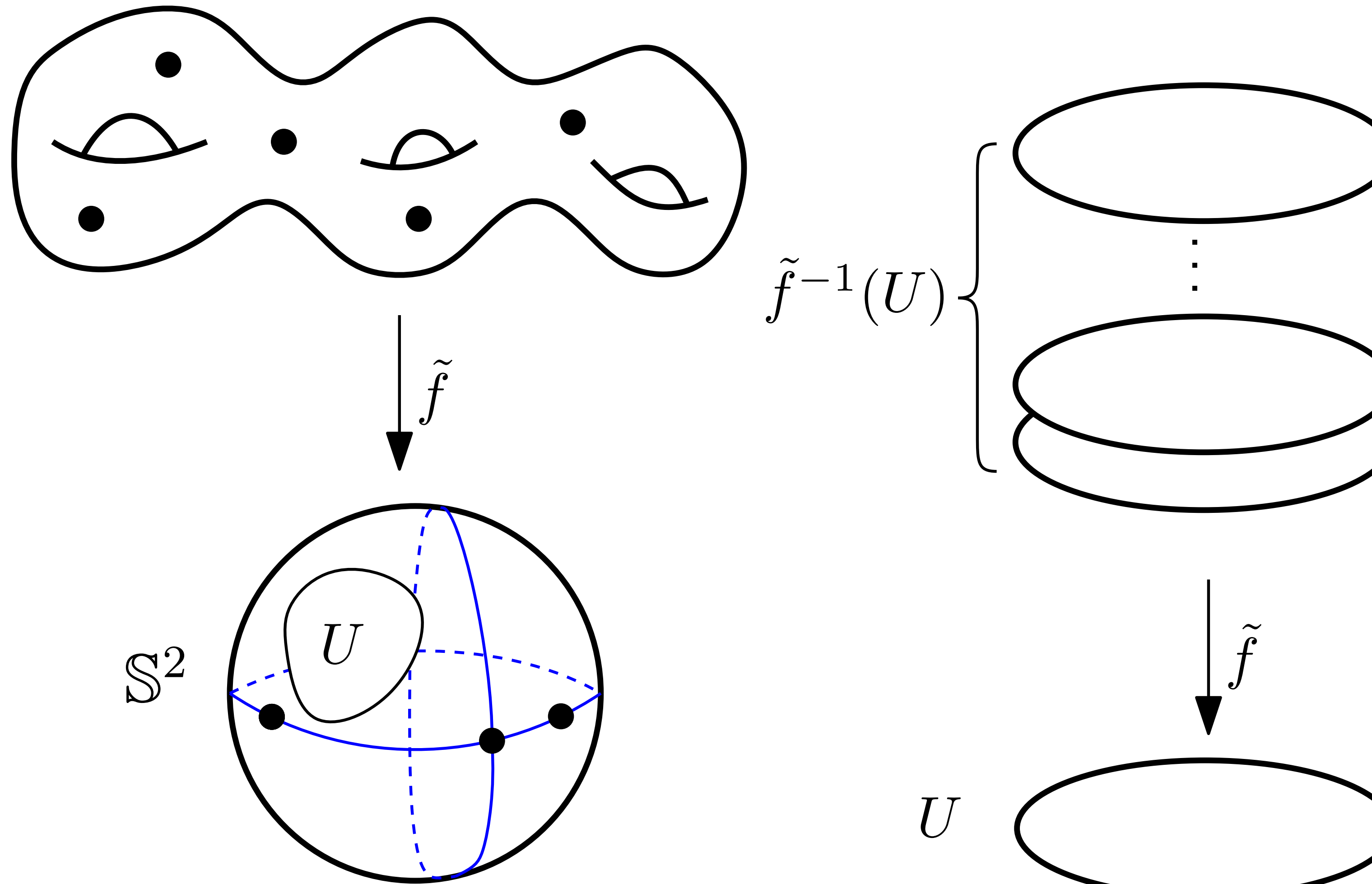

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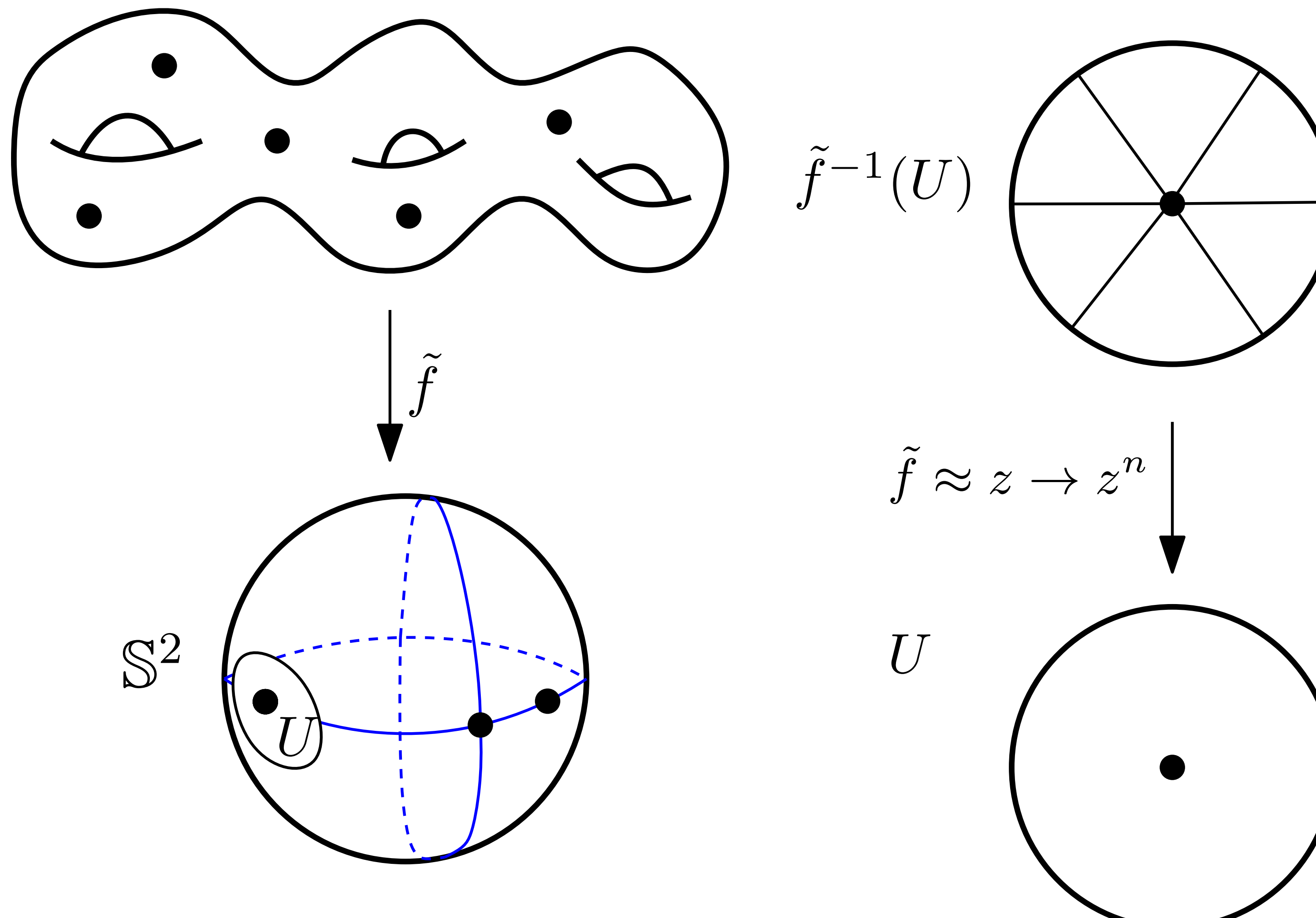

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Hurwitz's observation: Classifying/counting all the branched coverings of the sphere $\mathbb{S}^2 \equiv$ counting factorizations:

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$$\sigma_1 \cdots \sigma_k = \mathrm{id}$$

$$\tau_k^{(0)} := \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma_1 \cdots \sigma_{k+2} = \mathrm{id} \in \mathfrak{S}(n)} \prod_{i=1}^{k+2} \mathbf{p}^{(i)}(\sigma_i)$$

where $\mathbf{p}^i(\sigma) := \prod_{c:\text{ cycle in } \sigma} p^{(i)}_{\ell(c)}$

Example: $(1245)(3) \cdot (1)(23)(4)(5) \cdot (54321) = \mathrm{id}$

$$\mathbf{p}^{(1)}((1245)(3))\mathbf{p}^{(2)}(1)(23)(4)(5)\mathbf{p}^{(3)}(54321) = p_1^{(1)} p_4^{(1)} (p_1^{(2)})^3 p_2^{(2)} p_5^{(3)}$$

$\frac{td}{dt} \log \tau_k^{(0)}$ - g.f. of transitive k-factorizations modulo conjugation $\equiv$ g.f. of branched coverings

Branched coverings & symmetric functions

Power-sum and Schur symmetric functions: Recall that:

- $p_i := \sum_j x_j^i$ - power-sum symmetric function
- χ_λ - character of the irreducible repr. ρ_λ of the symmetric group
- $s_\lambda(\mathbf{p}) := \frac{1}{n!}\sum_{\sigma\in\mathfrak{S}(n)}\chi_\lambda(\sigma)\mathbf{p}(\sigma)$ - Schur symmetric function

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Frobenius character formula:

- C_μ - conjugacy class of permutatios of a cycle type μ, i.e. $\mathbf{p}(\sigma) = \prod_{i=1} p_{\mu_i}$.
- $c_\mu = \sum_{\sigma\in C_\mu} \sigma$

$$[\mathrm{id}]c_{\mu^1}\cdots c_{\mu^k} = \frac{1}{n!}\sum_\lambda \frac{\chi_\lambda(c_{\mu^1})\cdots\chi_\lambda(c_{\mu^k})}{\dim(\rho_\lambda)^{k-2}}$$

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Corollary (cool formula):

$$\tau_k^{(0)} = \sum_{n \geq 0} t^n \sum_{\lambda \vdash n} \frac{\dim(\rho_\lambda)}{n!}^2 \prod_{i=1}^{k+2} \tilde{s}_\lambda(\mathbf{p}^{(i)})$$ where $\tilde{s}_\lambda := \frac{n!}{\dim(\rho_\lambda)} s_\lambda$

Proof: Definition + Frobenius formula

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  Multiparametric tau function of Hurwitz numbers; tau function of the KP (or more generally, Toda) hierarchy, fundamental function in the field [Okounkov,Orlov,Pandariphande]

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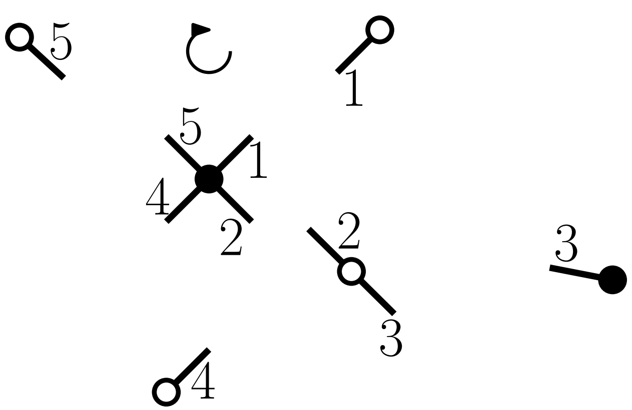

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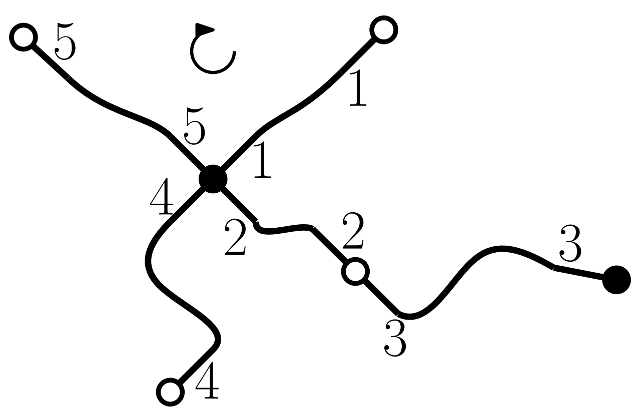

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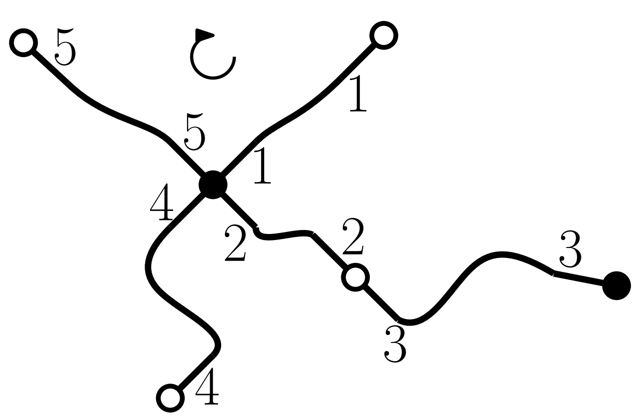

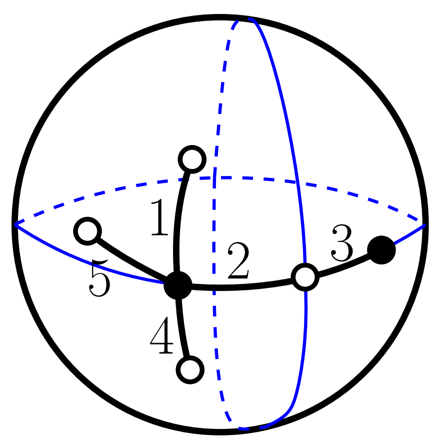

Labeled map

Map $\equiv$ graph embedded into a surface, such that it cuts this surface into simply connected pieces

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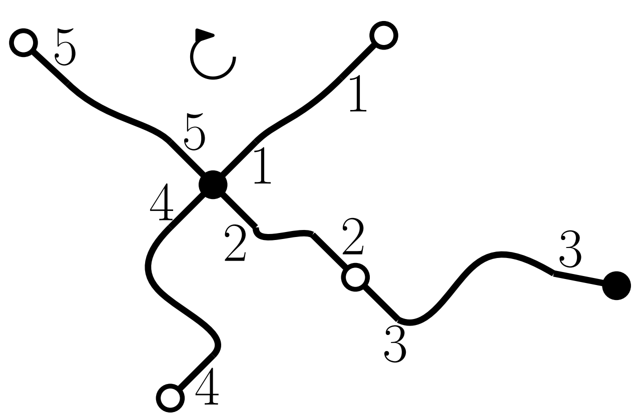

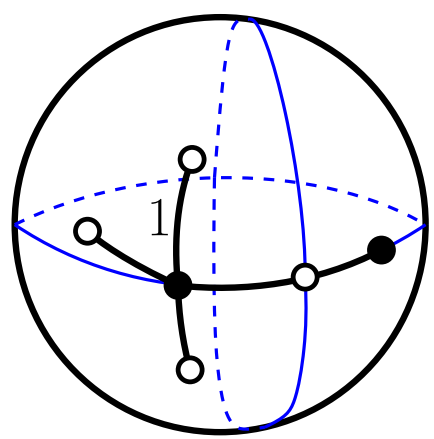

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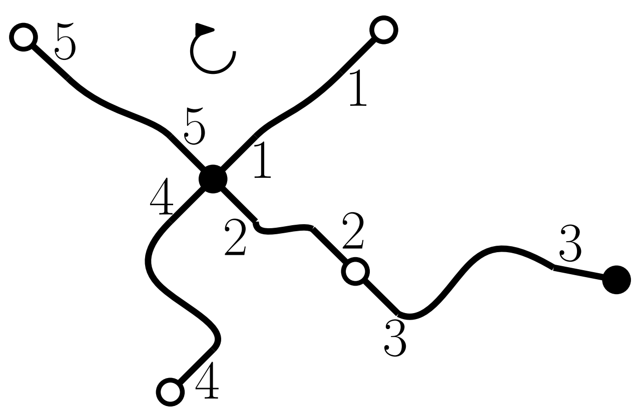

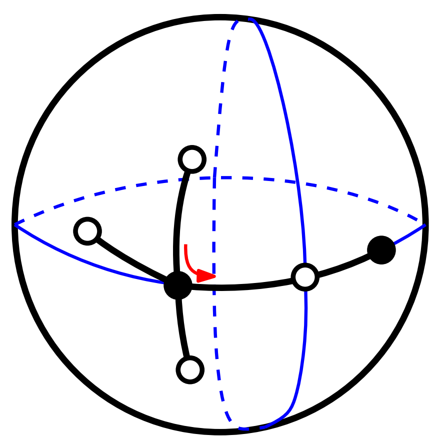

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Summary:

$$\tau_1^{(0)}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \sum_M \frac{t^{e(M)}}{e(M)!} \prod_{v_\bullet \in V_\bullet(M)} p_{\deg(v_\bullet)} \prod_{v_\circ \in V_\circ(M)} q_{\deg(v_\circ)} \prod_{f \in F(M)} r_{\deg(f)/2}$$

sum over **orientable**, labeled and possibly disconnected maps

$$\frac{td}{dt} \log \tau_1^{(0)}(\mathbf{p}, \mathbf{q}, \mathbf{r}) = \sum_M t^{e(M)} \prod_{v_\bullet \in V_\bullet(M)} p_{\deg(v_\bullet)} \prod_{v_\circ \in V_\circ(M)} q_{\deg(v_\circ)} \prod_{f \in F(M)} r_{\deg(f)/2}$$

sum over **orientable**, rooted and connected maps

What about **non-orientable** maps?

Non-oriented maps - representation theory & symmetric functions

Question: Can we encode non-oriented bipartite maps in an analogous way (representation theory/symmetric functions theory)?

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YES!

	orientable maps	non-oriented maps
rep. theory	Rep. theory of the symmetric group $\mathfrak{S}(n)$	Rep. theory of the Gelfand pair $(\mathfrak{S}(2n), H(n))$ [Hanlon, Stembridge, Stanley '92]
symmetric function	normalized Schur $\tilde{s}_\lambda$	Zonal polynomials Z_λ [Goulden, Jackson '96]

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$$2\frac{td}{dt}\log\tau_1^{(1)}(\mathbf{p},\mathbf{q},\mathbf{r}) :=$$

$$\sum_M t^{e(M)} \prod_{v_\bullet\in V_\bullet(M)} p_{\deg(v_\bullet)} \prod_{v_\circ\in V_\circ(M)} q_{\deg(v_\circ)} \prod_{f\in F(M)} r_{\deg(f)/2}$$

sum over **non-oriented**, rooted and connected maps

Then $\tau_1^{(1)}(\mathbf{p},\mathbf{q},\mathbf{r}) := \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{\dim(\rho_{2\lambda})}{(2n)!} Z_\lambda(\mathbf{p})Z_\lambda(\mathbf{q})Z_\lambda(\mathbf{r})$

Jack symmetric functions and the b-conjecture

So far we know that:

- $\frac{td}{dt}\log\tau_1^{(0)} := \frac{td}{dt}\log\sum_{n\geq 0} t^n \sum_{\lambda\vdash n}\frac{\dim(\rho_\lambda)^2}{(n)!^2}\tilde{s}_\lambda(\mathbf{p})\tilde{s}_\lambda(\mathbf{q})\tilde{s}_\lambda(\mathbf{r})$ g.f. of **orientable** maps
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If you are an expert in symmetric functions theory you can recognise that:

- $\tilde{s}_\lambda = J_\lambda^{(1)}, \qquad \|J_\lambda^{(1)}\|^2_{(1)} = \frac{\dim(\rho_\lambda)^2}{(n)!^2}$
- $Z_\lambda = J_\lambda^{(2)}, \qquad \|J_\lambda^{(2)}\|^2_{(2)} = \frac{\dim(\rho_{2\lambda})}{(2n)!}$

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Two-lines crash-course on Jack polynomials:

- $J_\lambda^{(\alpha)} = \mathrm{hook}_\alpha(\lambda) m_\lambda + \sum_{\mu<\lambda} a_\mu^\lambda(\alpha) m_\mu, \; a_\mu^\lambda(\alpha) \in \mathbb{Q}(\alpha)$ (uppertriangularity)
- $\langle J_\lambda^{(\alpha)}, J_\mu^{(\alpha)}\rangle_{(\alpha)} = \delta_{\mu,\lambda}\, \underline{\mathrm{hook}_\alpha(\lambda)}\; \underline{\mathrm{hook}_\alpha(\lambda)'}$ (orthogonality)

  where $\langle p_\lambda, p_\mu\rangle_{(\alpha)} := \delta_{\mu,\lambda}|C_\lambda|\alpha^{\ell(\lambda)}$

α-deformations of classical hook products

Think: Jack polynomials are symmetric functions obtained by applying Gram-Schmidt orthogonalization process to the monomial basis

Jack symmetric functions and the b-conjecture

So far we know that:

- $(1+0)\frac{td}{dt}\log\sum_{n\geq 0}t^n\sum_{\lambda\vdash n}\frac{J_\lambda^{(1+0)}(\mathbf{p})J_\lambda^{(1+0)}(\mathbf{q})J_\lambda^{(1+0)}(\mathbf{r})}{\|J_\lambda^{(1+0)}\|^2}$ g.f. of **orientable** maps
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Conjecture (the b-conjecture) [Goulden,Jackson '96]

Let

$$\tau_1^{(b)} := \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{J_\lambda^{(1+b)}(\mathbf{p})J_\lambda^{(1+b)}(\mathbf{q})J_\lambda^{(1+b)}(\mathbf{r})}{\|J_\lambda^{(1+b)}\|^2}$$

There exists a statistic MON (Measure Of Non-orientability) such that
$\mathrm{MON}(M) \in \mathbb{Z}_{\geq 0}$ and
$\mathrm{MON}(M) = 0$ if and only if M is orientable
and $\tau_1^{(b)}$ is the generating series of non-oriented maps:

$$(1+b)\frac{td}{dt}\log\tau_1^{(b)} = \sum_M t^{e(M)} b^{\mathrm{MON}(M)} \prod_{v_\bullet\in V_\bullet(M)} p_{\deg(v_\bullet)} \prod_{v_\circ\in V_\circ(M)} q_{\deg(v_\circ)} \prod_{f\in F(M)} r_{\deg(f)/2}$$

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There exists a statistic MON (Measure Of Non-orientability) such that $\mathrm{MON}(M)\in\mathbb{Z}_{\geq 0}$ and $\mathrm{MON}(M)=0$ if and only if M is orientable and $\tau_1^{(b)}$ is the generating series of non-oriented maps:

$$(1+{\color{blue}b})\frac{td}{dt}\log\tau_1^{({\color{blue}b})} = \sum_M t^{e(M)}{\color{blue}b}^{\mathrm{MON}(M)}\prod_{v_\bullet\in V_\bullet(M)}p_{\deg(v_\bullet)}\prod_{v_\circ\in V_\circ(M)}q_{\deg(v_\circ)}\prod_{f\in F(M)}r_{\deg(f)/2}$$

Many special cases proved: [Burchardt, D., Feŕay, Goulden, Jackson, Kanunnikov, La Croix, Promyslov, Vassilieva, Visentin], Still wide open in general...

b-deformed tau function

Recall the general tau-function of the Toda hierarchy of branched coverings with $k+2$ branch points:

$$\tau_k^{(0)} = \sum_{n\geq 0} t^n \frac{\dim(\rho_\lambda)}{n!}^2 \sum_{\lambda\vdash n} \tilde{s}_\lambda(\mathbf{p})\tilde{s}_\lambda(\mathbf{q}) \prod_{i=1}^k \tilde{s}_\lambda(\underline{u_i}),\text{ where } \underline{u_i} = (u_i, u_i, \dots)$$

Inspired by Goulden and Jackson's b-conjecture define the b-deformed tau function:

$$\tau_k^{(b)} = \sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{J_\lambda^{(1+b)}(\mathbf{p}) J_\lambda^{(1+b)}(\mathbf{q}) \prod_{i=1}^k J_\lambda^{(1+b)}(\underline{u_i})}{\| J_\lambda^{(1+b)} \|_{(1+b)}}.$$

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Theorem [Chapuy, D. '20]

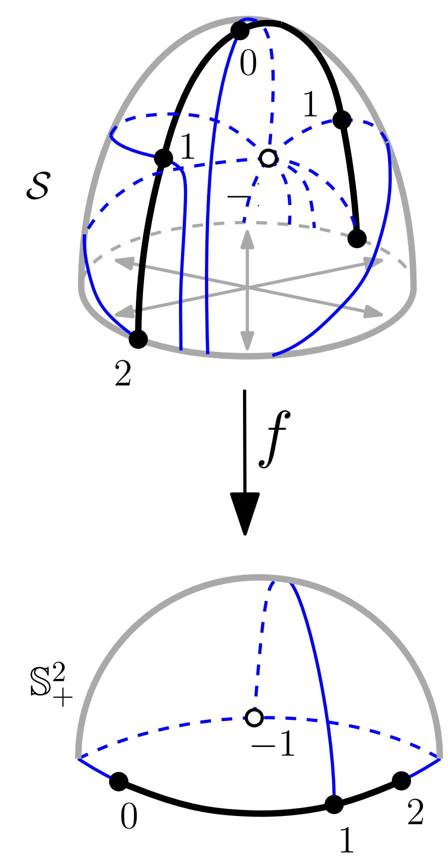
$$(1+b)\frac{td}{dt}\log\tau_k^{(b)} = \sum_{f:\mathcal{S}\to\mathbb{S}_+} \kappa(f) t^{|f|} b^{\mathrm{MON}(f)},$$

rooted generalized branched coverings f of the sphere $\mathbb{S}$ by a connected compact surface, orientable or not, with $k+2$ ramification points

$$\kappa(f) = p_{\lambda^{-1}(f)} q_{\lambda^0(f)} u_1^{v_1(f)} \dots u_k^{v_k(f)}$$

ramification profile of the first point

ramification profile of the second point



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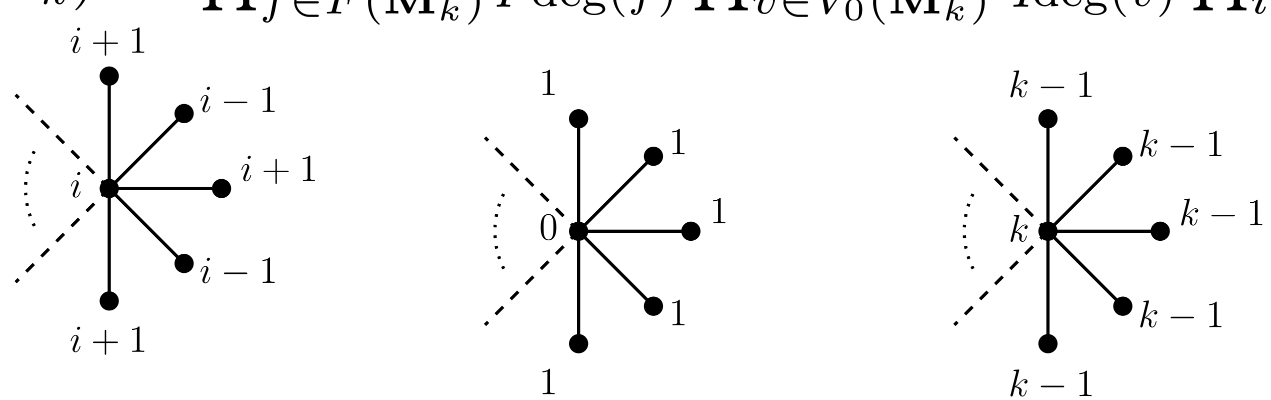
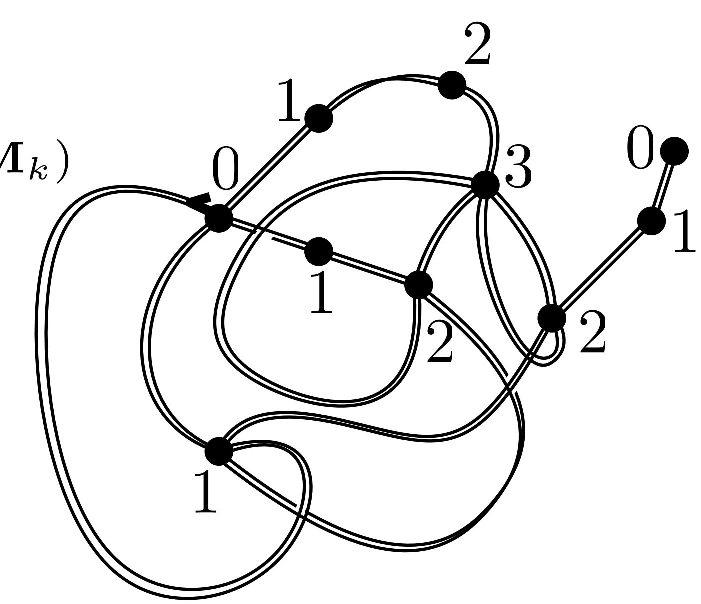
rooted k-constellations, orientable or not

$$\kappa(\mathbf{M}_k) := \prod_{f\in F(\mathbf{M}_k)} p_{\deg(f)} \prod_{v \in V_0(\mathbf{M}_k)} q_{\deg(v)} \prod_{i=1}^k u_i^{v_i(\mathbf{M}_k)}$$





Idea of the proof

Maps experts:

- remove the root edge and analyse how your map changed (classical ideas [Tutte '63, Lehman and Walsh '72])
- try to do the same with constallations: replace the root edge by a "rooted branch" $0 \to 1 \to \cdots \to k$ - analysis is (much) harder but still possible!
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A miracle!!!

- prove that PDE1 = PDE2 (very long and technical proof using heavy algebraic manipulations and lifting original operators to much bigger spaces by adding new variables to the picture)

Applications and problems

- we prove that the (logarithm of) tau function of weighted b-deformed Hurwitz numbers:

$$(1+b)\log\sum_{n\geq 0} t^n \sum_{\lambda\vdash n} \frac{J_\lambda^{(1+b)}(\mathbf{p})J_\lambda^{(1+b)}(\mathbf{q})\prod_{\square\in\lambda} G(c_b(\square))}{\|J_\lambda^{(1+b)}\|_{(1+b)}}.$$

  has nonnegative integer coefficients. It covers the case of

  • b-deformed classical Hurwitz numbers (single or double)

  • b-monotone Hurwitz numbers (aka topological expansion of b-deformed HCIZ integral)

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- Many questions: integrability (Virasoro constraints + BKP structure at $b = 1$ in some cases [Bonzom,Chapuy, D.])? the proof of the b-conjecture? geometric interpretation (which moduli space? the meaning of MON?)

THANK YOU!

References:

- arXiv:2004.07824
- arXiv:2109.01499
- arXiv:2110.12834

Measure Of Non-orientability (MON)

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Types of edges:

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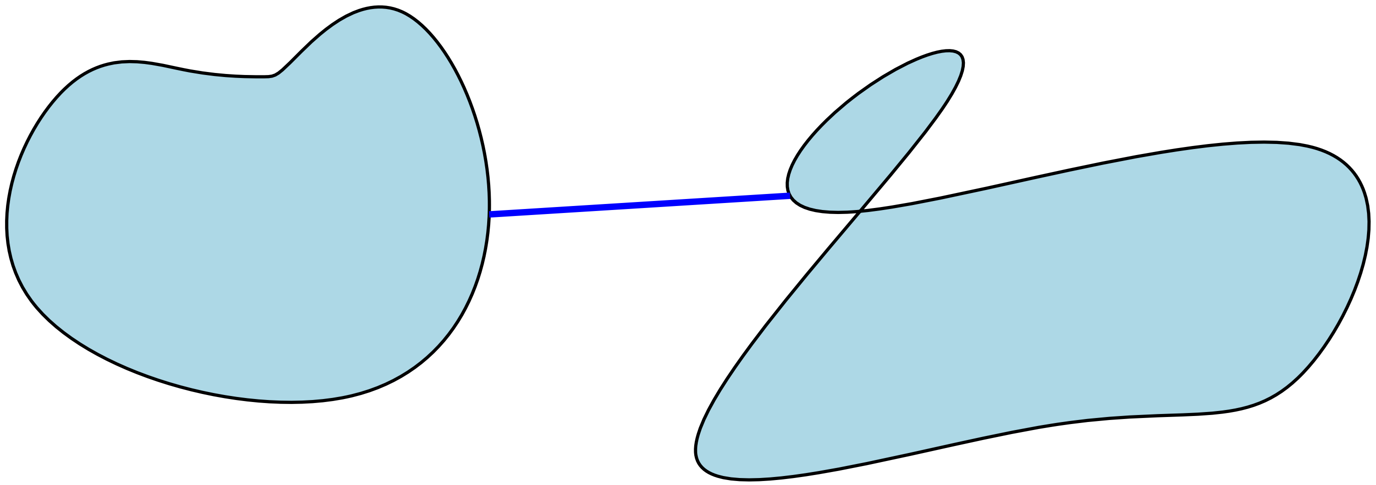

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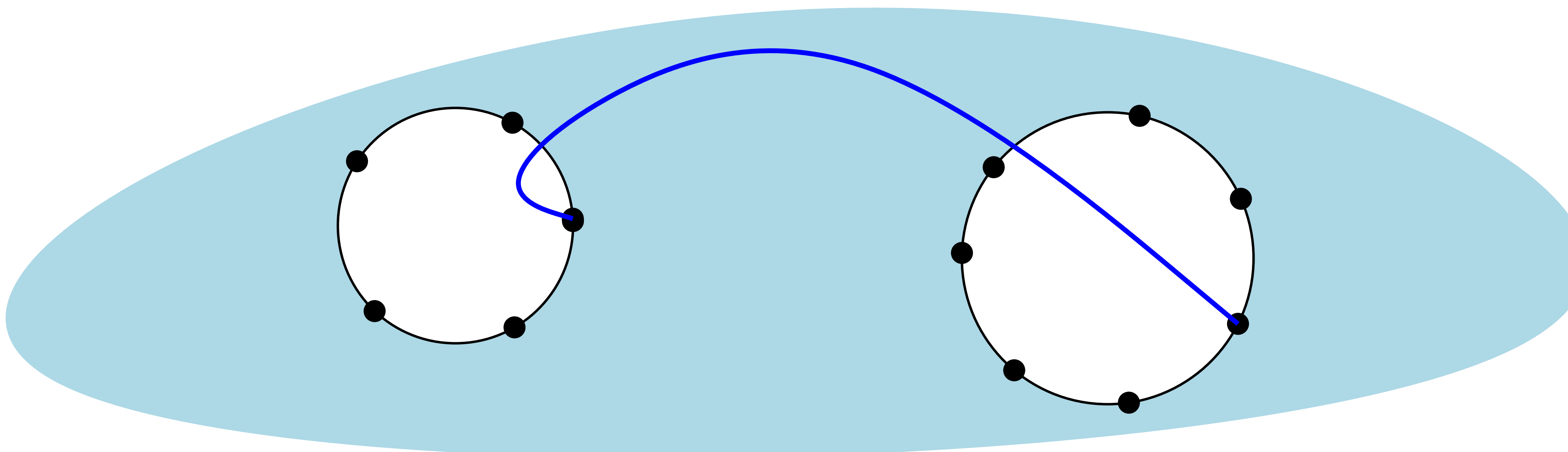

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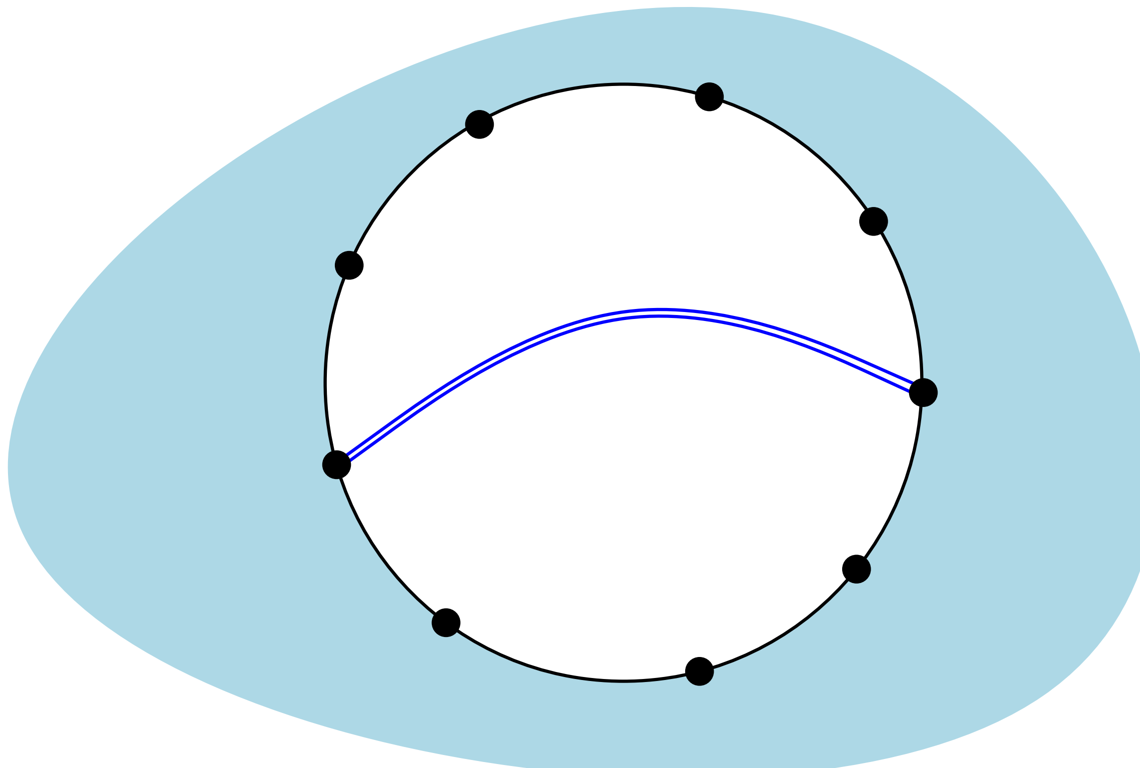

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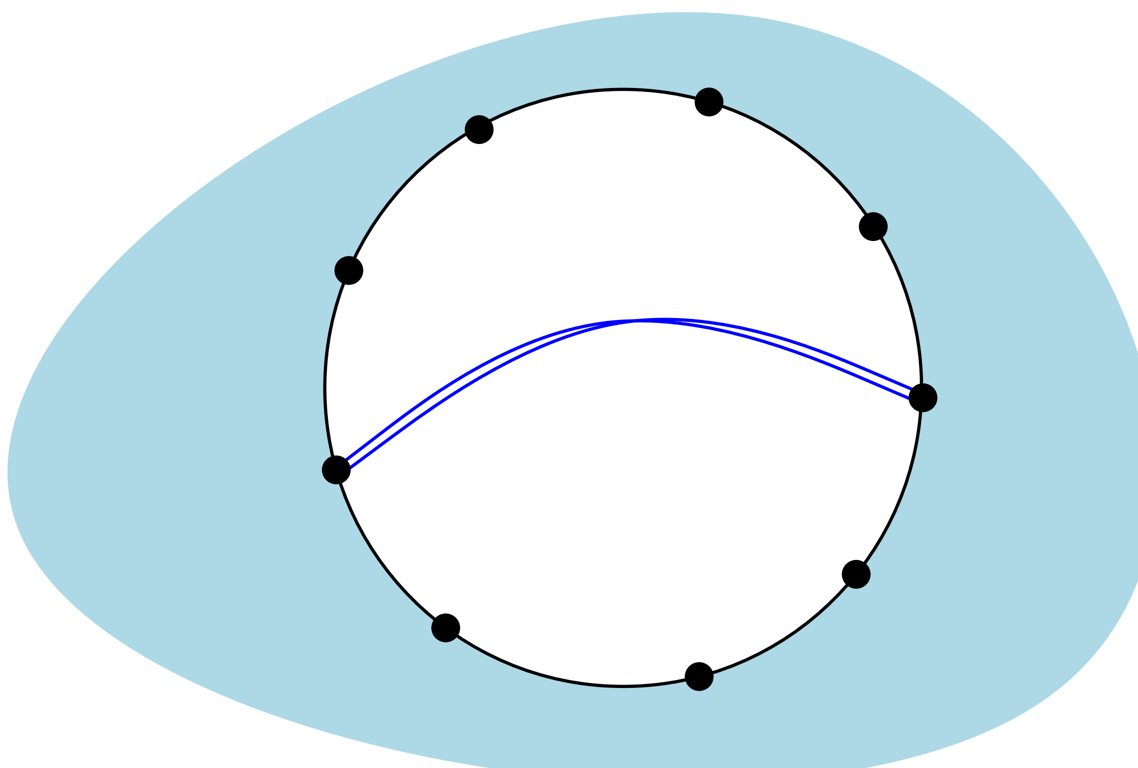

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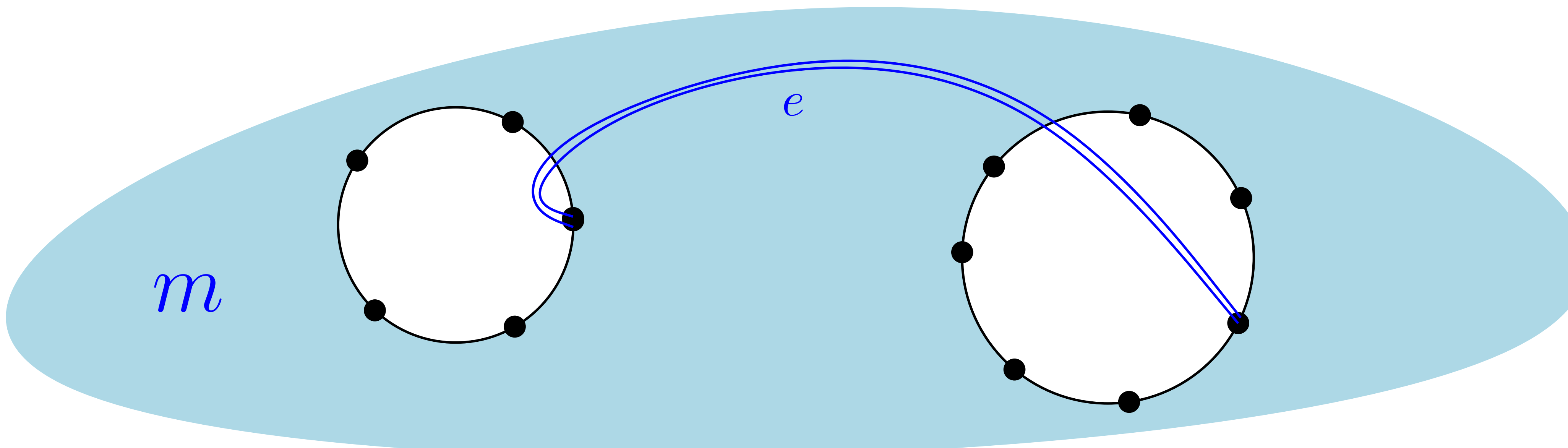

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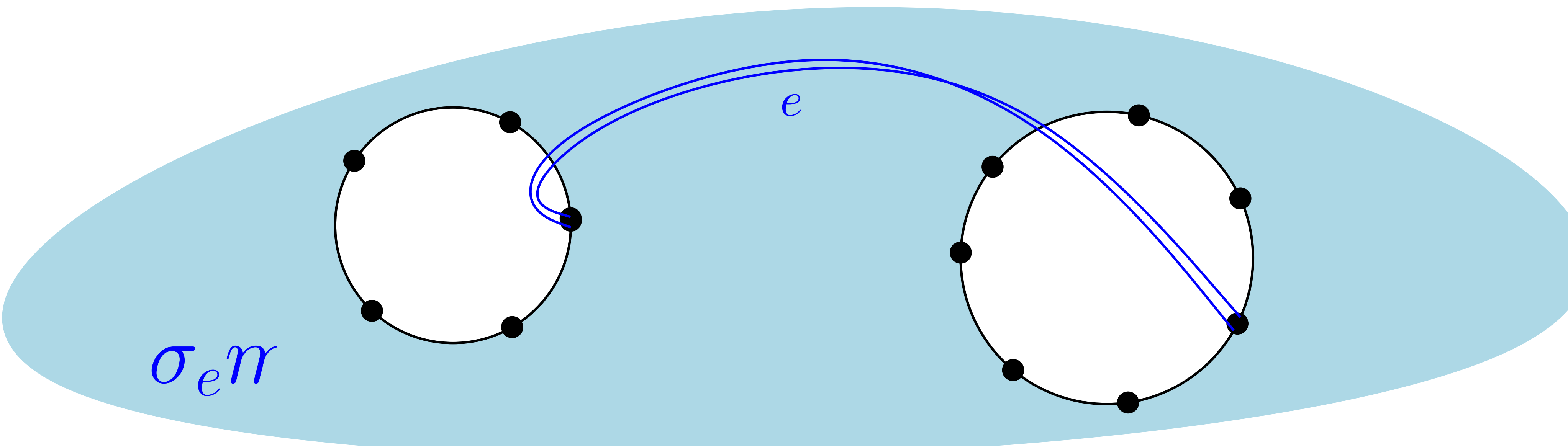

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