Central limit theorem for random Young diagrams with respect to Jack measure

(joint work with Valentin Féray)

Maciej Dołęga

LIAFA, Université Paris Diderot,
Instytut Matematyczny, Uniwersytet Wrocławski

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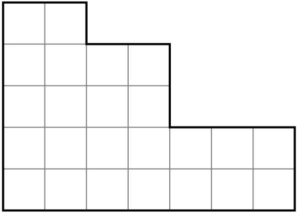
Young diagrams

Definition

A partition λ is a finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. It can be represented by a Young diagram λ. The size of the Young diagram λ is defined by $|\lambda| := \sum_i \lambda_i$.



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We want to investigate some asymptotic properties of Young diagrams as their size is tending to infinity. How to do it?

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Look on 'large Young diagrams' from a 'large perspective' and treat these discrete objects as continuous ones!

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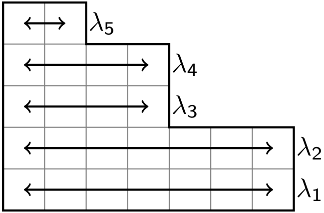

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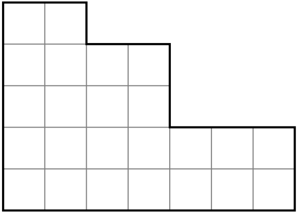

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Young diagrams as continuous objects

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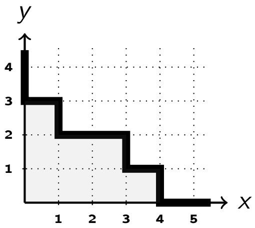

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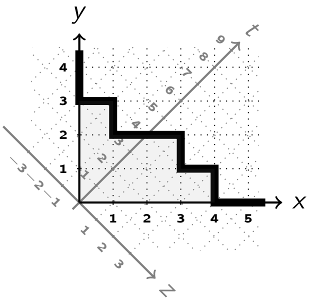

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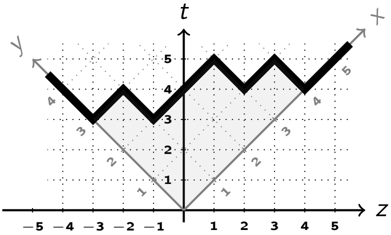

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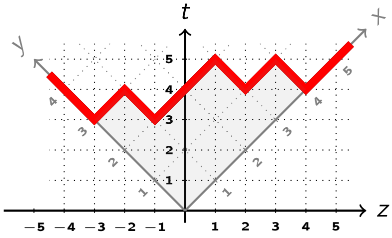

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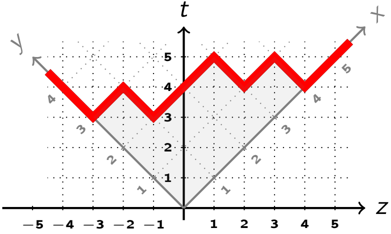

Definition

A profile of a Young diagram λ is a function $\omega(\lambda) : \mathbb{R} \to \mathbb{R}_+$ such that its graph is a profile of λ drawn in Russian convention.

Continuous Young diagrams

Definition

A continuous Young diagram is a function $\omega : \mathbb{R} \to \mathbb{R}_+$ such that

- $\omega(x) - |x|$ has compact support;
- $|\omega(x_1) - \omega(x_2)| \leq |x_1 - x_2|$ for any $x_1, x_2 \in \mathbb{R}$.

An area of a continuous Young diagram ω is given by:

$$\mathrm{Area}(\omega) := \frac{1}{2} \int_{\mathbb{R}} |\omega(x) - |x|| \, dx.$$

Remark

Let λ - Young diagram with $|\lambda| = n$. Then

$$\mathrm{Area}(\omega(\lambda)) = n.$$

Normalized Young diagrams

Problem

How to look on the 'large Young diagrams' from 'large perspective'?

Solution

Normalize them in a way that their areas are constant.

Definition

Let λ - Young diagram with $|\lambda| = n$. We define scaled (continuous) Young diagram

$$\omega(D_{\sqrt{n}^{-1}}(\lambda))(x) := \sqrt{n}^{-1}\omega(\lambda)(\sqrt{n}x).$$

Remark

$$\text{Area}\left(\omega(D_{\sqrt{n}^{-1}}(\lambda))\right) = 1.$$

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Asymptotic shape of large Young diagrams

Let $\left(\lambda_{(n)}\right)_{n\in\mathbb{N}_+}$ - sequence of Young diagrams with $|\lambda_{(n)}| = n$.

Definition

We say that $\left(\lambda_{(n)}\right)_{n\in\mathbb{N}_+}$ has a limit shape ω if

$$\left\|\omega(D_{\sqrt{n}^{-1}}(\lambda_{(n)})) - \omega\right\| \to 0,$$

as $n \to \infty$, where $\|f\| = \sup_{x\in\mathbb{R}} |f(x)|$.

Problem

Let us choose $\left(\lambda_{(n)}\right)_{n\in\mathbb{N}_+}$ randomly according with some 'nice' distribution. Does it have a limit shape with a high probability? Is it unique? Can we say more about it?

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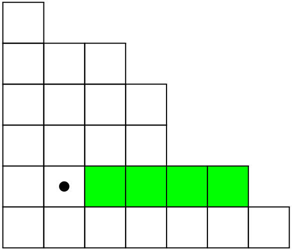
'Nice' distribution = Plancherel distribution



$a(\bullet)$ = number of boxes to the right of the given box

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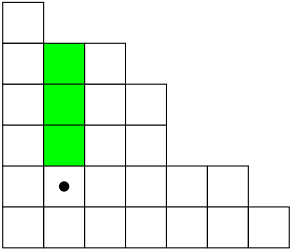

$\ell(\bullet)$ = number of boxes above the given box

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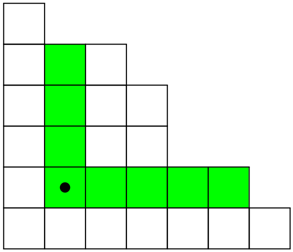

$$\text{hook}^{(1)}(\bullet) := a(\bullet) + \ell(\bullet) + 1.$$

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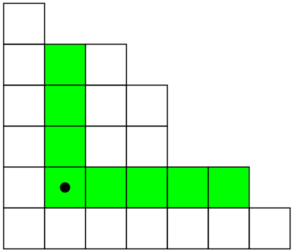

$$\mathbb{P}_n^{(1)}(\lambda) = \frac{\dim(\lambda)^2}{n!},$$

where (hook formula:)

$$\dim(\lambda) = \frac{n!}{\prod_{\square \in \lambda} \text{hook}(\square)}.$$

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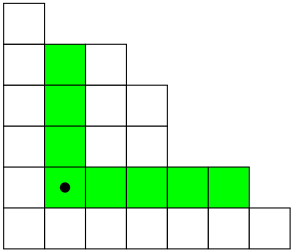

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Plancherel measure $\mathbb{P}_n^{(1)}$ is a probability measure on the set $\mathbb{Y}_n$ of Young diagrams of size n.

Vershik-Kerov, Logan-Shepp limit shape

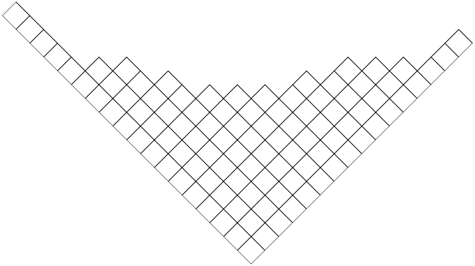

Figure: Scaled random Young diagram of size 100 distributed according with Plancherel measure

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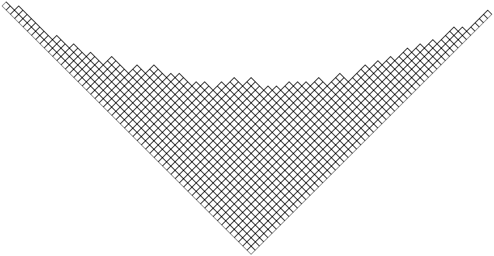

Figure: Scaled random Young diagram of size 1000 distributed according with Plancherel measure

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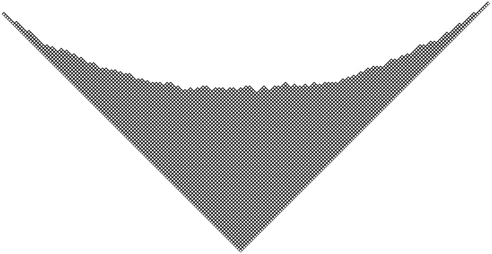

Figure: Scaled random Young diagram of size 5000 distributed according with Plancherel measure

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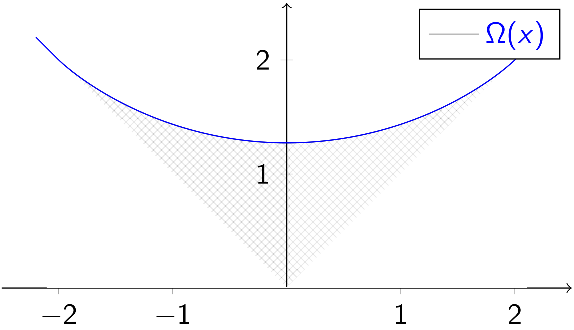
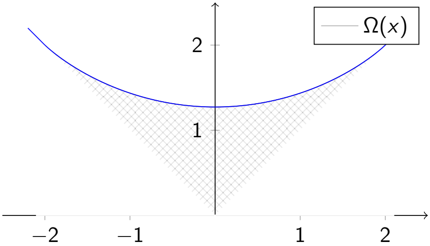


Figure: $\Omega(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin\frac{x}{2} + \sqrt{4-x^2}\right) & \text{otherwise.} \end{cases}$

First order asymptotic = 'law of large numbers'



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Theorem (Vershik-Kerov, Logan-Shepp '77)

Let $\lambda_{(n)}$ be a random Young diagram of size n distributed with Plancherel measure $\mathbb{P}_n^{(1)}$. Then, in probability, as $n \to \infty$

$$\left\| \omega\left(D_{1/\sqrt{n}}(\lambda_{(n)})\right) - \Omega \right\| \to 0.$$

Second order asymptotic

Problem

Can we describe the second order asymptotic? What does it really mean?

Solution

We should look on the fluctuations around the limit shape.

Let $\lambda_{(n)}$ - random Young diagram distributed according with $\mathbb{P}_n^{(1)}$.

- We know that $\left\|\omega\left(D_{1/\sqrt{n}}(\lambda_{(n)})\right) - \Omega\right\| \to 0$ in probability.
- We would like to investigate behaviour of random variables:

$$m_k(\lambda_{(n)}) := \int_{\mathbb{R}} x^k \Delta(\lambda_{(n)})(x)\, dx,$$

where

$$\Delta(\lambda)(x) := \sqrt{n}\frac{\omega\left(D_{1/\sqrt{n}}(\lambda)\right)(x) - \Omega(x)}{2}.$$

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Second order asymptotic = 'central limit theorem'

- $u_k(x) = U_k(x/2) = \sum_{0 \le j \le \lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} x^{k-2j}$;
- $u_k(2\cos(\theta)) = \frac{\sin((k+1)\theta)}{\sin(\theta)}$;
- $u_k(\lambda) = \int_{\mathbb{R}} u_k(x) \Delta(\lambda)(x)\ dx$.

Theorem (Kerov, 1993)

Choose a sequence $(\Xi_k)_{k=2,3,\dots}$ of independent standard Gaussian random variables and let $\lambda_{(n)}$ be a random Young diagram of size n distributed with Plancherel measure. As $n \to \infty$, we have:

$$\left(u_k(\lambda_{(n)})\right)_{k=1,2,\dots} \xrightarrow{d} \left(\frac{\Xi_{k+1}}{\sqrt{k+1}}\right)_{k=1,2,\dots}.$$

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Jack measure

Let us recall that

$$\mathrm{hook}(\square) = a(\square) + \ell(\square) + 1.$$

Let $\alpha \in \mathbb{R}_+$. α-deformations of hook length:

- $\mathrm{hook}^{(\alpha)}(\square) := \sqrt{\alpha}(a(\square) + 1) + \sqrt{\alpha}^{-1}\ell(\square)$,
- $\left(\mathrm{hook}^{(\alpha)}\right)'(\square) := \sqrt{\alpha}a(\square) + \sqrt{\alpha}^{-1}(\ell(\square) + 1)$.

Definition

is a probability measure $\mathbb{P}_n$ on the set $\mathbb{Y}_n$ defined by

$$\mathbb{P}_n(\lambda) := \frac{n!}{\prod_{\square\in\lambda}(\mathrm{hook}(\square)\,(\mathrm{hook})'(\square)}$$

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Plancherel measure is a probability measure $\mathbb{P}_n^{(1)}$ on the set $\mathbb{Y}_n$ defined by

$$\mathbb{P}_n^{(1)}(\lambda) := \frac{n!}{\prod_{\square\in\lambda}(\text{hook}^{(1)}(\square)\left(\text{hook}^{(1)}\right)'(\square)} = \frac{n!}{\prod_{\square\in\lambda}(\text{hook}(\square))^2}$$

.

- for $\alpha = 1$ Jack measure $\equiv$ Plancherel measure.

α-anisotropic Young diagrams

Let λ be a Young diagram.

Definition

α-anisotropic Young diagram $A_\alpha(\lambda)$ (for $\alpha \in \mathbb{R}_+$) - continuous Young diagram obtained from λ (considered in French convention) by a horizontal stretching of ratio $\sqrt{\alpha}$ and a vertical stretching of ratio $\sqrt{\alpha}^{-1}$.

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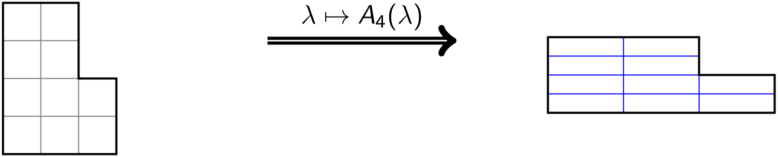

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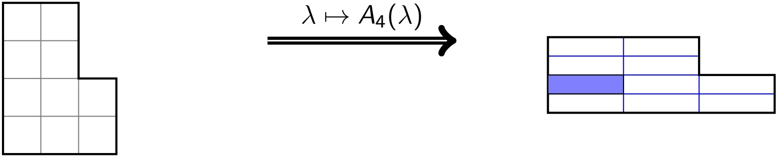

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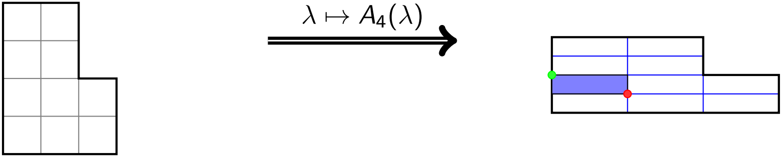

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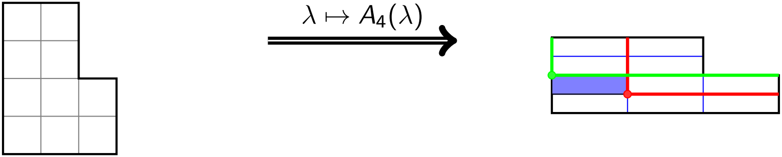

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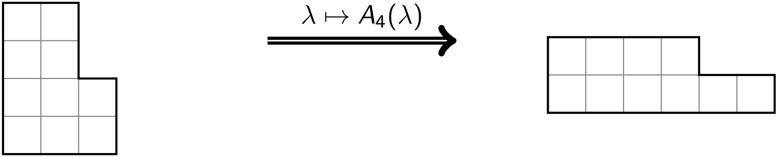

First order asymptotic = 'law of large numbers'

Theorem (D., Féray)

Let $\lambda_{(n)}$ be a random Young diagram of size n distributed with Jack measure $\mathbb{P}_n^{(\alpha)}$. Then, in probability, as $n \to \infty$

$$\left\| \omega\left(D_{1/\sqrt{n}}\left(A_\alpha\left(\lambda_{(n)}\right)\right)\right) - \Omega \right\| \to 0.$$

Remark

Plugging $\alpha = 1$ we recover Vershik-Kerov, Logan-Shepp limit shape for Plancherel measure.

Second order asymptotic = 'central limit theorem'

Theorem (D. Féray)

Choose a sequence $(\Xi_k)_{k=2,3,\ldots}$ *of independent standard Gaussian random variables and let* $\lambda_{(n)}$ *be a random Young diagram of size* n *distributed with Jack measure. As* $n \to \infty$*, we have:*

$$\left(u_k^{(\alpha)}(\lambda_{(n)})\right)_{k=1,2,\ldots} \xrightarrow{d} \left(\frac{\Xi_{k+1}}{\sqrt{k+1}} - \frac{\gamma}{k+1}[k \text{ is odd}]\right)_{k=1,2,\ldots},$$

where $u_k^{(\alpha)}(\lambda) = \int_{\mathbb{R}} u_k(x)\Delta(A_\alpha(\lambda))(x)\, dx$, $\gamma := \sqrt{\alpha} - \sqrt{\alpha}^{-1}$, *and we use the usual notation* [*condition*] *for the indicator function of the corresponding condition.*

Remark

Plugging $\alpha = 1$ *we recover central limit theorem of Kerov for Plancherel measure.*

Jack polynomials and Jack characters

Jack polynomials $J_\lambda^{(\alpha)}$:

- symmetric functions introduced by Jack;
- generalization of Schur symmetric function (for $\alpha = 1$);
- special case of Macdonald polynomials

Expand Jack polynomial in power-sum symmetric basis:

$$J_\lambda^{(\alpha)} = \sum_{\substack{\rho: \\ |\rho|=|\lambda|}} \theta_\rho^{(\alpha)}(\lambda)\, p_\rho.$$

We call quantities $\theta_\rho^{(\alpha)}(\lambda)$ Jack characters (for $\alpha = 1$ they coincide with the irreducible characters of the symmetric groups up to some normalization constant).

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Characterization of Jack measure

$$\mathbb{E}_{\mathbb{P}_n^{(\alpha)}}(\theta_\mu^{(\alpha)}) = \begin{cases} 1 & \text{if } \mu = 1^n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\lambda \in \mathbb{Y}_n$.

Proposition

$\int_{\mathbb{R}} x^k \Delta(A_\alpha(\lambda))(x)\, dx$ can be expressed as a function of

$$\theta^{(\alpha)}_{(1^n)}, \theta^{(\alpha)}_{(2,1^{n-2})}, \dots, \theta^{(\alpha)}_{(k-1,1^{n-k+1})}.$$

Corollary

Our central limit theorem has equivalent, algebraic version!

Algebraic central limit theorem

Theorem (D., Féray)

Choose a sequence $(\Xi_k)_{k=2,3,\dots}$ of independent standard Gaussian random variables. As $n \to \infty$, we have:

$$\left(\frac{\sqrt{k}\, \theta^{(\alpha)}_{(k,1^{n-k})}(\lambda_{(n)})}{n^{k/2}} \right)_{k=2,3,\dots} \xrightarrow{d} (\Xi_k)_{k=2,3,\dots},$$

where the distribution of $\lambda_{(n)}$ is Jack measure of size n and where $\xrightarrow{d}$ means convergence in distribution of the finite-dimensional law.

We can prove this theorem using algebraic methods (Jack characters after normalization span a very nice algebra)!

Polynomials functions

We define

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = \begin{cases} \alpha^{-\frac{|\mu|-\ell(\mu)}{2}} \binom{|\lambda|-|\mu|+m_1(\mu)}{m_1(\mu)} \ z_\mu \ \theta^{(\alpha)}_{\mu,1^{|\lambda|-|\mu|}}(\lambda) & \text{if } |\lambda| \geq |\mu|; \\ 0 & \text{if } |\lambda| < |\mu|, \end{cases}$$

where

- $z_\mu = \mu_1\mu_2 \cdots m_1(\mu)!m_2(\mu)! \cdots$,
- $m_i(\mu)$ - number of parts of μ equal to i.

Theorem (Lassalle, 2009)

The family $\left(\mathrm{Ch}_\mu^{(\alpha)}\right)_\mu$ span linearly an algebra $\Lambda_\star^{(\alpha)}$ of α-shifted symmetric functions.

What do we have and what do we miss?

In order to prove our main theorem:

- We want to estimate mixed moments of Jack characters;
- Expectation of the Jack characters is easy to compute;
- Suitably normalized Jack characters span linearly some nice algebra $\Lambda_\star^{(\alpha)}$;
- We want to expand a product:

$$\mathrm{Ch}_\mu^{(\alpha)}\,\mathrm{Ch}_\nu^{(\alpha)} = \sum_\rho g_{\mu,\nu;\pi}^{(\alpha)}\,\mathrm{Ch}_\pi^{(\alpha)}$$

as a linear combination of suitably normalized Jack characters.

Problem

What can we say about $g_{\mu,\nu;\pi}^{(\alpha)}$?

Main result for structure constants

Theorem (D., Féray)

Let

$$\mathrm{Ch}^{(\alpha)}_{\mu}\,\mathrm{Ch}^{(\alpha)}_{\nu} = \sum_{\rho} g^{(\alpha)}_{\mu,\nu;\pi}\,\mathrm{Ch}^{(\alpha)}_{\pi}.$$

Then, structure constants $g^{(\alpha)}_{\mu,\nu;\pi}$ are polynomials in $\gamma := \alpha^{1/2} - \alpha^{-1/2}$ of degree less than

$$\min_{i=1,2,3}\left(n_i(\mu) + n_i(\nu) - n_i(\pi)\right),$$

with rational coefficients, where $n_i(\lambda)$ - natural valued function of λ.

- It is crucial for proving central limit theorem;
- It is applicable to different problems.

Projection on the set of Young diagrams of a fixed size

Let $\mu, \nu, \pi \in \mathbb{Y}_n$.

$$\theta_\mu^{(\alpha)}(\lambda)\theta_\nu^{(\alpha)}(\lambda) = \sum_{|\pi|=n} c_{\mu,\nu;\pi}^{(\alpha)} \theta_\pi^{(\alpha)}.$$

Hence

$$c_{\mu,\nu;\pi}^{(\alpha)} = \frac{\alpha^{d(\mu,\nu;\pi)/2}}{z_{\tilde{\mu}} z_{\tilde{\nu}}} \sum_{0 \le i \le m_1(\pi)} g_{\tilde{\mu},\tilde{\nu};\tilde{\pi}1^i}^{(\alpha)} \cdot z_{\tilde{\pi}} \cdot i! \cdot \binom{n-|\tilde{\pi}|}{i},$$

where

- $\tilde{\mu}$ is created from μ by removing all parts equal to 1,
- $d(\mu,\nu;\pi) = |\mu| - \ell(\mu) + |\nu| - \ell(\nu) - (|\pi| - \ell(\pi))$.

$\alpha = 1$ - Structure contants of the $Z(\mathbb{C}[\mathfrak{S}_n])$

Let $\mathbb{C}[\mathfrak{S}_n] := \{f : f : \mathfrak{S}_n \to \mathbb{C}\}$ be a group algebra of the symmetric group. This is algebra with the multiplication defined by:

$$f \cdot g(\sigma) := \sum_{\sigma_1 \sigma_2 = \sigma} f(\sigma_1) g(\sigma_2).$$

Let

$$Z(\mathbb{C}[\mathfrak{S}_n]) := \{f \in \mathbb{C}[\mathfrak{S}_n] : \forall g \in \mathbb{C}[\mathfrak{S}_n], fg = gf\}$$

be the center of that algebra. It has a basis $(f_\mu)_{|\mu|=n}$, where

$$f_\mu(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ has cycle type } \mu, \\ 0 & \text{otherwise.} \end{cases}$$

$\alpha = 1$ - Structure contants of the $Z(\mathbb{C}[\mathfrak{S}_n])$

Let

$$f_\mu f_\nu = \sum_{|\rho|=n} c_{\mu,\nu;\rho} f_\rho.$$

Lemma

The structure constant $c_{\mu,\nu;\rho}$ is equal to the number of pairs of permutation (σ_1, σ_2) such that

- σ_1 *has cycle type* μ,
- σ_2 *has cycle type* ν,
- $\sigma_1\sigma_2 = \sigma$, *where* σ *is a fixed permutation of the cycle-type* ρ.

$\alpha = 1$ - Structure contants of the $Z(\mathbb{C}[\mathfrak{S}_n])$

One has a following relation:

$$c^{(1)}_{\mu,\nu;\rho} = c_{\mu,\nu;\rho}.$$

Remark

From the previous theorem and a relation between $c^{(\alpha)}$ and $g^{(\alpha)}$ one can deduce a classical result of Farahat and Higman: $c_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\rho 1^{n-|\rho|}}$ is a polynomial in n.

$\alpha = 2$ - Structure contants of the Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

Let $\mathfrak{S}_{2n}$ acts on the set $X_n := \{1, \bar{1}, \dots, n, \bar{n}\}$ by permutations and let

$$\mathfrak{S}_{2n} > H_n := \{\sigma \in \mathfrak{S}_{2n} : \forall i \in X_n \ \sigma(\bar{i}) = \overline{\sigma(i)}\}$$

be a hyperoctahedral subgroup.

Hecke algebra $\mathbb{C}[H_n \backslash \mathfrak{S}_{2n} / H_n] < \mathbb{C}[\mathfrak{S}_{2n}]$ of the pair $(\mathfrak{S}_{2n}, H_n)$ is defined by:

$$\mathbb{C}[H_n \backslash \mathfrak{S}_{2n} / H_n] := \{x \in \mathbb{C}[\mathfrak{S}_{2n}] : hxh' = x \forall h, h' \in H_n\}.$$

Double-cosets: equivalence classes for the relation $x \sim hxh'$ (for $x \in \mathfrak{S}_{2n}$ and $h, h' \in H_n$)

- naturally indexed by partitions of size n;
- $F_\mu = \sum_{x \text{ of type } \mu} \delta_x$ - linear basis of $\mathbb{C}[H_n \backslash \mathfrak{S}_{2n} / H_n]$.

$\alpha = 2$ - Structure contants of the Hecke algebra of $(\mathfrak{S}_{2n}, H_n)$

Let

$$F_\mu F_\nu = \sum_{|\rho|=n} h_{\mu,\nu;\rho} F_\rho.$$

Then

$$c^{(2)}_{\mu,\nu;\rho} = \frac{h_{\mu,\nu;\rho}}{2^n n!}.$$

Remark

From the previous theorem and a relation between $c^{(\alpha)}$ and $g^{(\alpha)}$ one can deduce a result of Tout (2013):

$$\frac{h_{\mu 1^{n-|\mu|},\nu 1^{n-|\nu|};\pi 1^{n-|\pi|}}}{n!\ 2^n}$$

is a polynomial in n.

Fin

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