Dual combinatorics of Jack polynomials via maps

(joint work with Valentin Féray and Piotr Śniady)

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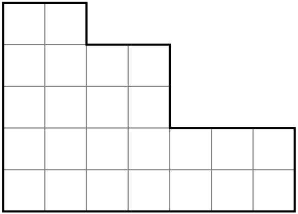
Plan for today

Young diagrams

Definition

- A partition λ of the integer n ($\lambda \vdash n$): finite non-increasing sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$, such that $|\lambda| := \sum_i \lambda_i = n$;
- Graphical representation by a Young diagram of size n (with n boxes).



Young diagrams are important objects in

- symmetric functions theory,
- representation theory.

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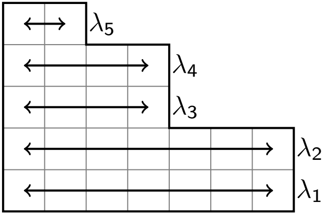

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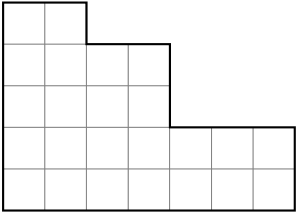
Young diagrams

Example

- $\lambda = (7, 7, 4, 4, 2) \vdash 24$,
- $m_7(\lambda) = 2, m_4(\lambda) = 2, m_2(\lambda) = 1, m_i(\lambda) = 0$ for $i \notin \{2, 4, 7\}$, where $m_i(\lambda)$ denotes the number of parts of λ equal to i,
- $\ell(\lambda) = 5$, where $\ell(\lambda)$ denotes the number of rows of λ.



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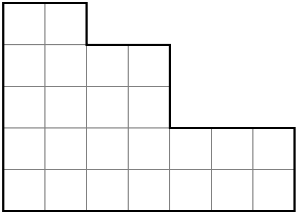

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Symmetric functions

Definition

A symmetric function f is a symmetric polynomial in infinitely many variables $x_1, x_2, \dots,$ i.e.

- $f = \sum_{\substack{J=(j_1,j_2,\dots)\in\mathbb{N}^{\mathbb{N}} \\ |J|:=j_1+j+2+\cdots<K}} c_J x^J, \quad c_J \in \mathbb{Q}, \quad x^J = x_1^{j_1} x_2^{j_2} \cdots;$
- $f(x_1, x_2, \dots, x_k, 0, 0, \dots)$ is a symmetric polynomial, i. e. for any permutation $\sigma \in \mathfrak{S}_k$ polynomial $f(x_1, x_2, \dots, x_k, 0, 0, \dots)$ is equal to $f(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}, 0, 0, \dots)$.

Example

$$\begin{aligned} f = (x_1^2 + x_2^2 + x_3^2 + \cdots)(x_1 + x_2 + x_3 + \cdots) \\ = x_1^3 + x_2^3 + x_3^3 + \cdots \\ + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots. \end{aligned}$$

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Example

$$\begin{aligned} p_{(2,1)} &= (x_1^2 + x_2^2 + x_3^2 + \cdots)(x_1 + x_2 + x_3 + \cdots) \\ &= x_1^3 + x_2^3 + x_3^3 + \cdots (= m_{(3)}) \\ &\quad + x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + \cdots (= m_{(2,1)}). \end{aligned}$$

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Example

Monomial symmetric functions m_λ:

$$m_\lambda = \sum_{J\in\mathbb{N}^{\mathbb{N}}} x^J,$$

summation over all $J \in \mathbb{N}^{\mathbb{N}}$ equal to λ after reordering its parts.

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Example

- $m_{(1)} = \sum_i x_i$,
- $m_{(2)} = \sum_i x_i^2$,
- $m_{(1,1)} = \sum_{i<j} x_i x_j$.

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Example

Power-sum symmetric functions p_λ:

$$p_k = \sum_i x_i^k, \quad p_\lambda = \prod_i p_{\lambda_i}.$$

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Example

- $p_{(1)} = \sum_i x_i = m_{(1)}$,
- $p_{(2)} = \sum_i x_i^2 = m_{(2)}$,
- $p_{(1,1)} = (\sum_i x_i)^2 = \sum_i x_i^2 + 2\sum_{i<j} x_i x_j = m_{(2)} + 2m_{(1,1)}$.

Schur symmetric functions

Definition

Hall scalar product:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda,$$

where $z_\lambda = \prod_i m_i(\lambda)! i^{m_i(\lambda)}$.

Schur symmetric functions s_λ:

- obtained from monomial symmetric functions ordered by lexicographic order by Gram-Schmidt orthonormalization process.

Example

$(1,1) < (2)$.

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- $s_{(1,1)} = \frac{m_{(\mathbf{1},\mathbf{1})}}{\|m_{(\mathbf{1},\mathbf{1})}\|} = \frac{m_{(\mathbf{1},\mathbf{1})}}{\|1/2(p_{(\mathbf{1},\mathbf{1})} - p_{(\mathbf{2})})\|} = m_{(1,1)}$

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Representation theory of the symmetric groups

Definition

Let G be a finite group and V be a finite dimensional linear space over $\mathbb{C}$.

- A homomorphism $\rho : G \to \mathrm{End}(V)$ is called representation. Representation ρ is called irreducible (**irrep** for short) if $\rho(G)W \not\subseteq W$ for any $0 \subsetneq W \subsetneq V$ linear subspace.
- A function $\chi : G \to \mathbb{C}$ given by $\chi(g) = \mathrm{Tr}(\rho(g))$ is called character (conjugacy invariant).

Fact

There is a one to one correspondence between:

- *partitions of n and conjugacy classes of a permutation group $\mathfrak{S}_n$;*
- *partitions of n and irreps of a permutation group $\mathfrak{S}_n$;*

Character $\chi_\lambda(\mu) := \mathrm{Tr}(\rho_\lambda(\pi_\mu))$ is indexed by a pair (λ, μ) of partitions of n, where π_μ is any permutation from the conjugacy class given by μ.

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Representation theory vs. symmetric functions theory

Let us consider an expansion of Schur symmetric function of degree n in power-sum basis:

$$s_\lambda = \sum_{\mu \vdash n} c_\mu(\lambda) p_\mu.$$

Theorem (Frobenius formula)

For any pair (λ, μ) of partitions of n one has

$$c_\mu(\lambda) = \frac{\chi_\lambda(\mu)}{z_\mu},$$

where $z_\mu = \prod_i m_i(\mu)! i^{m_i(\mu)}$ is a standard numerical factor.

Jack symmetric functions

Definition

Deformation of Hall scalar product:

$$\langle p_\lambda, p_\mu \rangle_\alpha = \alpha^{\ell(\lambda)} \delta_{\lambda,\mu} z_\lambda,$$

where $z_\lambda = \prod_i m_i(\lambda)! i^{m_i(\lambda)}$.

Jack symmetric functions $J_\lambda^{(\alpha)}$:

- obtained from monomial symmetric functions ordered by lexicographic order by Gram-Schmidt orthonormalization process and multiplied by explicit constant $c^{(\alpha)}(\lambda)$;
- for any pair (λ, μ) of partitions of n we define Jack characters $\theta_\mu^{(\alpha)}(\lambda)$ by

$$J_\lambda^{(\alpha)} = \sum_{\mu \vdash n} \theta_\mu^{(\alpha)}(\lambda) p_\mu.$$

Jack symmetric functions

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Jack symmetric functions for $\alpha = 1$:

- obtained from monomial symmetric functions ordered by lexicographic order by Gram-Schmidt orthonormalization process and multiplied by explicit constant $c^{(1)}(\lambda) = \frac{n!}{\dim(\lambda)}$;
- for any pair (λ, μ) of partitions of n we define Jack characters $\theta^{(1)}_\mu(\lambda)$ by
$$\frac{n!}{\dim(\lambda)} s_\lambda = J^{(1)}_\lambda = \sum_{\mu \vdash n} \theta^{(1)}_\mu(\lambda) p_\mu;$$
- hence $\theta^{(1)}_\mu(\lambda) = \frac{n!}{z_\mu} \frac{\chi_\lambda(\pi_\mu)}{\dim(\lambda)}$, where π_μ - any permutation of type μ.

Idea of dual picture

Typically, character $\chi_\lambda(\mu)$ is considered as a function of variable μ, while λ is fixed. Kerov and Olshanski introduced the dual combinatorics of characters:

Definition (Kerov, Olshanski ($\alpha = 1$))

Let $\mu \vdash k$ does not contain parts equal to 1. Then

$$\mathrm{Ch}^{(1)}_\mu(\lambda) = \begin{cases} \frac{n!}{(n-k)!} \frac{\chi_\lambda(\pi_{\mu,\mathbf{1}^{n-k}})}{\dim(\lambda)} = z_\mu \, \theta^{(1)}_{\mu,1^{n-k}}(\lambda) & \text{if } |\lambda| = n \geq k; \\ 0 & \text{if } |\lambda| < k. \end{cases}$$

Problem

It seems that these objects have a rich and complicated combinatorial structure. What can we say about this structure?

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Definition (Kerov, Olshanski ($\alpha = 1$), Lassalle (general α))

Let $\mu \vdash k$ does not contain parts equal to 1. Then

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = \begin{cases} \alpha^{-\frac{k-\ell(\mu)}{2}} \, z_\mu \, \theta^{(\alpha)}_{\mu,1^{n-k}}(\lambda) & \text{if } |\lambda| = n \geq k; \\ 0 & \text{if } |\lambda| < k. \end{cases}$$

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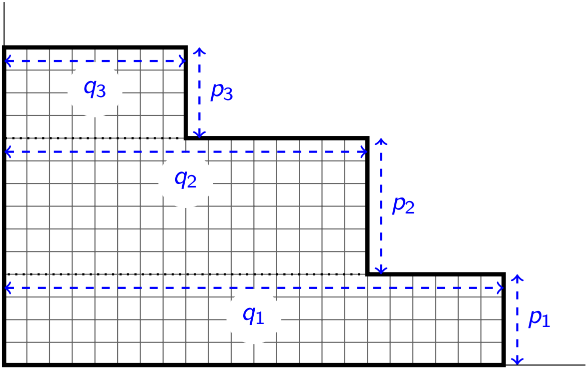
Stanley coordinates and first conjecture of Lassalle

Let $\boldsymbol{p} = (p_1, \dots, p_k)$ and $\boldsymbol{q} = (q_1, \dots, q_k)$ denotes two lists of positive integers (where $\boldsymbol{q}$ is non-increasing). We define a multirectangular Young diagram:

$$\lambda(\boldsymbol{p}, \boldsymbol{q}) = (\underbrace{q_1, \dots, q_1}_{p_1 \text{ times}}, \dots, \underbrace{q_\ell, \dots, q_\ell}_{p_\ell \text{ times}}).$$



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Proposition (Lassalle)

Let us fix $\mu \vdash k$. Then $\mathrm{Ch}_\mu^{(\alpha)}(\lambda(\boldsymbol{p}, \boldsymbol{q}))$ is a polynomial in $(p_1, p_2, \dots, q_1, q_2, \dots)$ of degree $k + \ell(\mu)$.

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Conjecture (Lassalle)

Let us fix $\mu \vdash k$. Then $\alpha^{\frac{k-\ell(\mu)}{2}} (-1)^k \operatorname{Ch}_\mu^{(\alpha)}(\lambda(\boldsymbol{p}, \boldsymbol{q}))$ is a polynomial in $(p_1, p_2, \dots, -q_1, -q_2, \dots, \alpha - 1)$ with non-negative, integer coefficients..

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Partial result: it is polynomial in $(p_1, p_2, \dots, -q_1, -q_2, \dots, \alpha - 1)$ with rational coefficients (D., Féray 2012). Positivity and integrality suggests combinatorial interpretation!

Free cumulants

Proposition (Lassalle)

Let λ be given by Stanley coordinates: $\lambda = (\mathbf{p}, \mathbf{q})$. For any positive integer $n \geq 1$ we define dilated Young diagram $D_n(\lambda) := (n\mathbf{p}, n\mathbf{q}) = (np_1, \dots, np_k, nq_1, \dots, nq_k)$.

- *free cumulant $R_k^{(\alpha)}(\lambda)$ is defined by $R_k^{(\alpha)}(\lambda) := \lim_{n\to\infty} \frac{\mathrm{Ch}_{(k)}^{(\alpha)}(D_n(\lambda))}{n^{k+1}}$;*
- *for any $\mu \vdash m$ there is a polynomial $K_\mu^{(\alpha)}(R_2^{(\alpha)}, R_3^{(\alpha)}, \dots)$ such that*

$$K_\mu^{(\alpha)}(R_2^{(\alpha)}(\lambda), R_3^{(\alpha)}(\lambda), \dots) = \mathrm{Ch}_\mu^{(\alpha)}(\lambda).$$

Kerov polynomials and second conjecture of Lassalle

Polynomials $K_\mu^{(\alpha)}$ are called Kerov polynomials. Kerov polynomials for one-part partitions μ:

$$
\begin{aligned}
K_{(1)}^{(\alpha)} &= R_2, \\
K_{(2)}^{(\alpha)} &= R_3 + \gamma R_2, \\
K_{(3)}^{(\alpha)} &= R_4 + 3\gamma R_3 + (1 + 2\gamma^2) R_2, \\
K_{(4)}^{(\alpha)} &= R_5 + 6\gamma R_4 + \gamma R_2^2 + (5 + 11\gamma^2) R_3 + (7\gamma + 6\gamma^3) R_2, \\
K_{(5)}^{(\alpha)} &= R_6 + 10\gamma R_5 + 5\gamma R_3 R_2 + (15 + 35\gamma^2) R_4 + (5 + 10\gamma^2) R_2^2 \\
&\quad + (55\gamma + 50\gamma^3) R_3 + (8 + 46\gamma^2 + 24\gamma^4) R_2,
\end{aligned}
$$

where $\gamma = \sqrt{\alpha}^{-1} - \sqrt{\alpha}$.

Kerov polynomials and second conjecture of Lassalle

Polynomials $K_\mu^{(\alpha)}$ are called Kerov polynomials. Kerov polynomials for one-part partitions μ:

Conjecture (Lassalle)

Let $k \geq 1$ be a positive integer. Then $K_{(k)}^{(\alpha)}$ is a polynomial in $\gamma, R_2^{(\alpha)}, R_3^{(\alpha)}, \ldots$ with positive, integer coefficients.

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Partial result: it is polynomial in $\gamma, R_2^{(\alpha)}, R_3^{(\alpha)}, \dots$ with rational coefficients (D., Féray 2012). Positivity and integrality suggests combinatorial interpretation!

Remark

Originally, conjecture of Lassalle was stated rather vaguely, since he used a different normalization of Kerov polynomials and he suggested that there is a was to write it as a polynomial in free cumulants and $\alpha, \beta := 1 - \alpha$ with non-negative, integer coefficients.

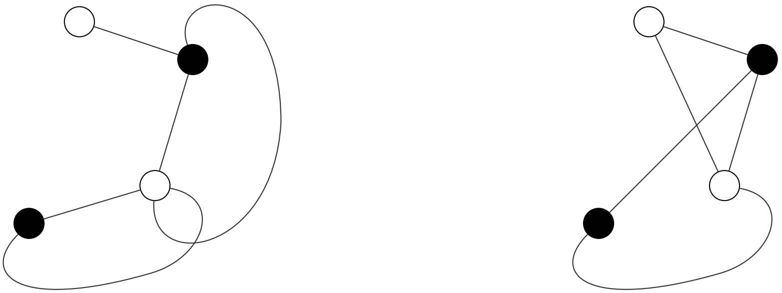
Embeddings of bipartite graphs into Young diagrams

Idea: in order to understand a structure of Jack characters we have to describe them using very simple functions of Young diagrams.

Embeddings of bipartite graphs into Young diagrams

Definition

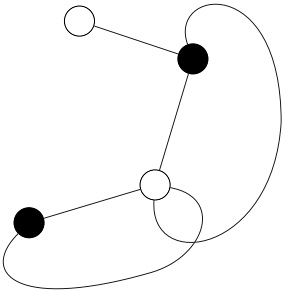
Bipartite graph is a graph G with the set of vertices $V = V_\circ \cup V_\bullet$ being a disjoin sum of white vertices $V_\circ$ and black vertices $V_\bullet$ such that each edge have endpoints with two different colors.

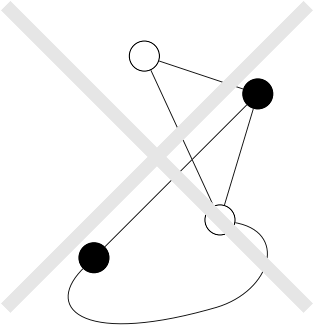


Embeddings of bipartite graphs into Young diagrams

Definition

Bipartite graph is a graph G with the set of vertices $V = V_\circ \cup V_\bullet$ being a disjoin sum of white vertices $V_\circ$ and black vertices $V_\bullet$ such that each edge have endpoints with two different colors.

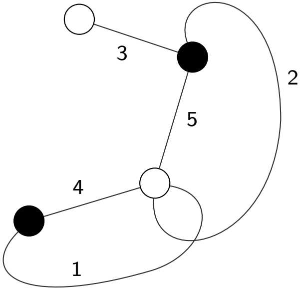


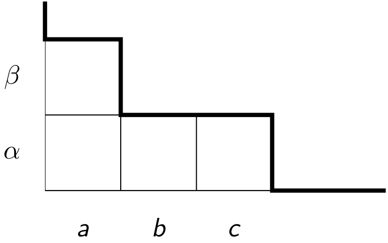


Embeddings of bipartite graphs into Young diagrams

Definition

An embedding of the bipartite graph G into Young diagram λ is a function $h : V_\circ \cup V_\bullet \to \mathbb{N}$ such that $(h(v_1), h(v_2)) \in \lambda$ whenever $(v_1, v_2) \in V_\circ \times V_\bullet$ is an edge in G.





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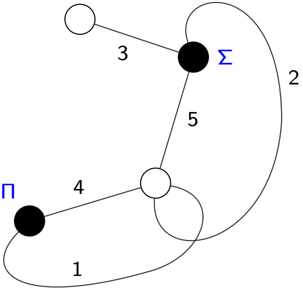

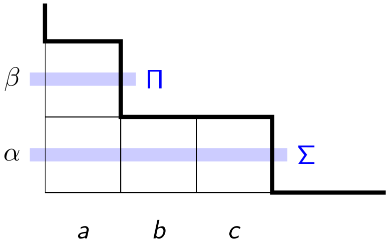

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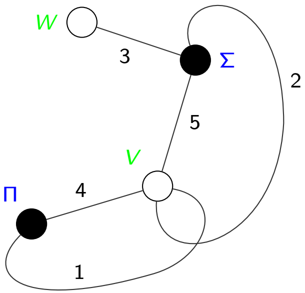

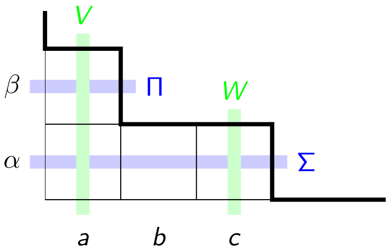

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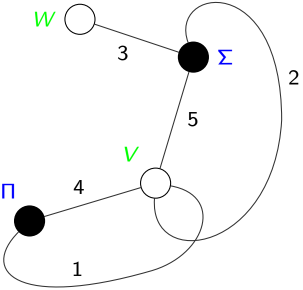
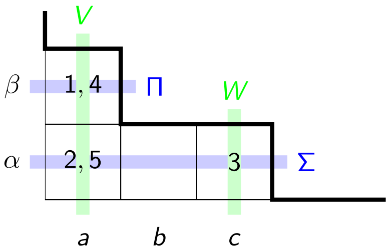




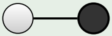
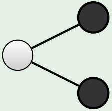
Simple function = number of embeddings

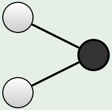
Definition

Let G be a bipartite graph. We define a function N_G on the set of Young diagrams, by setting:

$$N_G(\lambda) = \text{number of embeddings of } G \text{ into } \lambda.$$

Example

- Let $G =$  . Then $N_G(\lambda(\boldsymbol{p},\boldsymbol{q})) = \sum_i p_i q_i = |\lambda(\boldsymbol{p},\boldsymbol{q})|$.
- Let $G =$  . Then

  $$N_G(\lambda(\boldsymbol{p},\boldsymbol{q})) = 2\sum_i \sum_{j<i} q_i p_i p_j + \sum_i q_i p_i^2.$$
- Let $G =$  . Then $N_G(\lambda(\boldsymbol{p},\boldsymbol{q})) = \sum_i p_i q_i^2$.

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Proposition

We know that, there exists some collection of bipartite graphs $\mathcal{G}$ such that

$$\mathrm{Ch}_{\mu}^{(\alpha)}(\lambda) = (-1)^{\ell(\mu)} \sum_{G \in \mathcal{G}} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_{\bullet}(G)|} \left(\sqrt{\alpha}\right)^{|V_{\circ}(G)|} f_G(\gamma) N_G(\lambda),$$

where f_G is a polynomial with rational coefficients and $\gamma = \sqrt{\alpha}^{-1} - \sqrt{\alpha}$.

Jack characters and number of embeddings

Theorem (D. Féray, Śniady)

Assume, that there exists some collection of bipartite graphs $\mathcal{G}$ such that

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = (-1)^{|\ell(\mu)} \sum_{G \in \mathcal{G}} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_\bullet(G)|} (\sqrt{\alpha})^{|V_\circ(G)|} f_G(\gamma) N_G(\lambda),$$

where f_G is a polynomial with positive, integer coefficients. Then, answers for both conjectures of Lassalle are positive and there is a combinatorial interpretation of those coefficients.

Problem

Number of embeddings are not linearly independent, hence there are many possibilities for choosing a class $\mathcal{G}$ and polynomials f_G. Is there some canonical candidate?

Maps

- (Bipartite) map M is a connected (bipartite) graph embedded into a surface in a way that the complement of the image is homeomorphic to the collection of open discs called faces.
- Map is rooted if there is a ditingueshed corner of the map.
- Length $\ell(F)$ of the face F is the number of edges lying on its border. Type of the bipartite map with n edges and k faces is a partition $\lambda \vdash n, \ell(\lambda) = k$ given by $\lambda = (\ell(F_1)/2, \ldots, \ell(F_k))$.

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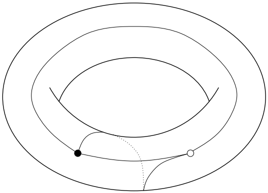
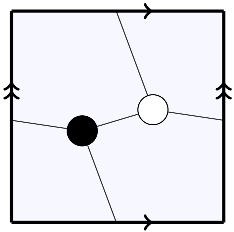
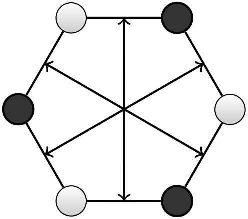




Orientable vs. non-orientable

Map M is orientable if the underlying surface is orientable.
Special case - $\alpha = 1$:

Theorem (Féray, Śniady)

Let us fix partition $\mu \vdash k$. Then

$$\mathrm{Ch}_{\mu}^{(1)}(\lambda) = (-1)^{\ell(\mu)} \sum_{M} (-1)^{|V_{\bullet}(M)|} N_M(\lambda),$$

where the summation is over all orientable bipartite maps M with face-type μ.

Remark

Both conjectures of Lassalle holds true for $\alpha = 1$.

Orientable vs. non-orientable

Map M is non-orientable if the underlying surface is non-orientable.
Special case - $\alpha = 2$:

Theorem (Féray, Śniady)

Let us fix partition $\mu \vdash k$. Then

$$\mathrm{Ch}^{(2)}_{\mu}(\lambda) = (-1)^{\ell(\mu)} \sum_{M} \left(-\frac{1}{\sqrt{2}}\right)^{|V_{\bullet}(M)|} \left(\sqrt{2}\right)^{|V_{\circ}(M)|} \cdot \left(-\frac{1}{\sqrt{2}}\right)^{|\pi|+\ell(\pi)-|V(M)|} N_M(\lambda),$$

where the summation is over all (orientable and non-orientable) bipartite maps M with face-type μ.

Remark

Both conjectures of Lassalle holds true for $\alpha = 2$.

Orientable vs. non-orientable

General case:

Conjecture (D.,Féray, Śniady)

Let us fix partition $\mu \vdash k$. Then

$$\mathrm{Ch}_{\mu}^{(\alpha)}(\lambda) = (-1)^{\ell(\mu)} \sum_{M} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_{\bullet}(M)|} \sqrt{\alpha}^{|V_{\circ}(M)|} f_M(\gamma) N_M(\lambda),$$

where the summation is over all bipartite maps M with face-type μ and $f_M(\gamma)$ is a polynomial with non-negative rational coefficients (and some extra restrictions for the degree).

Remark

Then both conjectures of Lassalle holds true for general α.

Rectangular Young diagrams

Theorem (D., Féray, Śniady)

For any bipartite map M there exists polynomial $f_M(\gamma)$ that satisfies conditions from the previous conjecture and the formula

$$\mathrm{Ch}_{\mu}^{(\alpha)}(\lambda) \;=\; (-1)^{\ell(\mu)} \sum_{M} \left(-\frac{1}{\sqrt{\alpha}}\right)^{|V_{\bullet}(M)|} \sqrt{\alpha}^{|V_{\circ}(M)|} f_M(\gamma) N_M(\lambda),$$

holds for any rectangular *Young diagram $\lambda(\mathbf{p}, \mathbf{q}) = (p, q)$.*

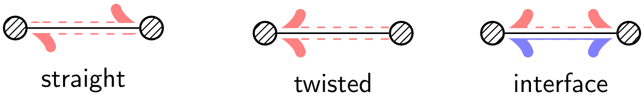
Idea: polynomial f_M measures the non-orientability of a map M.

Remark

Computer experiments showed that those polynomials f_M do not work in general case. The smallest counterexample is $\mu = (9)$ and $\lambda(\mathbf{p}, \mathbf{q}) = (p_1, p_2, p_3, q_1, q_2, q_3)$!

How to measure non-orientability?

There are three types of edges:



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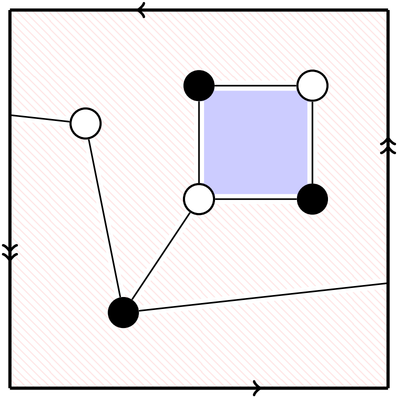

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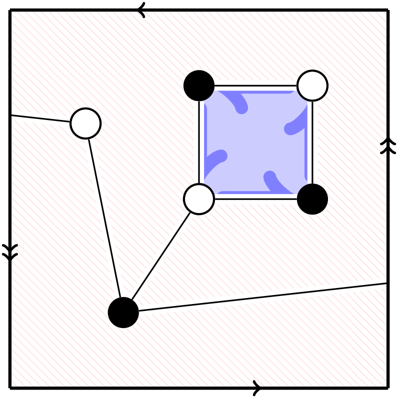

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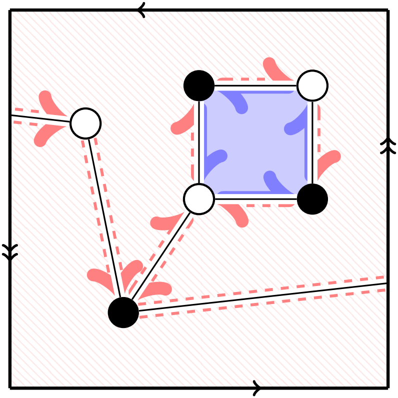

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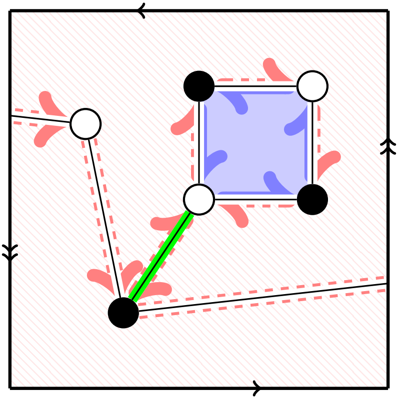

straight

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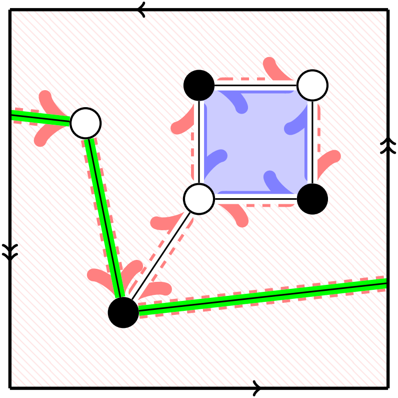

twisted

How to measure non-orientability?

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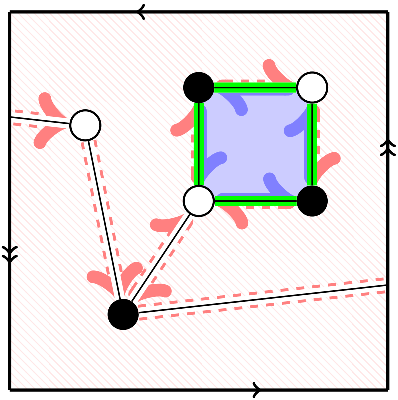
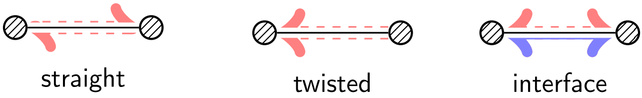


interface

How to measure non-orientability?

There are three types of edges:



There exists twisted edge in map $M \Rightarrow$ map M is non-orientable.

Idea: how many twisted edges do we have to erase from a given map M until we obtain an orientable map.

Measure of non-orientability

Let $e \in E(M)$.

$$\mathrm{mon}_M(e) = \begin{cases} 1 & \text{if } e \text{ is straight,} \\ \gamma & \text{if } e \text{ is twisted,} \\ \frac{1}{2} & \text{if } e \text{ is interface.} \end{cases}$$

For a map M with n edges, we define a random variable:

$$X_M(e_1, \dots, e_n) = \prod_{1 \le i \le n} \mathrm{mon}_{M_i}(e_i),$$

where $M_1 = M$, $M_i = M_{i-1} \setminus e_{i-1}$ and $(e_1, \dots, e_n)$ is a random vector of pairwise distinct edges of M chosen uniformly ($n!$ configurations).

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Theorem (D., Féray, Śniady)

For any bipartite map M there exists polynomial $f_M(\gamma) = \mathbb{E}(X_M)$ with positive rational coefficients such that for any rectangular Young diagram $\lambda(\boldsymbol{p}, \boldsymbol{q}) = (p, q)$ the following formula holds:

$$\mathrm{Ch}_\mu^{(\alpha)}(\lambda) = (-1)^{\ell(\mu)} \sum_M \left(-\frac{1}{\sqrt{\alpha}} \right)^{|V_\bullet(M)|} \sqrt{\alpha}^{|V_\circ(M)|} f_M(\gamma) N_M(\lambda(\boldsymbol{p}, \boldsymbol{q})).$$

Perspectives

Many things to do!

- Testing other weights (non-random ones);
- Try to find a representation theoretic framework (seems hard!);
- Understand a relation between our and others, very similar problems (*matching-Jack conjecture, b-conjecture*);
- General case = "interpolation" between orientable and non-orientable cases?