A positive combinatorial formula for symplectic Kostka–Foulkes polynomials I: Rows

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Kostka-Foulkes polynomials

- $\mathfrak{g}$ - complex semisimple Lie algebra
- $R = R_+ \cup R_-$ - root system
- W - Weyl group
- P_+ - set of dominant weights
- $P(\lambda)$ - set of weights of the irreducible representation $V(\lambda)$, $\lambda \in P_+$

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$\lambda \in P_+, \mu \in P(\lambda)$, $\quad V(\lambda) = \bigoplus_{\mu \in P(\lambda)} V_\mu^{\oplus K^R_{\lambda,\mu}}$, $\quad K^R_{\lambda,\mu}$ - multiplicity of μ in $V(\lambda)$

$$\frac{\sum_{\sigma \in W} (-1)^{\ell(\sigma)} x^{\sigma(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+} (1-x^{-\alpha})} = \sum_{\mu \in P(\lambda)} K^R_{\lambda,\mu} x^\mu.$$

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$$\frac{\sum_{\sigma \in W} (-1)^{\ell(\sigma)} x^{\sigma(\lambda+\rho)-\rho}}{\prod_{\alpha \in R_+} (1 - q x^{-\alpha})} = \sum_{\mu \in P(\lambda)} K_{\lambda,\mu}^R(q) x^\mu.$$

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Kostka-Foulkes polynomials are affine Kazhdan-Lusztig polynomials
$\Rightarrow K^R_{\lambda,\mu} \in \mathbb{Z}_{\geq}[q]$ [Kato '81]

Problem:

Let $\lambda \in P_+, \mu \in P_+(\lambda), \mathfrak{K}^R_{\lambda,\mu}$-parametrizes multiplicity of μ in $V(\lambda)$. Find

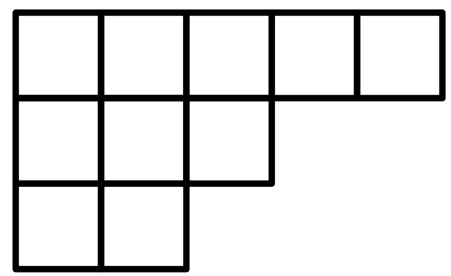
$$charge : \mathfrak{K}^R_{\lambda,\mu} \to \mathbb{Z}_{\geq 0}$$

such that

$$K^R_{\lambda,\mu}(q) = \sum_{T \in \mathfrak{K}^R_{\lambda,\mu}} q^{charge(T)}.$$

Kostka-Foulkes polynomials in type A

- $R = A_{n-1}, W = S(n)$
- P_+ = Young diagrams with at most n rows



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- $\mathfrak{K}^{A_{n-1}}_{\lambda,\mu} = SSYT(\lambda, \mu)$

$T =$

1	1	1	3	3
2	2	2		
4	4			

$\in \mathfrak{K}^{A_4}_{(5,3,2,0,0),(3,3,2,2,0)}$

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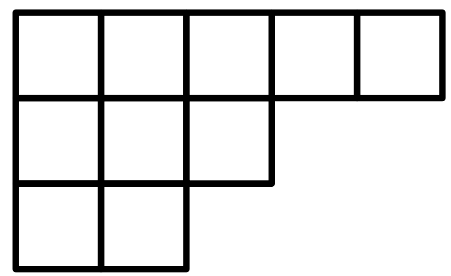

Theorem [Lascoux-Schutzenberger '78]

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$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 2 & \multicolumn{2}{c}{} \\ \cline{1-3} 4 & 4 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

$$w(T) = \ 4 \ 4_1 2 \ 2 \ 2_0 1 \ 1_0 1_0 3 \ 3_1$$

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Theorem [Lascoux-Schutzenberger '78]

$$K^{A_{n-1}}_{\lambda,\mu}(q) = \sum_{T \in SSYT(\lambda,\mu)} q^{charge(T)}.$$

Kostka-Foulkes polynomials in type C

- $R = C_n, W = H(n)$
- $P_+ =$ Young diagrams with at most n rows
- $\mathfrak{K}^{C_{n-1}}_{\lambda,\mu} = SympTab(\lambda, \mu)$

$$\begin{array}{|c|c|c|}\hline \overline{3} & \overline{3} & \overline{2} \\ \hline \overline{1} & 1 & \\ \cline{1-2} \end{array} \in SympTab_3((3,2),(2,1,0))$$

- SSYT in alphabet $\{\overline{n} < \cdots < \overline{1} < 1 < \ldots, n\}$
- shape $= \lambda$
- $\mu_{n+1-i} = \#\overline{i} - \#i$
- $+$ symplectic conditions

Kostka-Foulkes polynomials in type C

Conjecture [Lecouvey '05]

$$K^{C_n}_{\lambda,\mu}(q) = \sum_{T \in SympTab_n(\lambda,\mu)} q^{charge^{C_n}(T)}.$$

where $charge^{C_n}$ is defined through the symplectic cocyclage.

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Symplectic insertion: [Lecouvey '05]

1. Single column:

$$\boxed{\overline{4}} \;\rightarrow\; \begin{array}{|c|}\hline \overline{4} \\ \hline \overline{2} \\ \hline 2 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \;=$$

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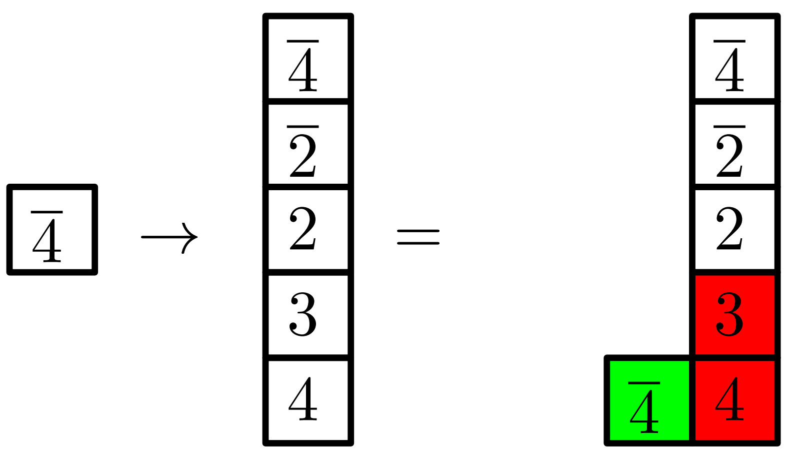

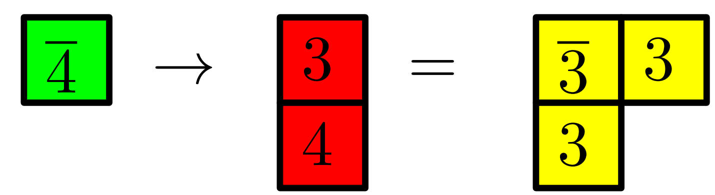

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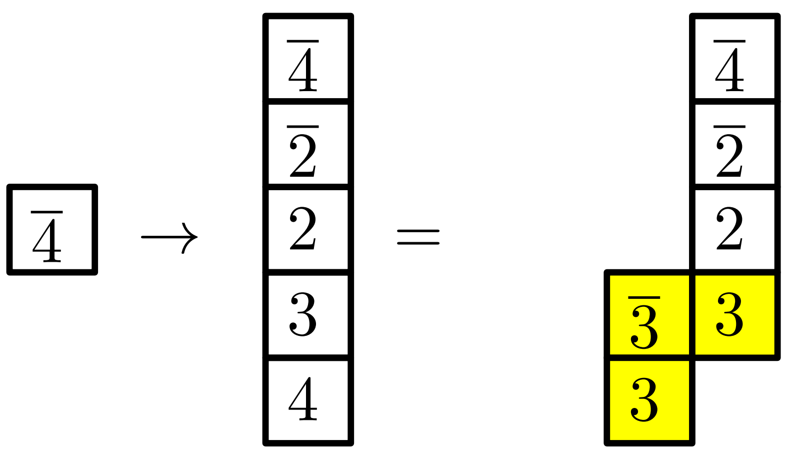

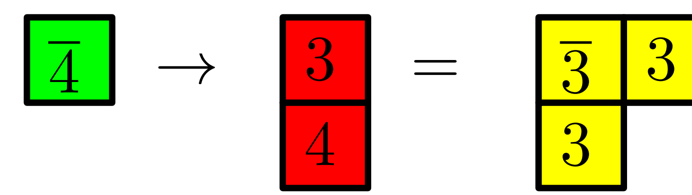

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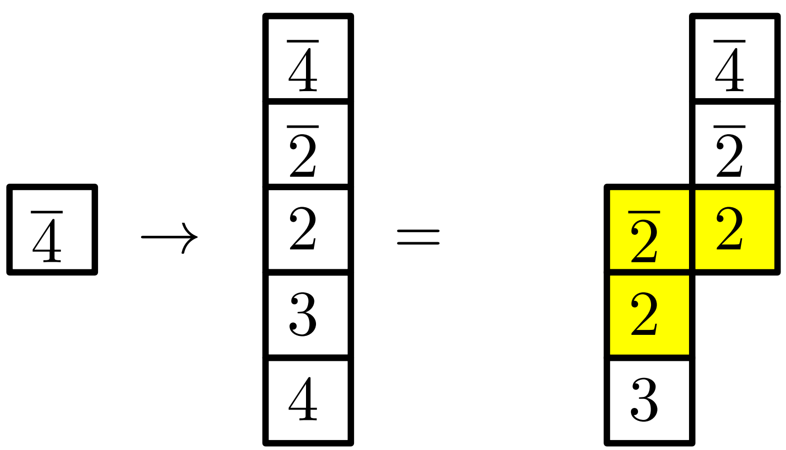

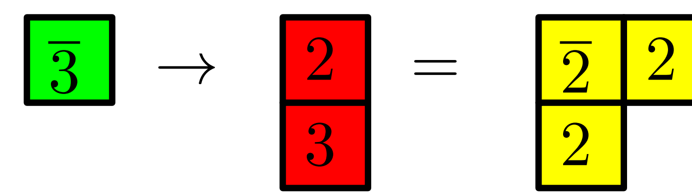

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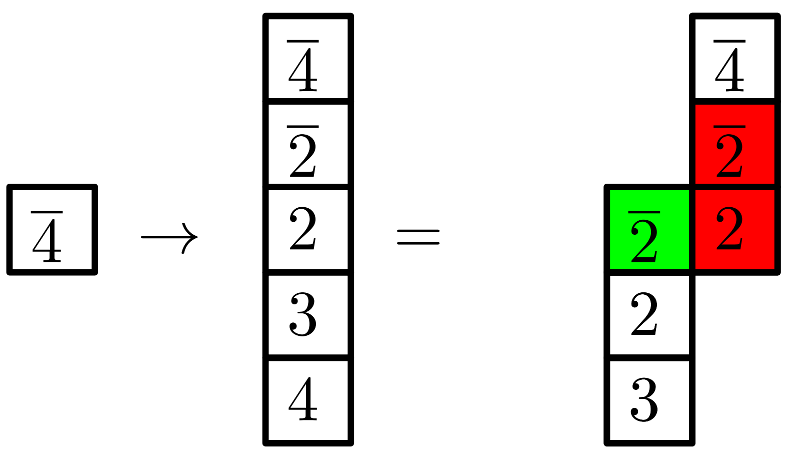

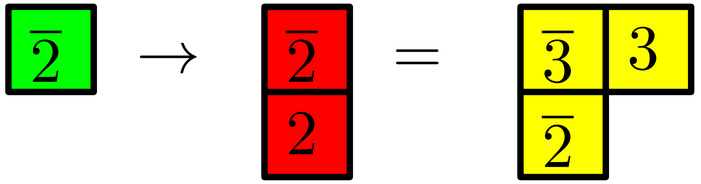

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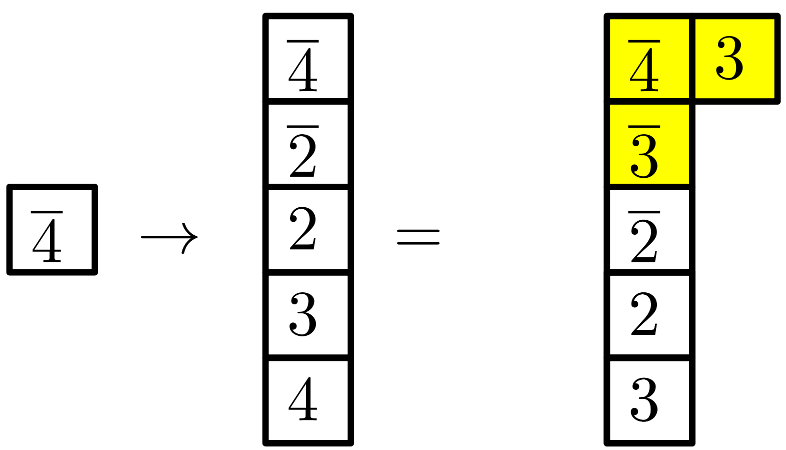

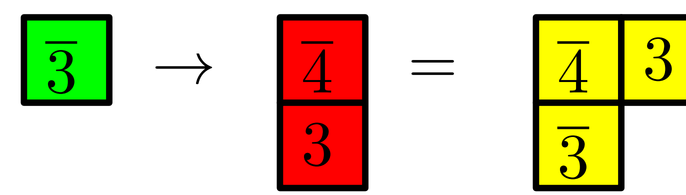

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2. Arbitrary shape:

$$\boxed{1} \to \begin{array}{|c|c|c|}\hline \bar{1} & 1 & 2 \\ \hline 1 & 2 & \multicolumn{1}{c}{} \\ \cline{1-2} 3 & \multicolumn{2}{c}{} \\ \cline{1-1}\end{array} \quad =$$

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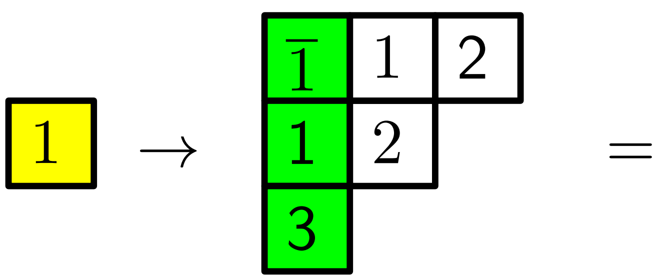

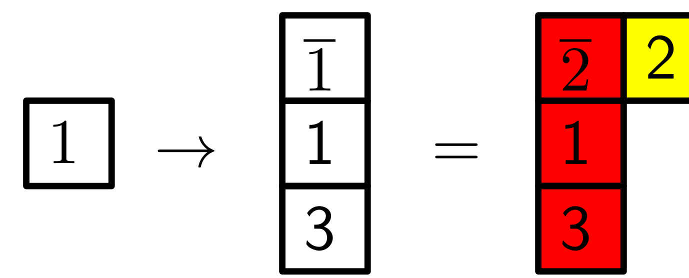

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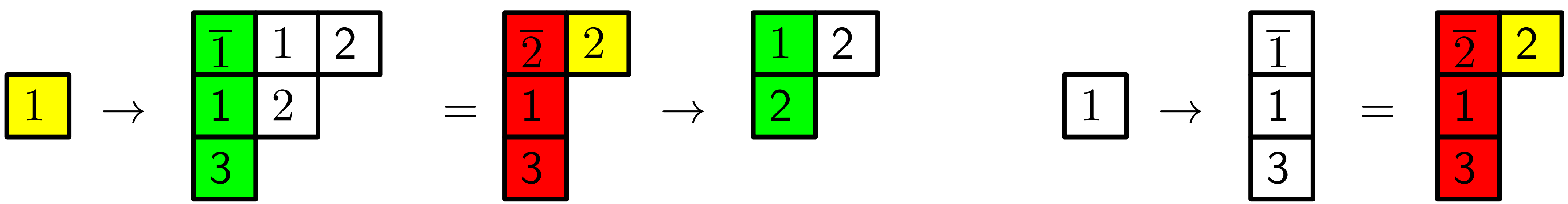

Kostka-Foulkes polynomials in type C

Conjecture [Lecouvey '05]

$$K^{C_n}_{\lambda,\mu}(q) = \sum_{T \in SympTab_n(\lambda,\mu)} q^{charge^{C_n}(T)}.$$

where $charge^{C_n}$ is defined through the symplectic cocyclage.

Symplectic insertion: [Lecouvey '05]

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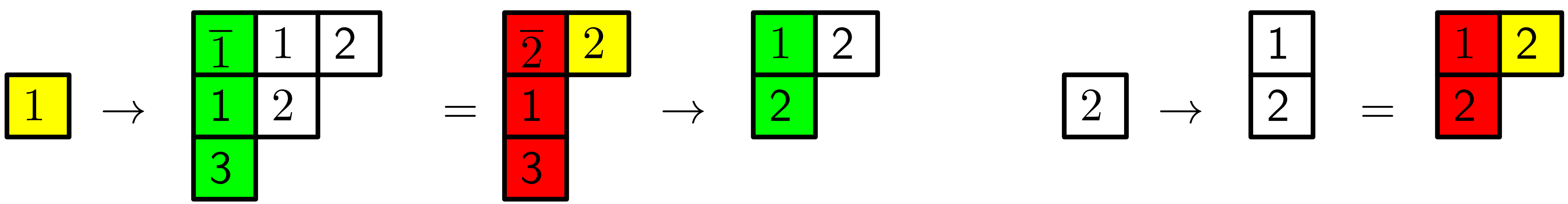

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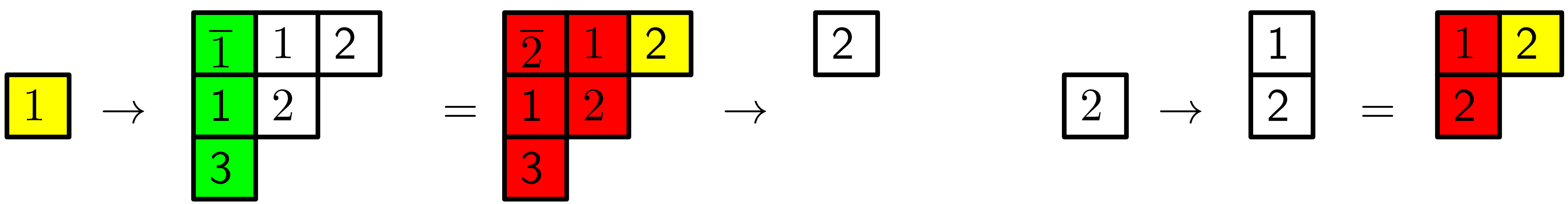

Kostka-Foulkes polynomials in type C

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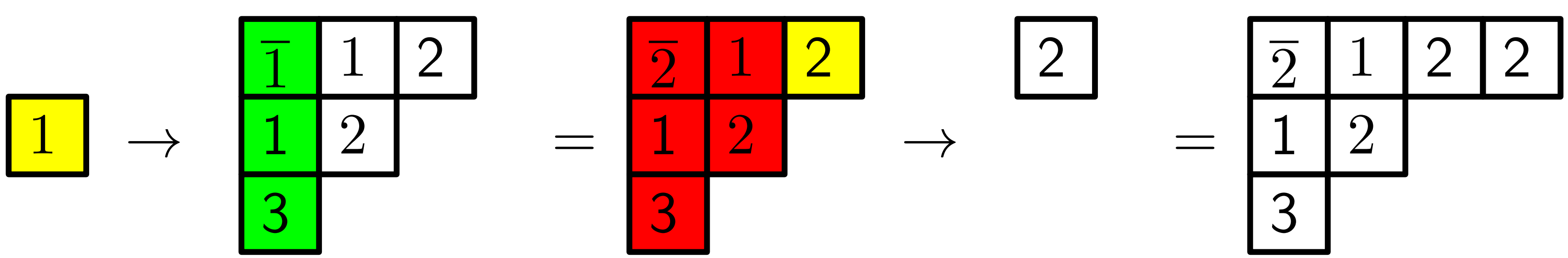

Kostka-Foulkes polynomials in type C

Conjecture [Lecouvey '05]

$$K_{\lambda,\mu}^{C_n}(q) = \sum_{T \in SympTab_n(\lambda,\mu)} q^{charge^{C_n}(T)}.$$

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Symplectic insertion: [Lecouvey '05]

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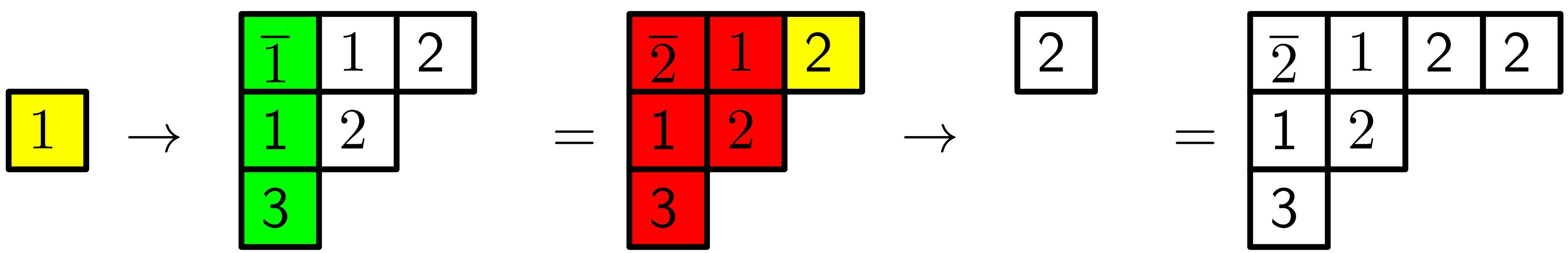
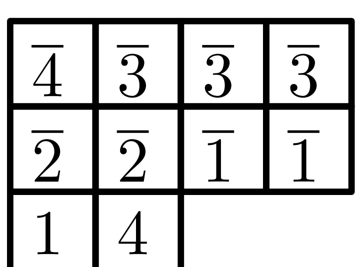


Symplectic cocyclage: [Lecouvey '05]

$CoCyc$  $=$

Kostka-Foulkes polynomials in type C

Conjecture [Lecouvey '05]

$$K^{C_n}_{\lambda,\mu}(q) = \sum_{T \in SympTab_n(\lambda,\mu)} q^{charge^{C_n}(T)}.$$

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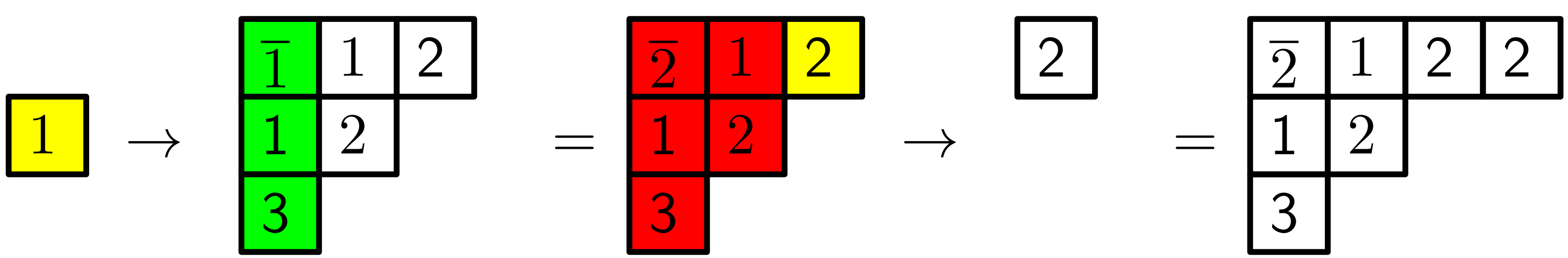
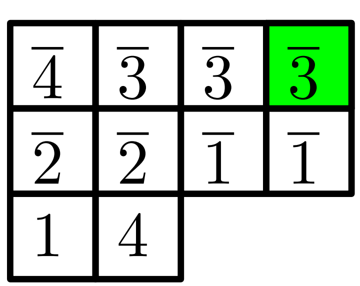
Symplectic insertion: [Lecouvey '05]

2. Arbitrary shape:



Symplectic cocyclage: [Lecouvey '05]

$CoCyc$



$=$

Kostka-Foulkes polynomials in type C

Conjecture [Lecouvey '05]

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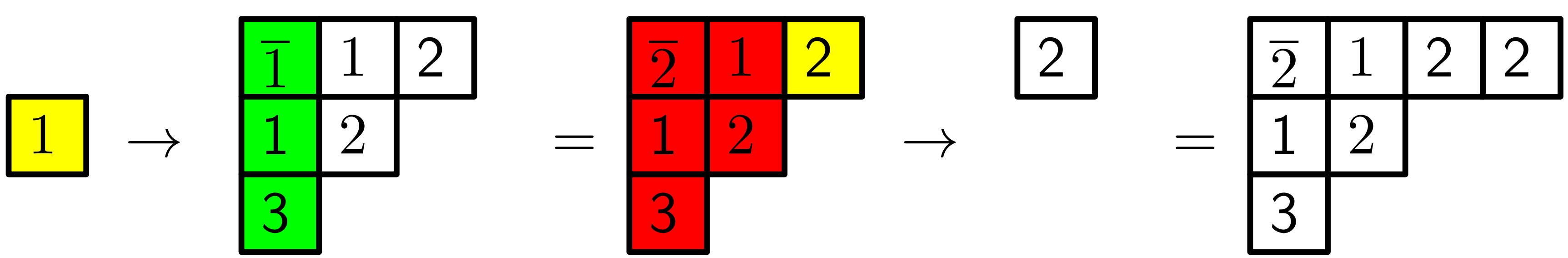
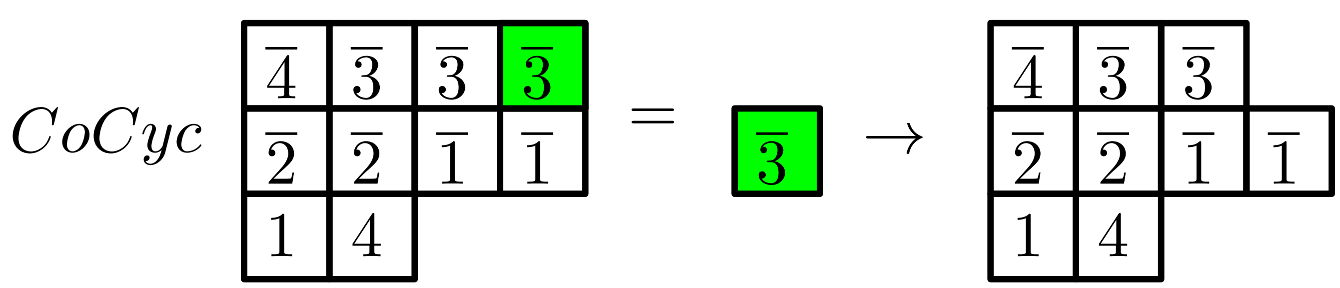


Symplectic cocyclage: [Lecouvey '05]



Kostka-Foulkes polynomials in type C

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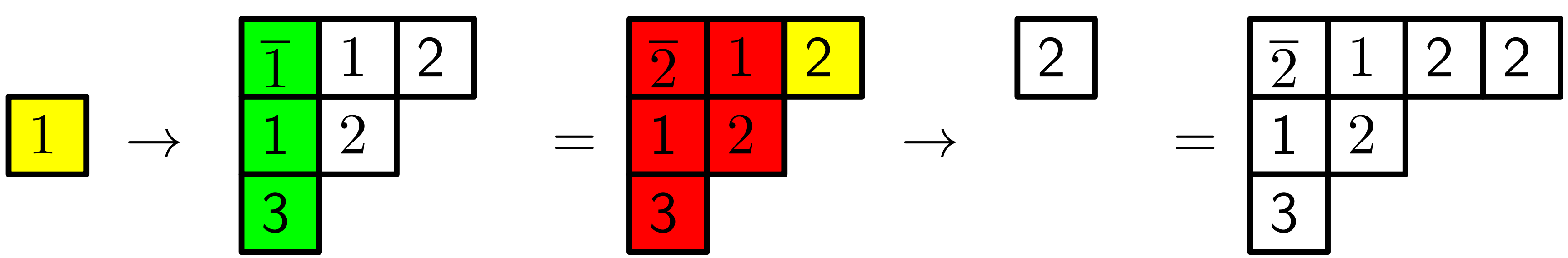
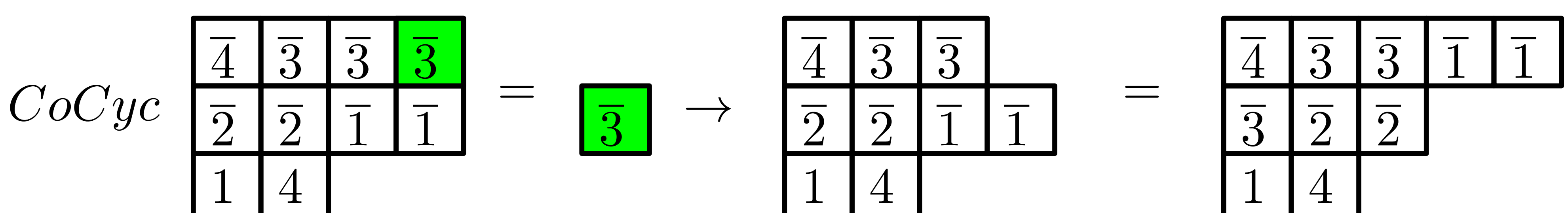


Symplectic cocyclage: [Lecouvey '05]



Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$ $\quad$ $charge^{C_n}(T) = ???$

$$T = \boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{3}}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\overline{2}}\boxed{\overline{1}}\boxed{\overline{1}}\boxed{1}\boxed{2} \in SympTab_3((10),(3,2,1))$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$ $\quad$ $charge^{C_n}(T) = ???$

$$CoCyc(T) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \overline{3} & \overline{3} & \overline{3} & \overline{2} & \overline{2} & \overline{2} & \overline{1} & \overline{1} & 1 \\ \hline 2 \\ \cline{1-1} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^2(T) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \overline{3} & \overline{3} & \overline{3} & \overline{2} & \overline{2} & \overline{2} & \overline{1} & \overline{1} \\ \hline 1 & 2 \\ \cline{1-2} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$CoCyc^3(T) =$

$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{2}$	$\overline{2}$	$\overline{2}$	$\overline{1}$
$\overline{1}$	1	2				

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^4(T) = \begin{array}{|c|c|c|c|c|c|} \hline \overline{3} & \overline{3} & \overline{3} & \overline{2} & \overline{2} & \overline{2} \\ \hline \overline{1} & \overline{1} & 1 & 2 \\ \cline{1-4} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$CoCyc^5(T) =$

$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{2}$
$\overline{2}$	$\overline{1}$	$\overline{1}$	1	3

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^6(T) = \begin{array}{|c|c|c|c|c|c|}\hline \overline{3} & \overline{3} & \overline{3} & \overline{3} & 1 & 3 \\ \hline \overline{2} & \overline{2} & \overline{1} & \overline{1} & & \\ \cline{1-4} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$CoCyc^7(T) =$

$\overline{3}$	$\overline{3}$	$\overline{3}$	$\overline{3}$	1
$\overline{2}$	$\overline{2}$	$\overline{1}$	$\overline{1}$	
3				

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^8(T) = \begin{array}{|c|c|c|c|} \hline \overline{4} & \overline{3} & \overline{3} & \overline{3} \\ \hline \overline{2} & \overline{2} & \overline{1} & \overline{1} \\ \hline 1 & 4 \\ \cline{1-2} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^9(T) = \begin{array}{|c|c|c|c|c|} \hline \overline{4} & \overline{3} & \overline{3} & \overline{1} & \overline{1} \\ \hline \overline{3} & \overline{2} & \overline{2} \\ \cline{1-3} 1 & 4 \\ \cline{1-2} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{10}(T) = \begin{array}{|c|c|c|c|} \hline \overline{4} & \overline{3} & \overline{3} & \overline{1} \\ \hline \overline{3} & \overline{2} & \overline{2} \\ \cline{1-3} \overline{1} & 1 & 4 \\ \cline{1-3} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{11}(T) = \begin{array}{|c|c|c|c|}\hline \overline{4} & \overline{3} & \overline{3} & 4 \\ \hline \overline{3} & \overline{2} & \overline{2} \\ \cline{1-3} \overline{1} & \overline{1} & 1 \\ \cline{1-3} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{12}(T) = \begin{array}{|c|c|c|}\hline \overline{4} & \overline{3} & \overline{3} \\ \hline \overline{3} & \overline{2} & \overline{2} \\ \hline \overline{1} & \overline{1} & 1 \\ \hline 4 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$CoCyc^{12}(T) =$

$\overline{4}$	$\overline{3}$	$\overline{3}$
$\overline{3}$	$\overline{2}$	$\overline{2}$
$\overline{1}$	$\overline{1}$	1
4		

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{12}(T) = \begin{array}{|c|c|c|}\hline \overline{4} & \overline{3} & \overline{3} \\ \hline \overline{2} & \overline{2} & 2 \\ \hline 4 \\ \cline{1-1} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{13}(T) = \begin{array}{|c|c|c|c|}\hline \overline{4} & \overline{3} & \overline{2} & 2 \\ \hline \overline{3} & \overline{2} \\ \cline{1-2} 4 \\ \cline{1-1} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{14}(T) = \begin{array}{|c|c|c|}\hline \overline{5} & \overline{3} & \overline{2} \\ \hline \overline{3} & \overline{2} & \\ \cline{1-2} 2 & 5 & \\ \cline{1-2} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$CoCyc^{15}(T) =$

$\overline{5}$	$\overline{3}$	5
$\overline{3}$	$\overline{2}$	
$\overline{2}$	2	

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$CoCyc^{16}(T) =$

$\overline{5}$	$\overline{3}$
$\overline{3}$	$\overline{2}$
$\overline{2}$	2
5	

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$CoCyc^{16}(T) =$

$\overline{5}$	$\overline{3}$
$\overline{3}$	$\overline{2}$
$\overline{2}$	2
5	

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{16}(T) = \begin{array}{|c|c|} \hline \overline{5} & \overline{3} \\ \hline \overline{3} & 3 \\ \hline 5 & \\ \cline{1-1} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{17}(T) = \begin{array}{|c|c|c|}\hline \overline{5} & \overline{3} & 3 \\ \hline \overline{3} \\ \cline{1-1} 5 \\ \cline{1-1}\end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{18}(T) = \begin{array}{|c|c|}\hline \overline{6} & \overline{3} \\ \hline \overline{3} & 6 \\ \hline 3 \\ \cline{1-1}\end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{19}(T) = \begin{array}{|c|c|c|} \hline \overline{6} & 4 & 6 \\ \hline \overline{4} \\ \cline{1-1} \overline{3} \\ \cline{1-1} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{20}(T) = \begin{array}{|c|c|} \hline \overline{6} & 4 \\ \hline \overline{4} & \\ \cline{1-1} \overline{3} & \\ \cline{1-1} 6 & \\ \cline{1-1} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$$CoCyc^{21}(T) = \begin{array}{|c|c|}\hline \overline{7} & 7 \\ \hline \overline{4} & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

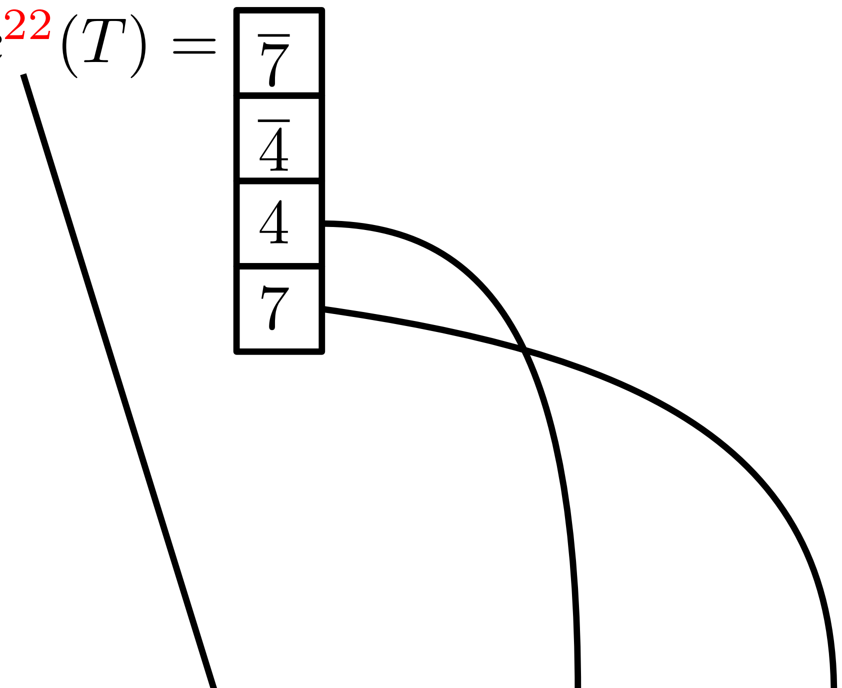
$$CoCyc^{22}(T) = \begin{array}{|c|}\hline \overline{7} \\ \hline \overline{4} \\ \hline 4 \\ \hline 7 \\ \hline \end{array}$$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$CoCyc^{22}(T) =$



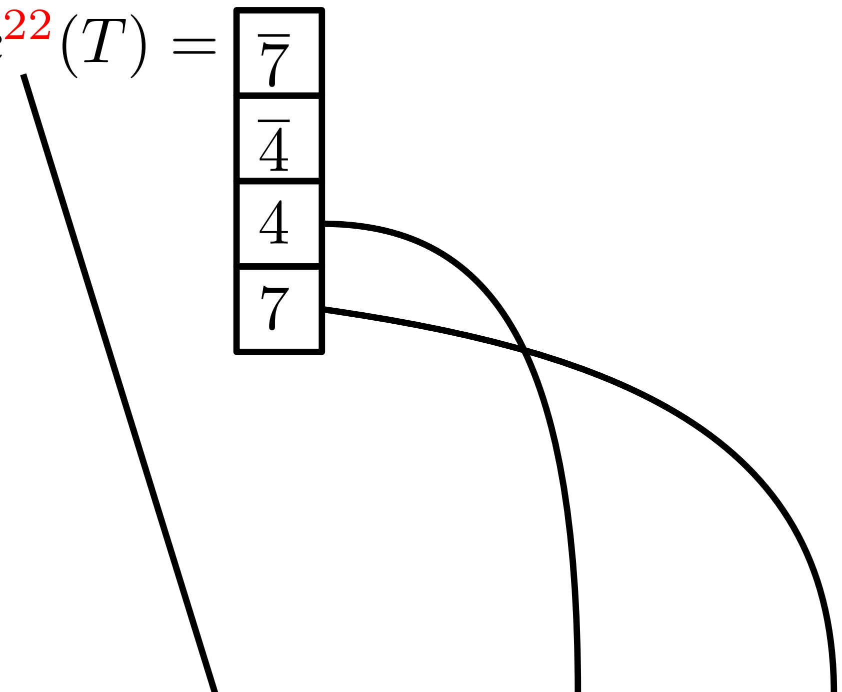
$charge^{C_3}(T) = 22 + 2\big((3-4) + (3-7)\big) = 12$

Cocyclage & charge in type C (Lecouvey 2005)

$T \in SympTab_n(\lambda, \mu)$

$charge^{C_n}(T) = ???$

$CoCyc^{22}(T) =$

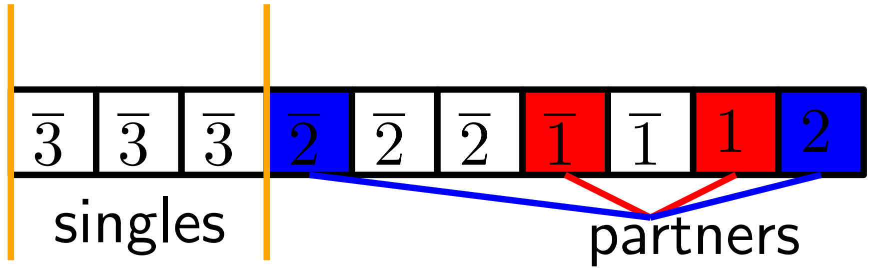


$charge^{C_3}(T) = 22 + 2\big((3-4) + (3-7)\big) = 12$

Problem Computing charge = going through the **whole** cocyclage.
$CoCyc^{n+1}(T)$ depends heavily on $CoCyc^{n}(T)$ + local constraints!

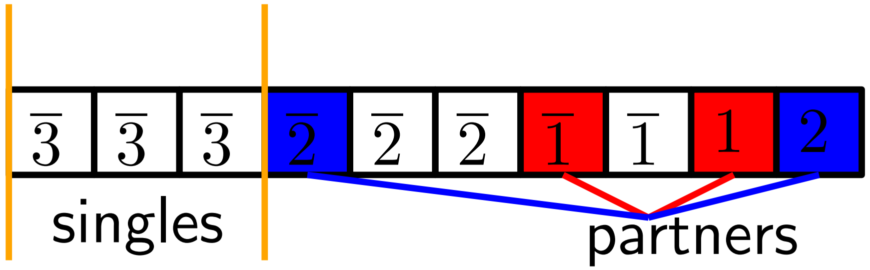
Main result - new algorithm for CoCyclage!

$T =$



$$CoCyc^{14}(T) = ???$$

Main result - new algorithm for CoCyclage!

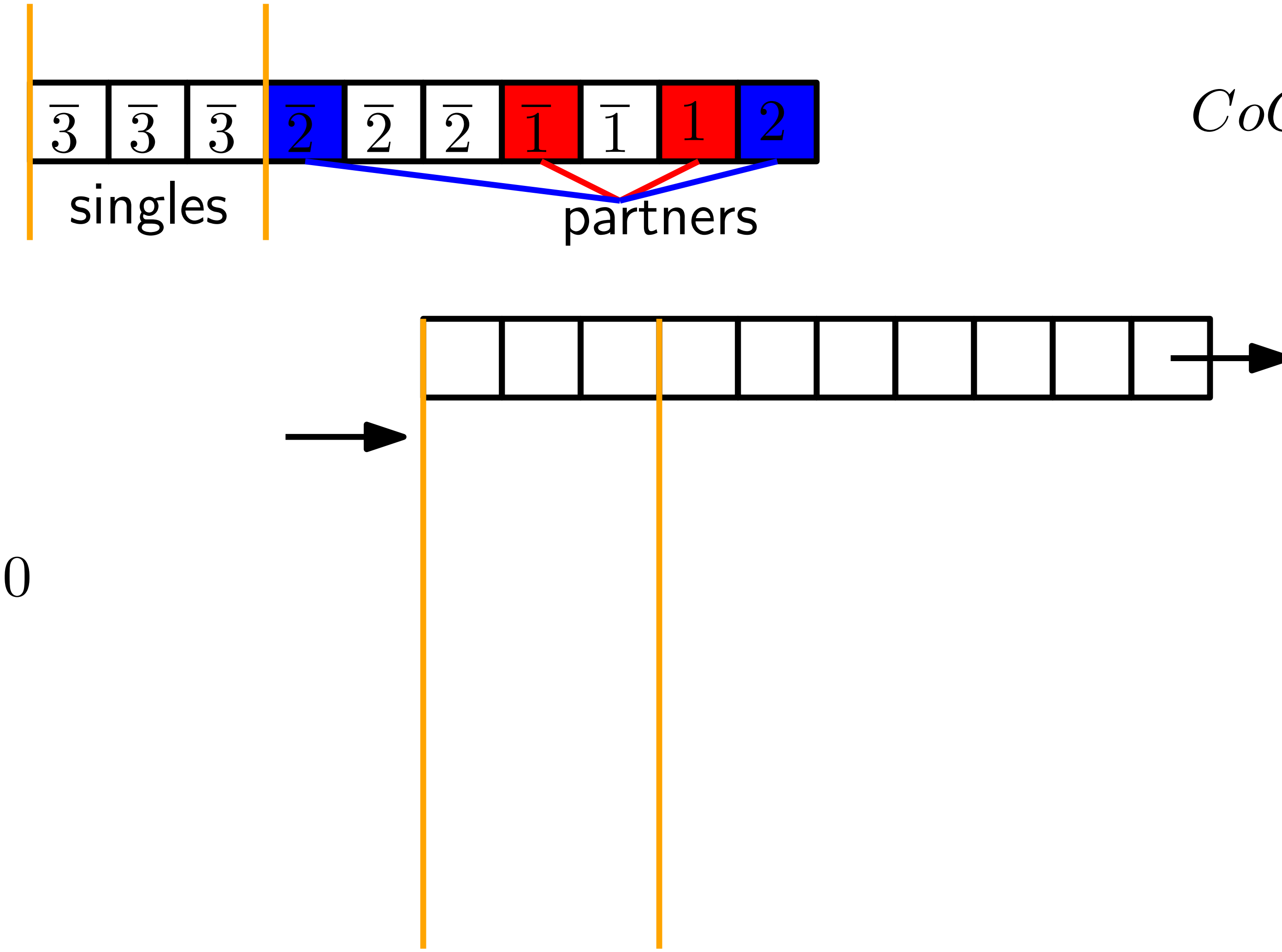
$T =$ 

$CoCyc^{14}(T) = ???$

Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

$T =$



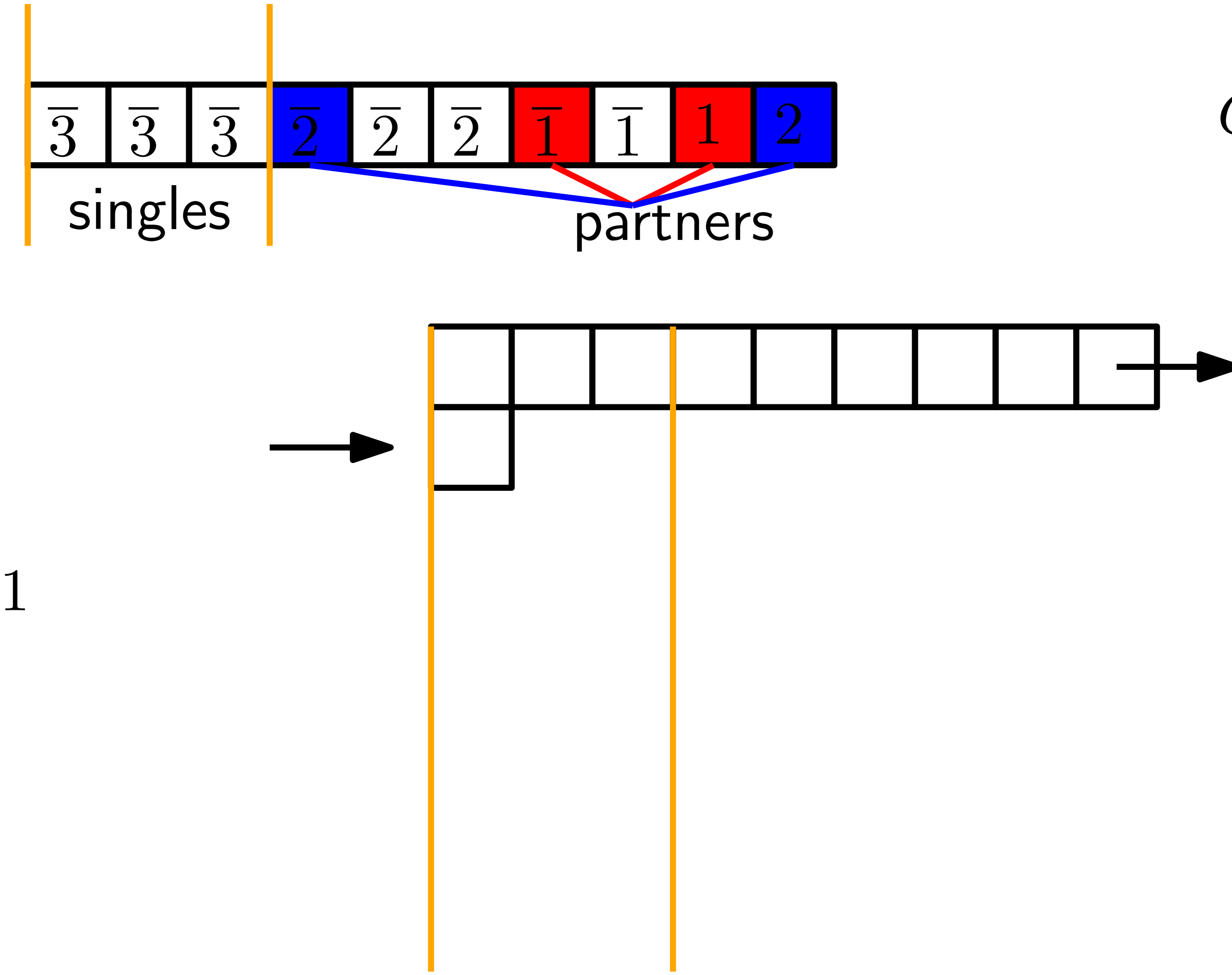
$CoCyc^{14}(T) = ???$

Step 0

Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

$T =$



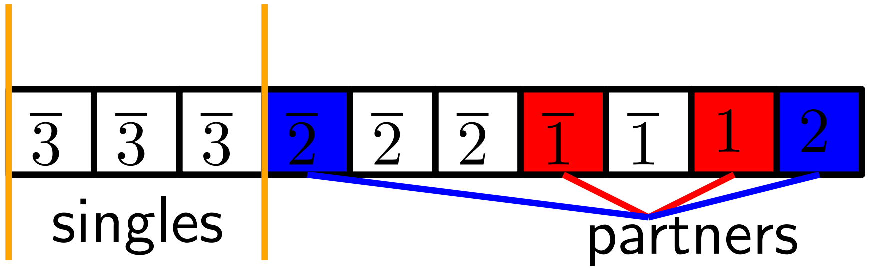
$CoCyc^{14}(T) = ???$

Step 1

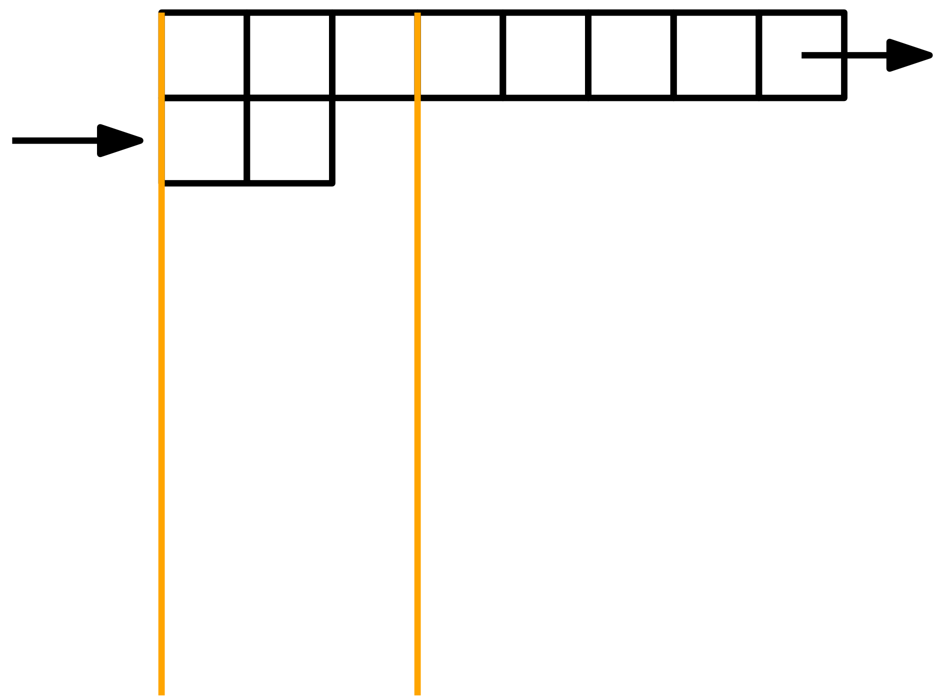
Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

$T =$



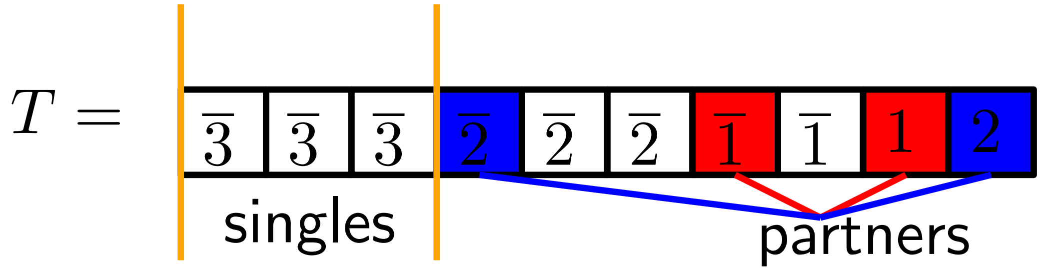
$CoCyc^{14}(T) = ???$



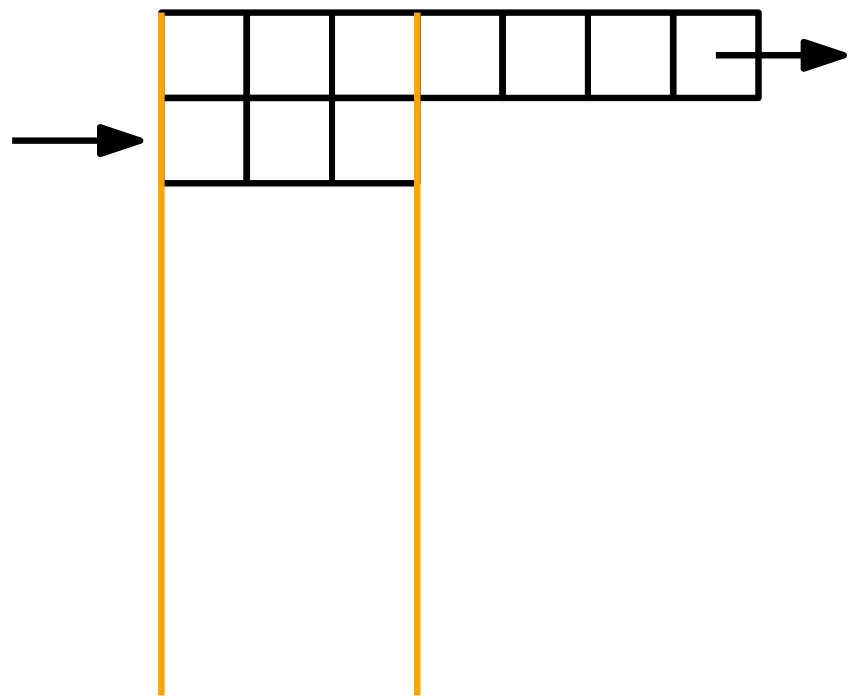
Step 2

Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!



$CoCyc^{14}(T) = ???$

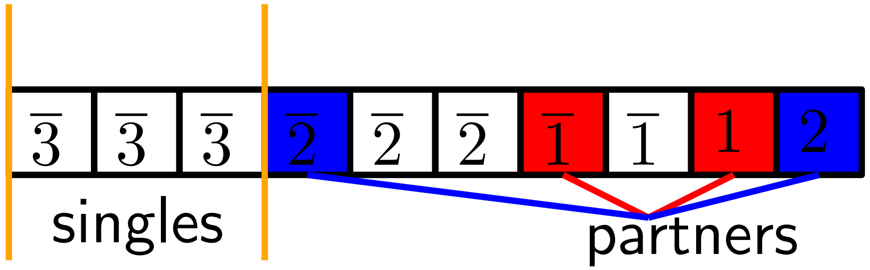


Step 3

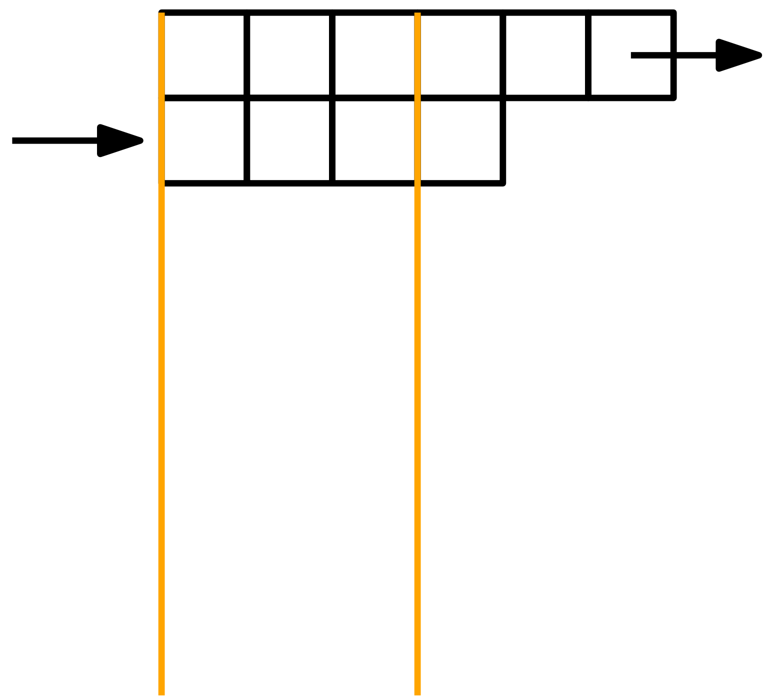
Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

$T =$



$CoCyc^{14}(T) = ???$

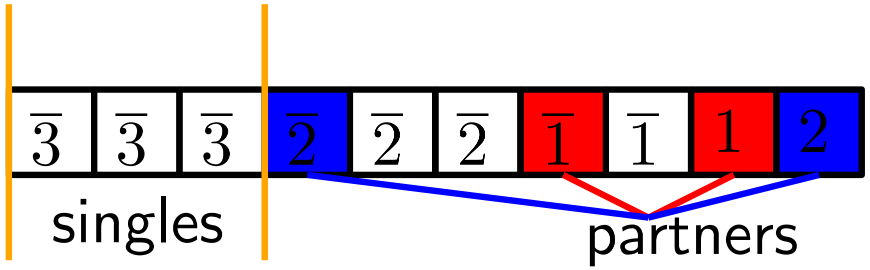


Step 4

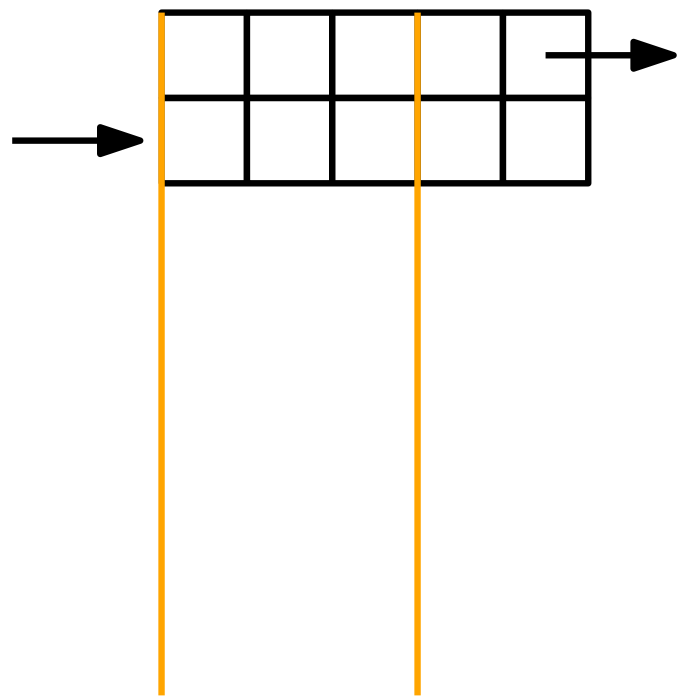
Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

$T =$



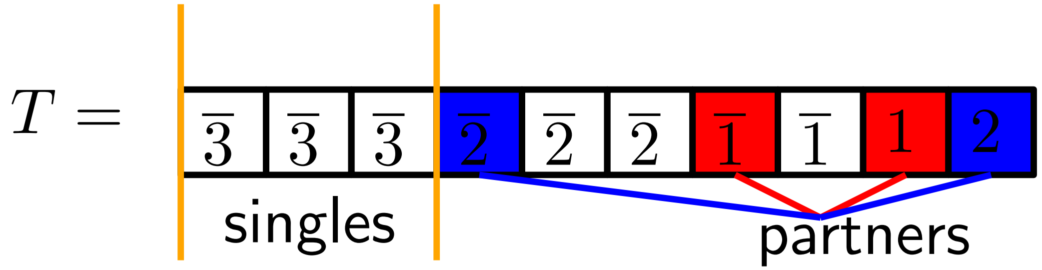
$CoCyc^{14}(T) = ???$



Step 5

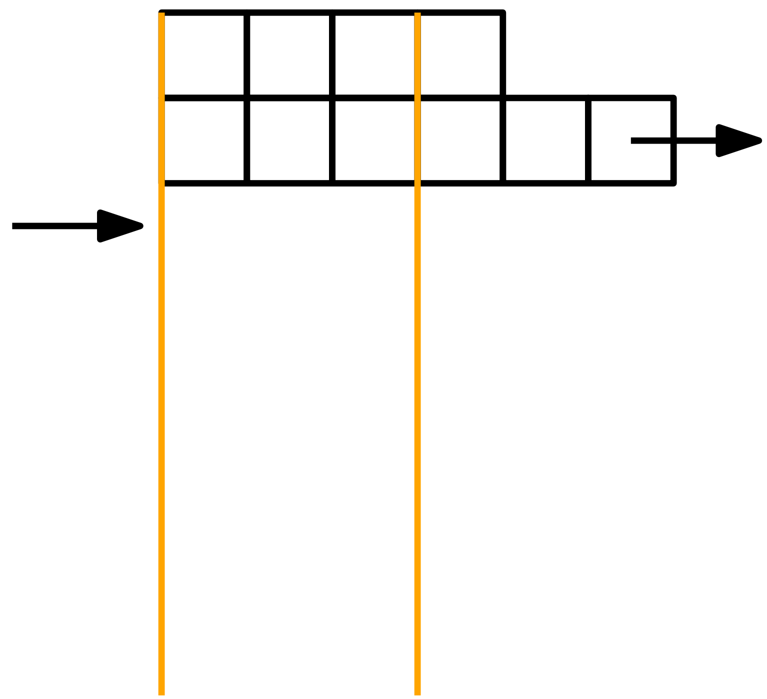
Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!



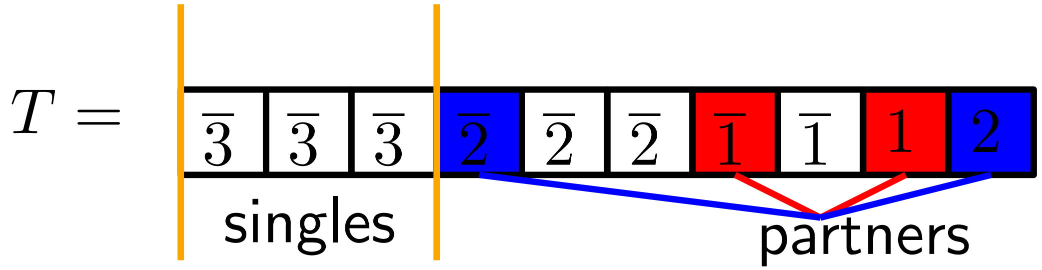
$CoCyc^{14}(T) = ???$

Step 6



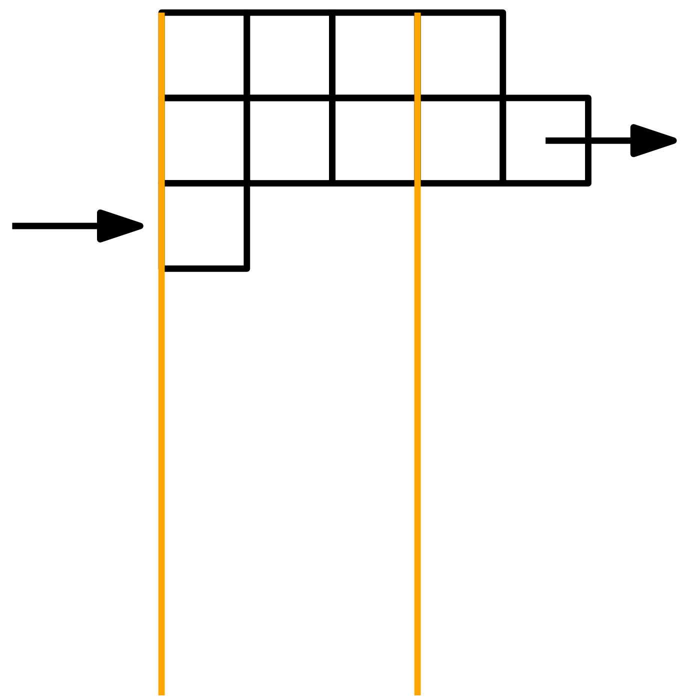
Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!



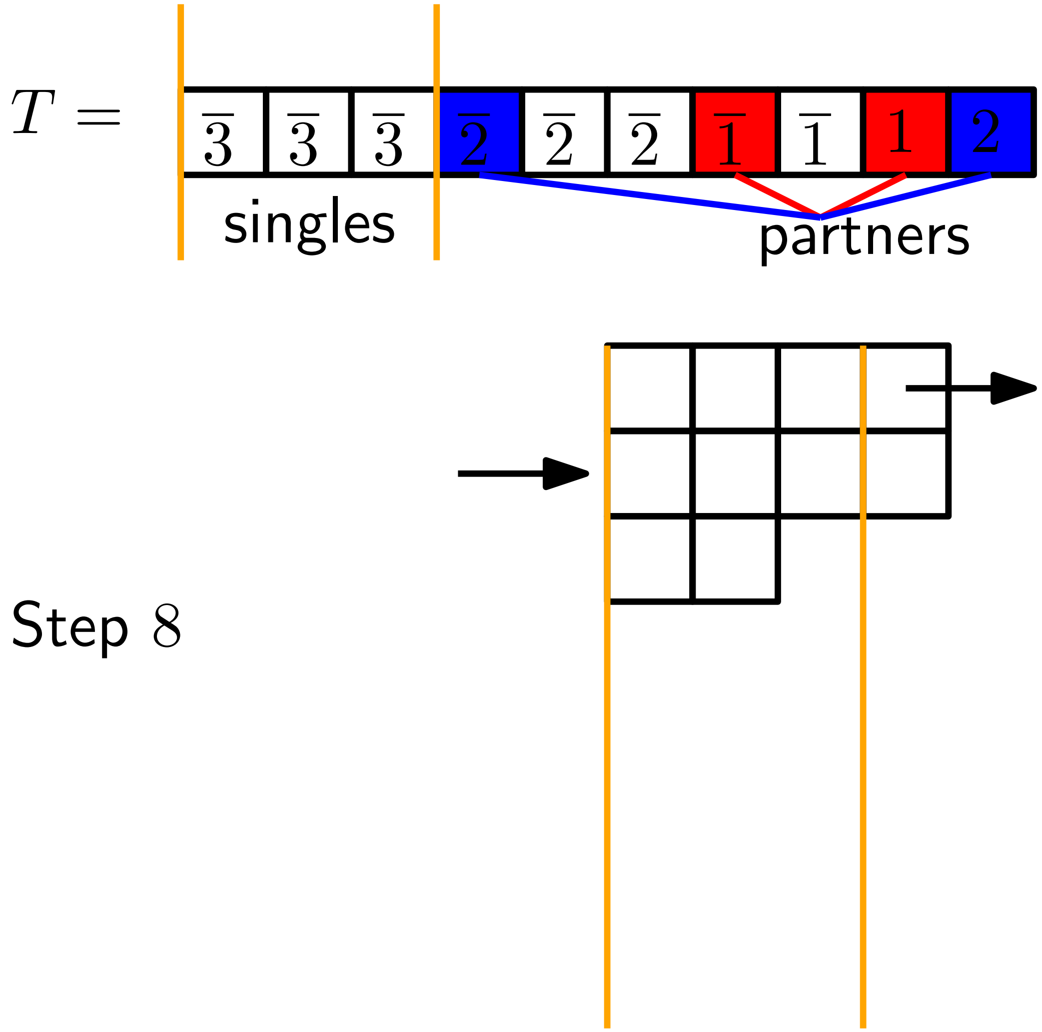
$CoCyc^{14}(T) = ???$

Step 7



Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!



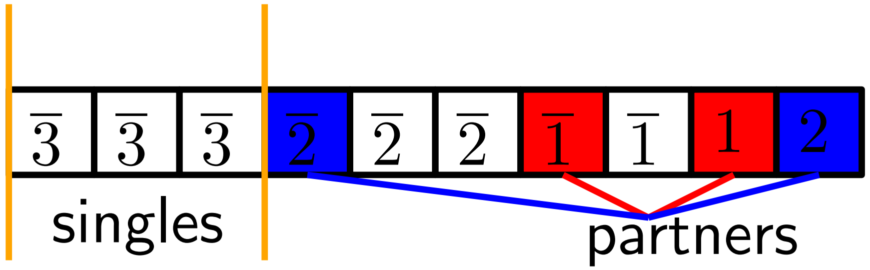
$CoCyc^{14}(T) = ???$

Step 8

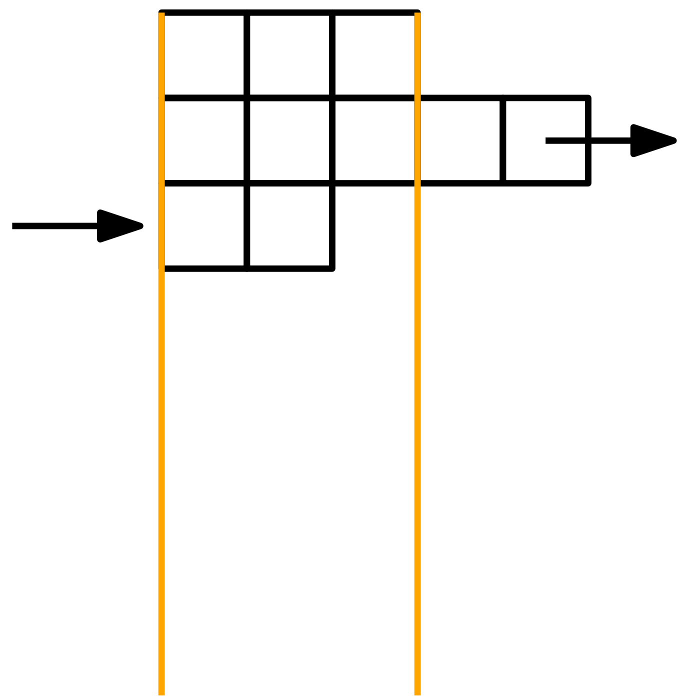
Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

$T =$



$CoCyc^{14}(T) = ???$

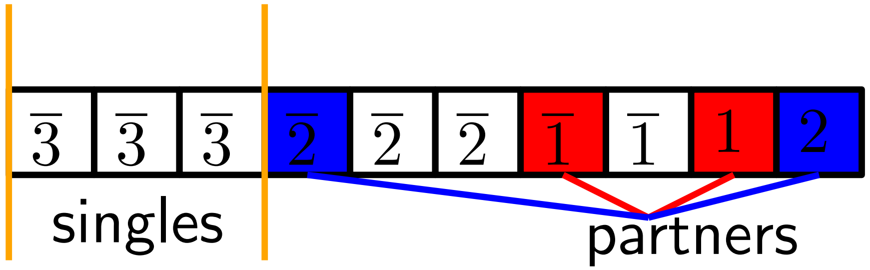


Step 9

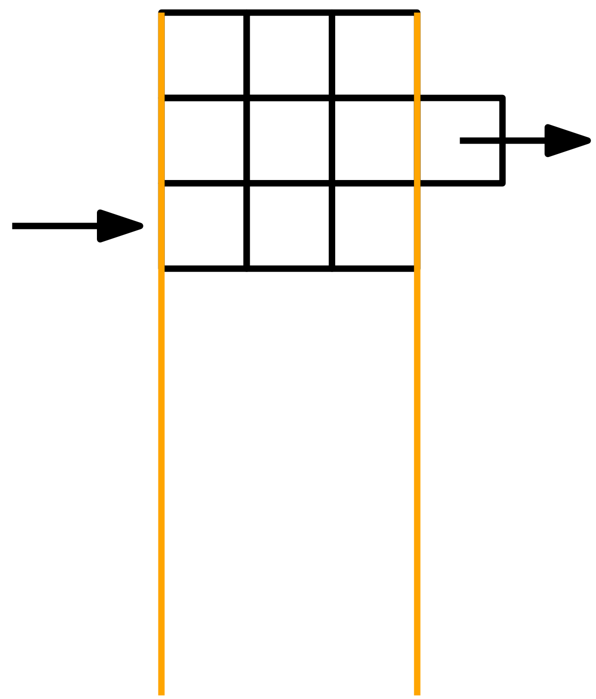
Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

$T =$



$CoCyc^{14}(T) = ???$

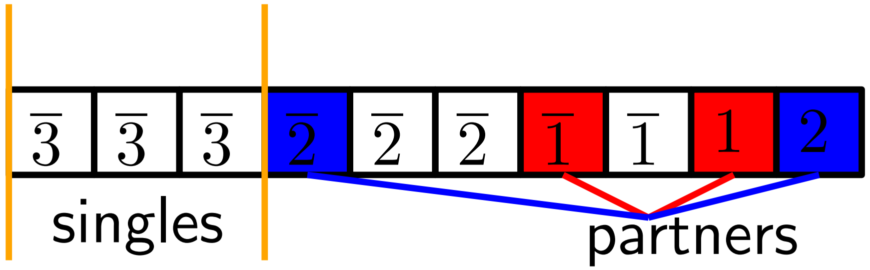


Step 10

Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

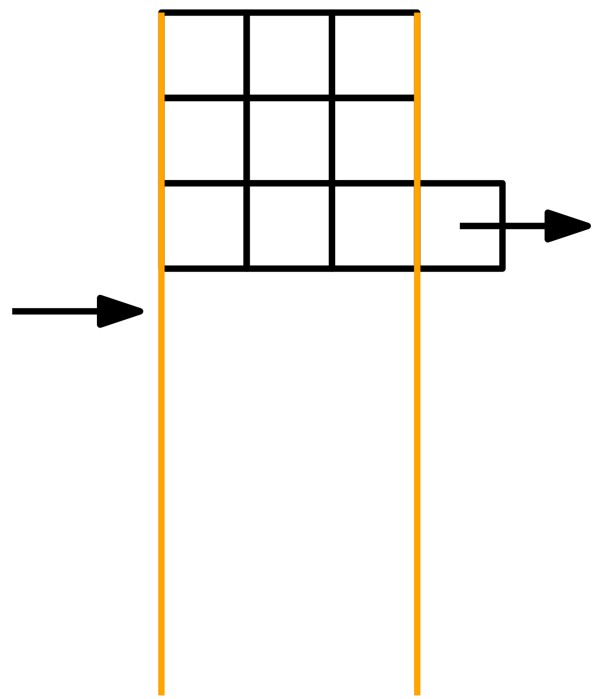
Main result - new algorithm for CoCyclage!

$T =$



$CoCyc^{14}(T) = ???$

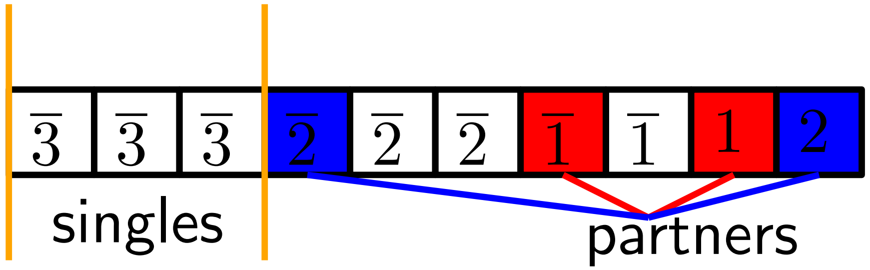
Step 11



Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

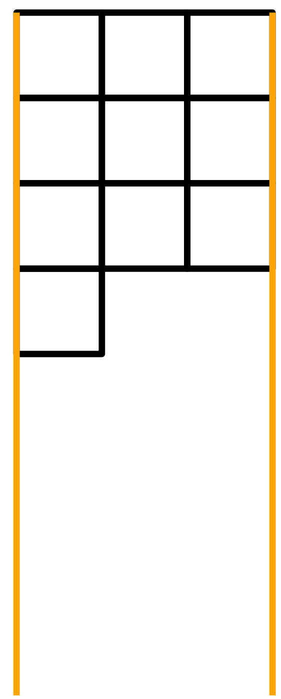
Main result - new algorithm for CoCyclage!

$T =$



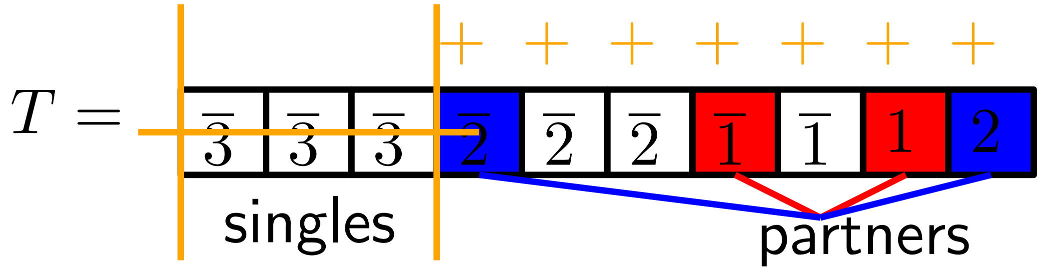
$CoCyc^{14}(T) = ???$

Step 12



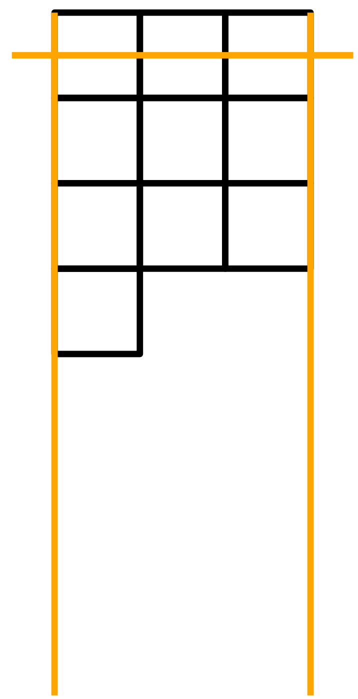
Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!



$CoCyc^{14}(T) = ???$

Step 12

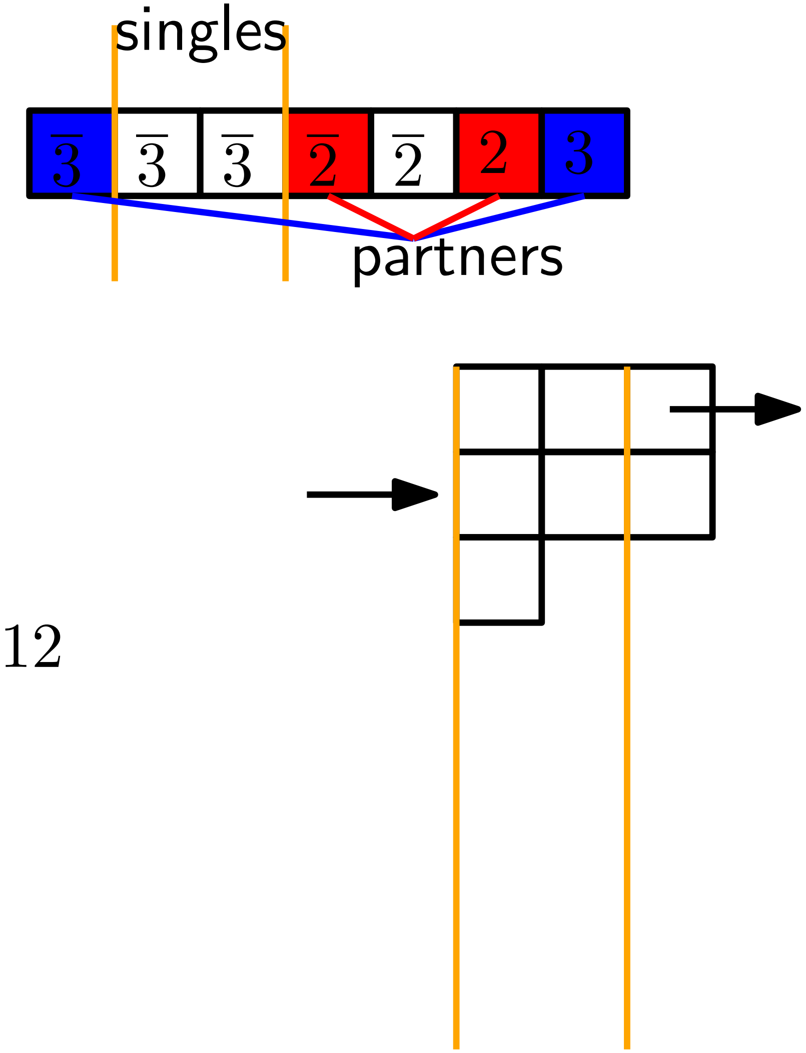


Reduction: The whole diagram is contained between orange lines -> update T (remove orange singles, increase everything by 1)

Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

$T =$



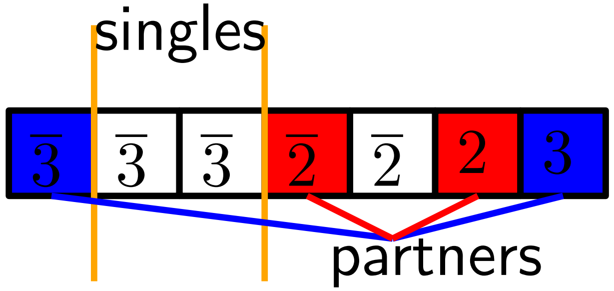
$CoCyc^{14}(T) = ???$

Step 12

Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

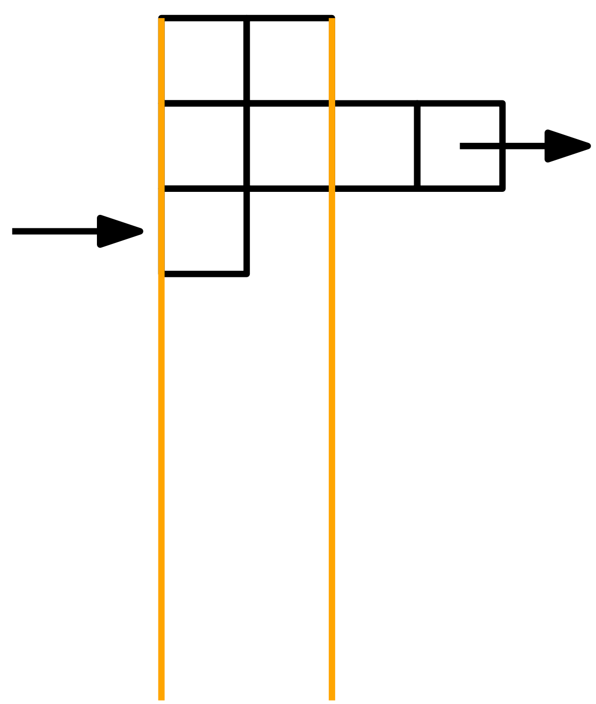
Main result - new algorithm for CoCyclage!

$T =$



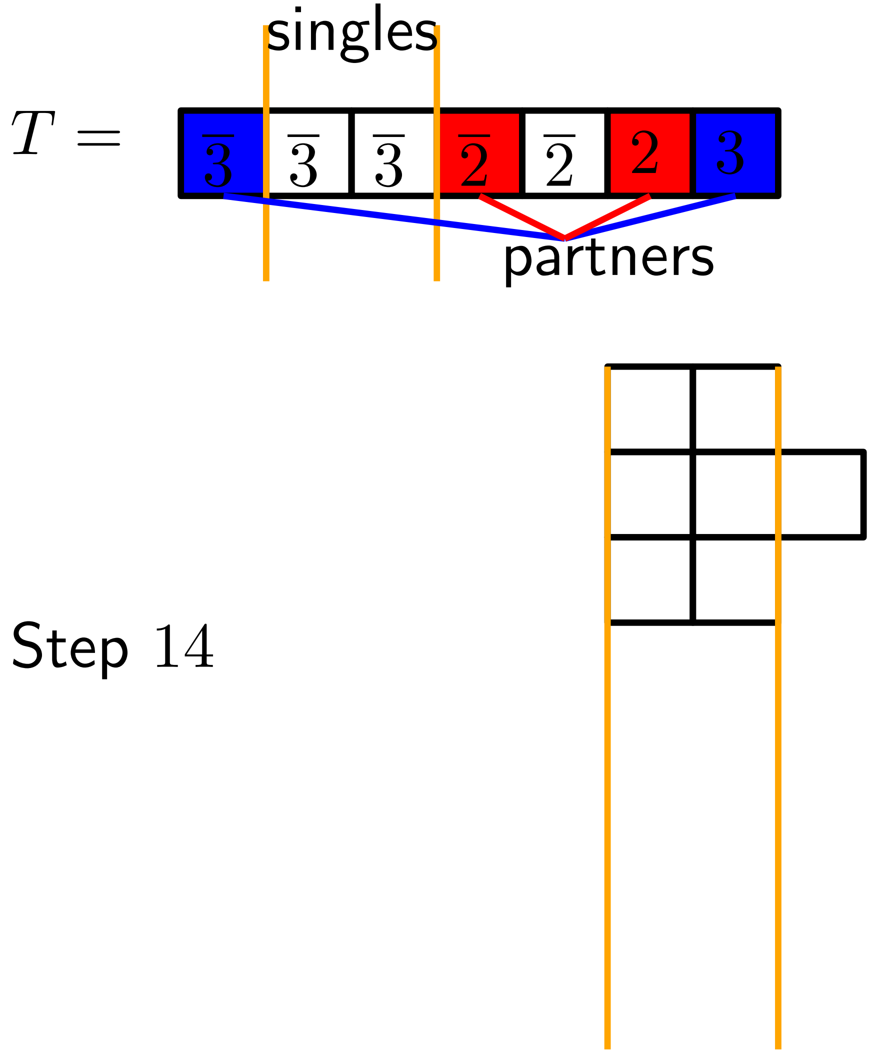
$CoCyc^{14}(T) = ???$

Step 13



Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

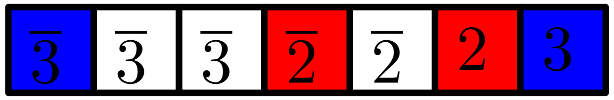


$CoCyc^{14}(T) = ???$

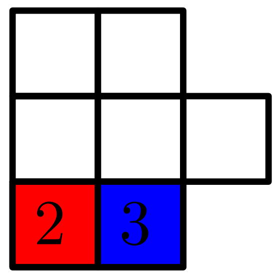
Step 14

Find a shape by shifting: remove the far right box from the longest row and insert it to the next row by shifting. **Remove singles** when the shape is contained between orange lines.

Main result - new algorithm for CoCyclage!

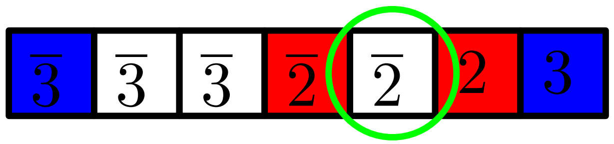
$T =$ 

$CoCyc^{14}(T) = ???$

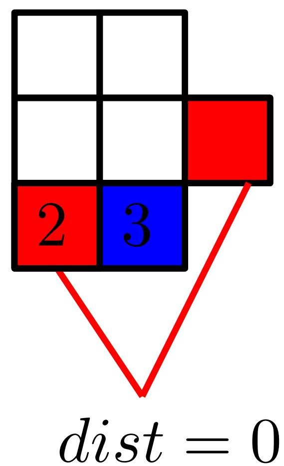


Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!

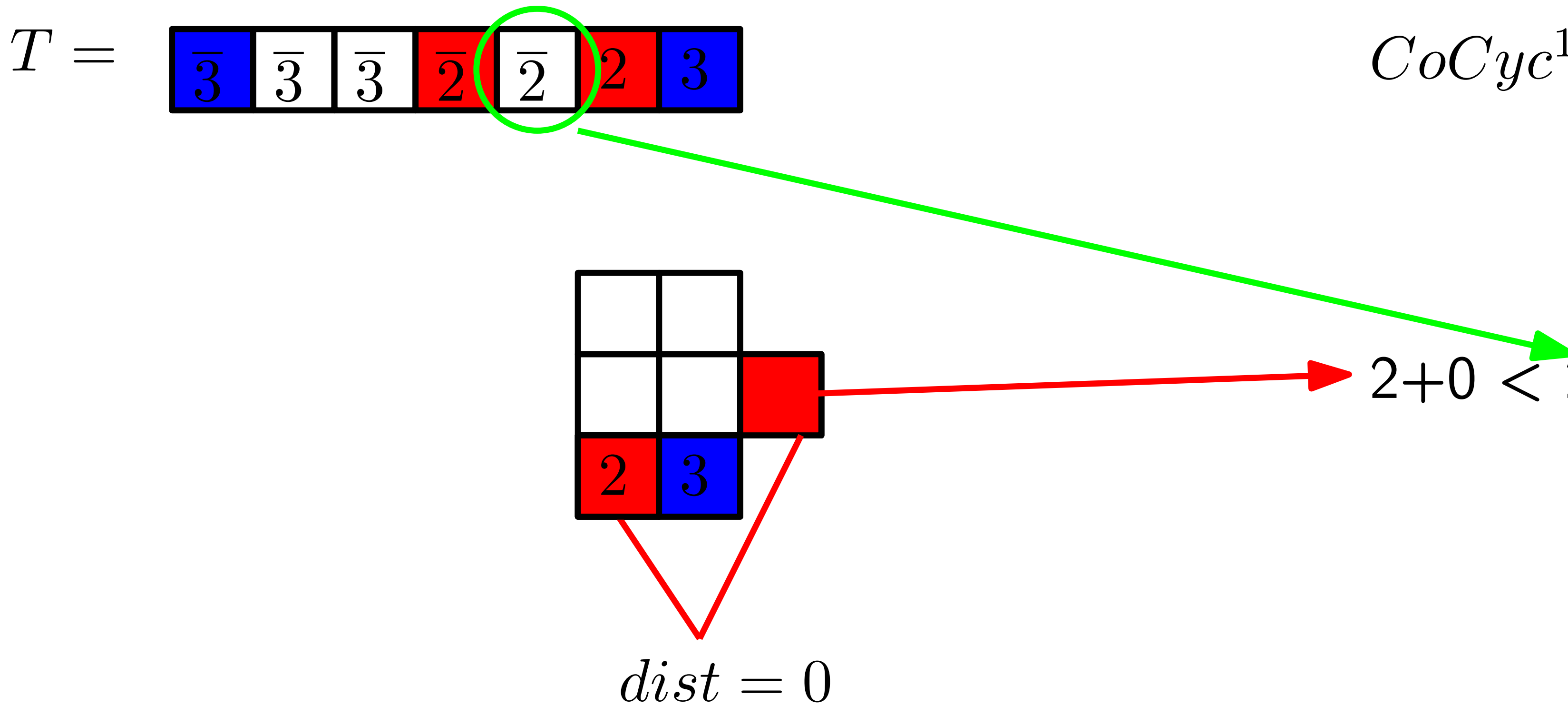
$T =$ 

$CoCyc^{14}(T) =???$



Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!



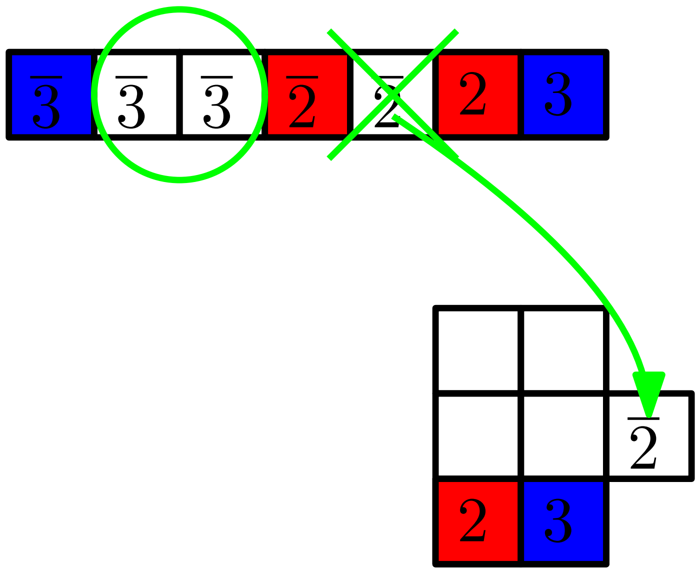
$CoCyc^{14}(T) = ???$

2+0 < 2?? - **NO!**

Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!

$T =$

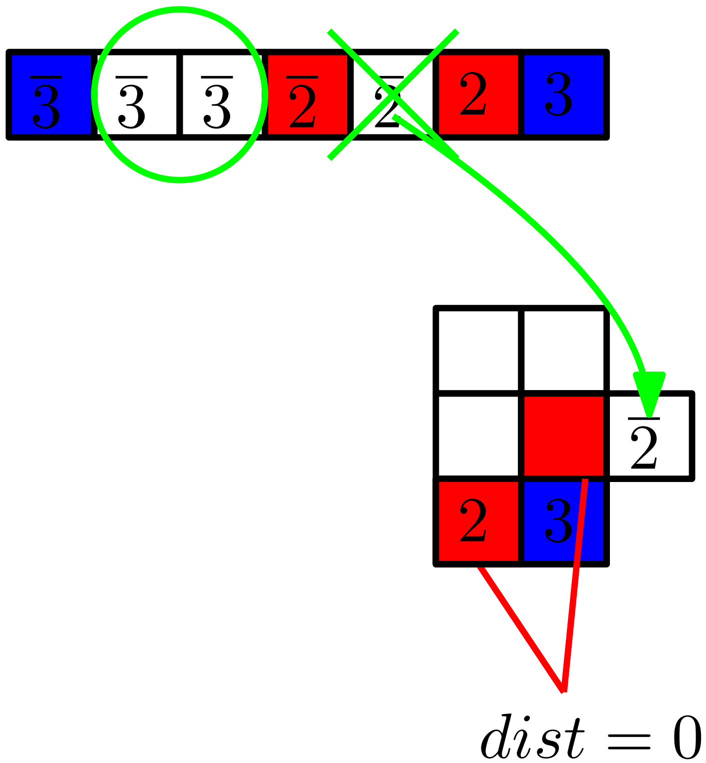


$CoCyc^{14}(T) = ???$

Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!

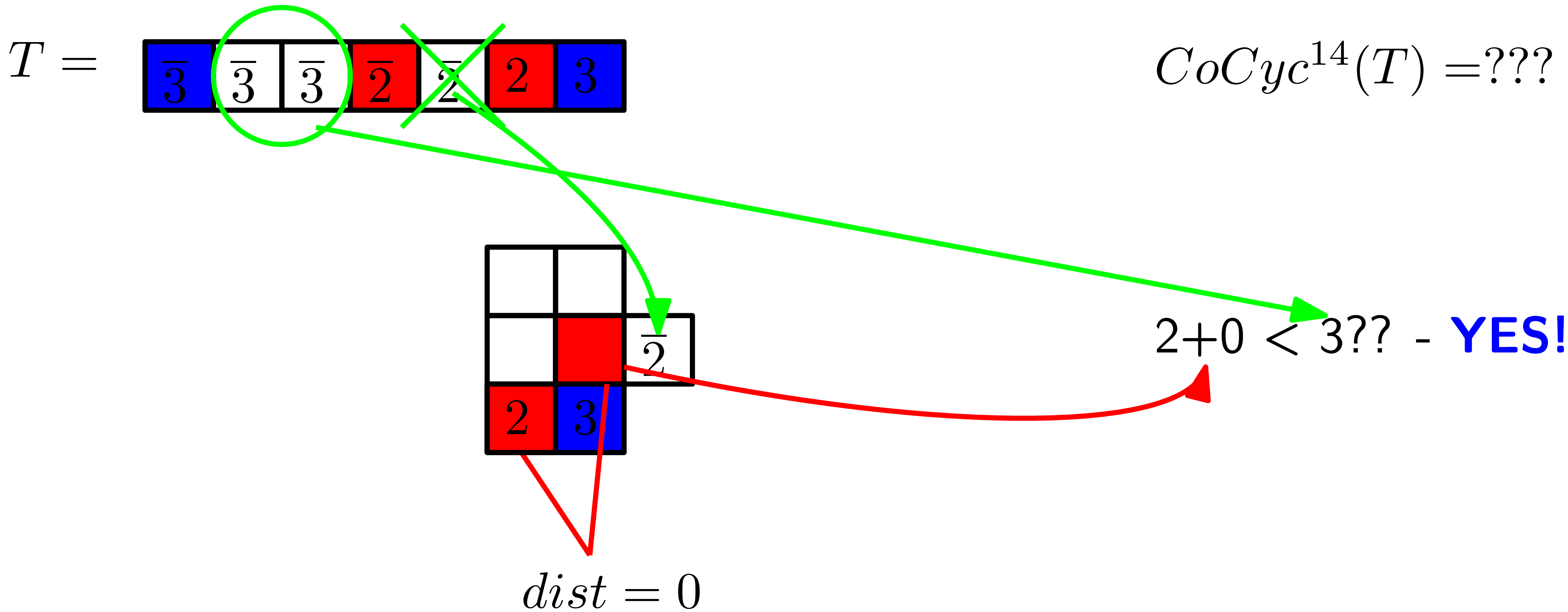
$T =$



$CoCyc^{14}(T) =???$

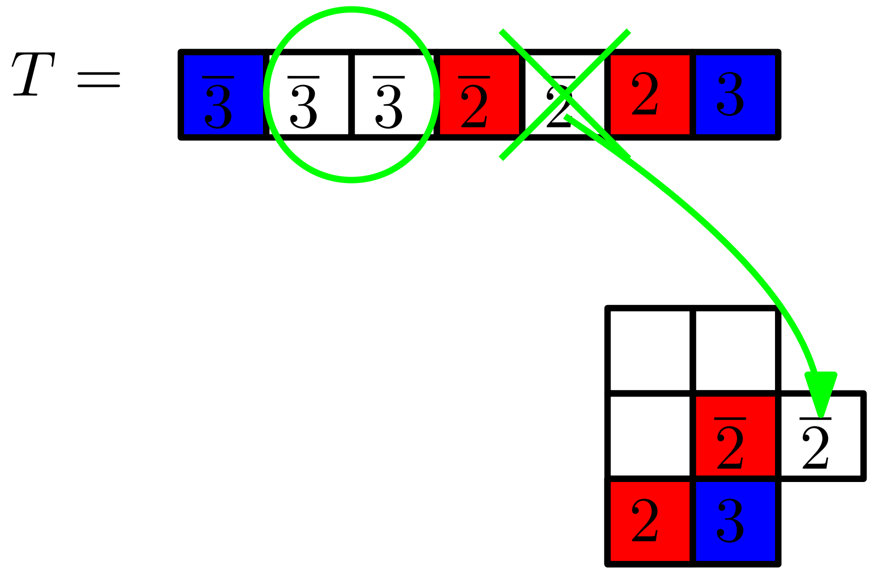
Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!



Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!

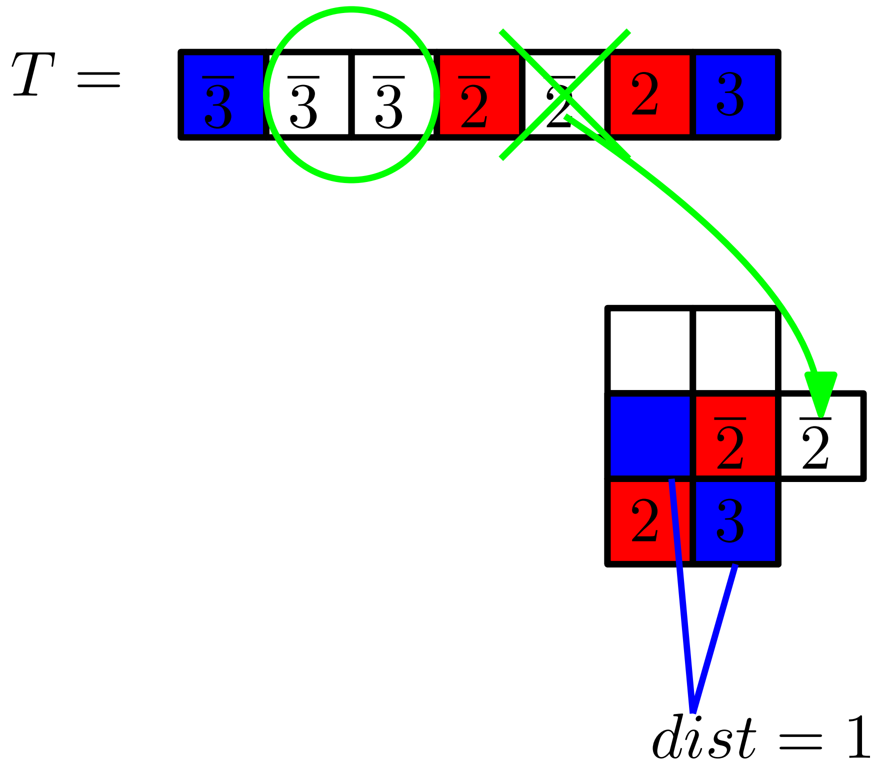


$CoCyc^{14}(T) =???$

Red partner found

Find a partner: compute the distance between the first available box and compare its value with the value of singles.

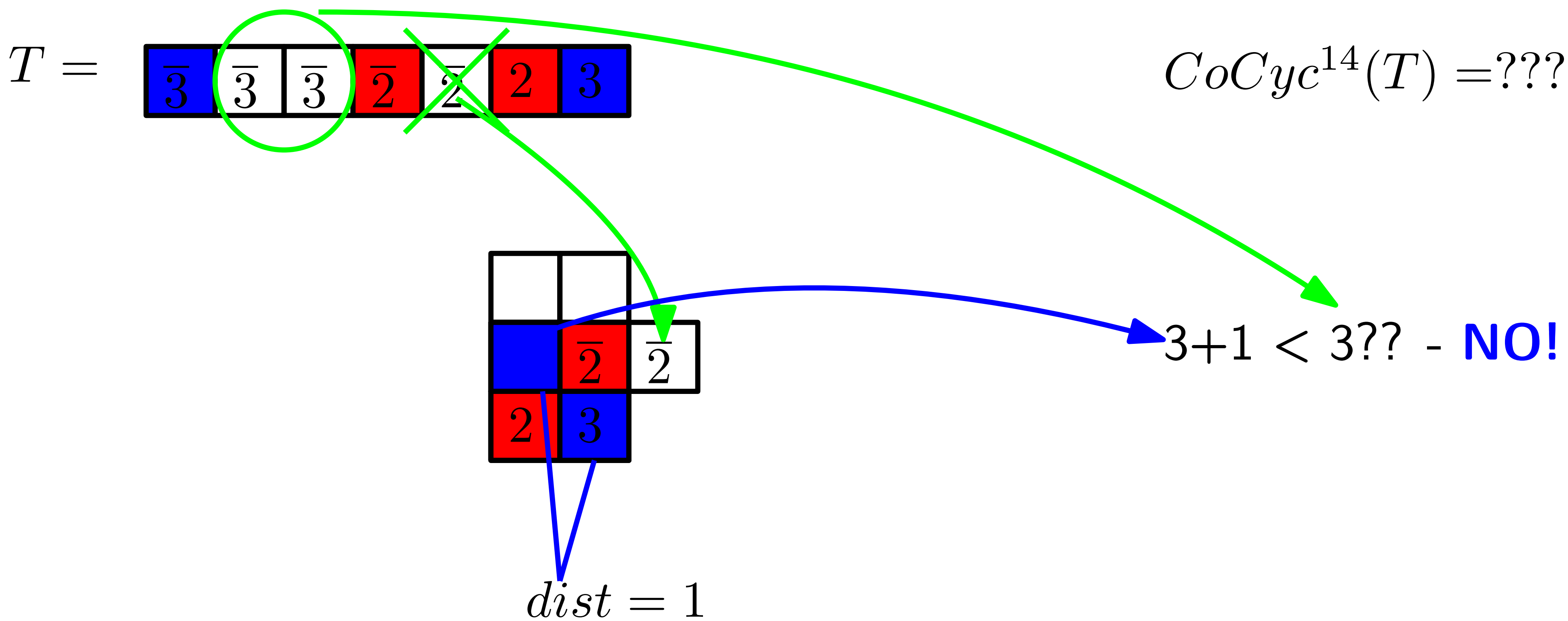
Main result - new algorithm for CoCyclage!



$CoCyc^{14}(T) = ???$

Find a partner: compute the distance between the first available box and compare its value with the value of singles.

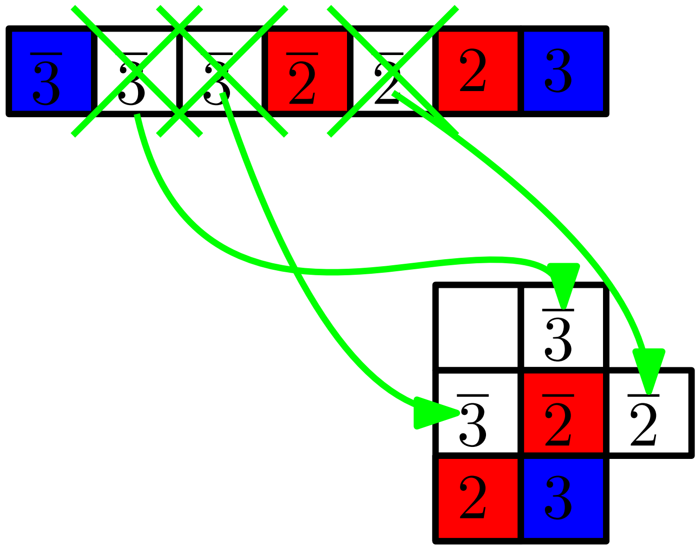
Main result - new algorithm for CoCyclage!



Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!

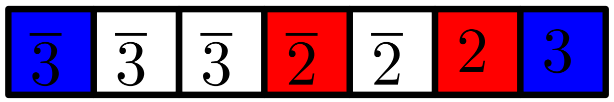
$T =$



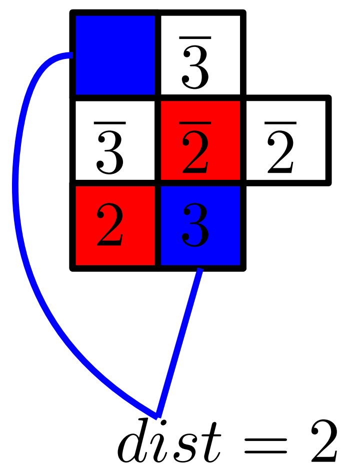
$CoCyc^{14}(T) = ???$

Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!

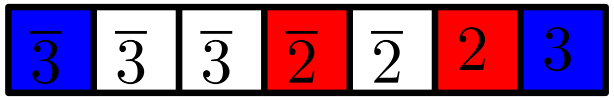
$T =$ 

$CoCyc^{14}(T) = ???$

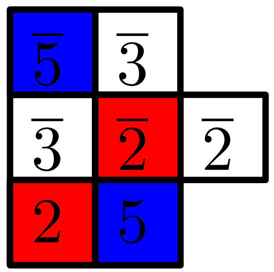


Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!

$T =$ 

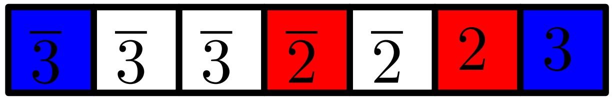
$CoCyc^{14}(T) = ???$



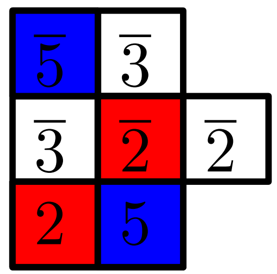
Blue partner found

Find a partner: compute the distance between the first available box and compare its value with the value of singles.

Main result - new algorithm for CoCyclage!

$T =$ 

$CoCyc^{14}(T) = ???$



Blue partner found

Corollary: [Dołęga, Gerber, Torres '20]
Lecouvey's conjecture is true for arbitrary n, p, μ and $\lambda = (p)$.

Theorem [Dołęga, Gerber, Torres '20]

$$K^{C_n}_{\lambda,\mu}(q) = \sum_{T \in SympTab_n(\lambda,\mu)} q^{charge^{C_n}(T)}.$$

for $\lambda = (p)$, where $charge^{C_n}$ is defined through the symplectic cocyclage.