Bijections for maps on non-oriented surfaces

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NARODOWE CENTRUM NAUKI

Random Geometry, Luminy - 17th of January 2022.

I. Maps

Maps

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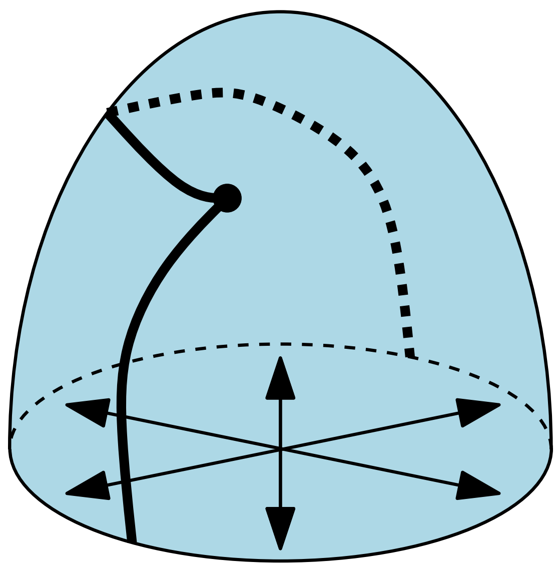

This is an map on the projective plane

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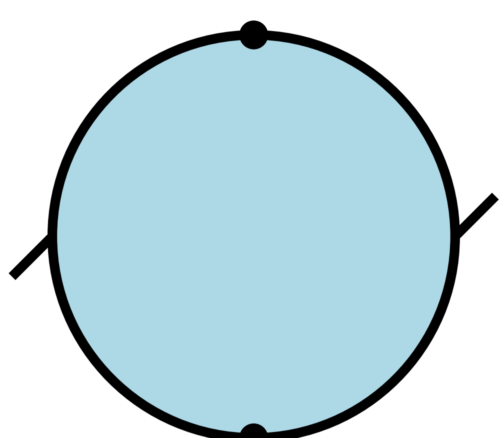

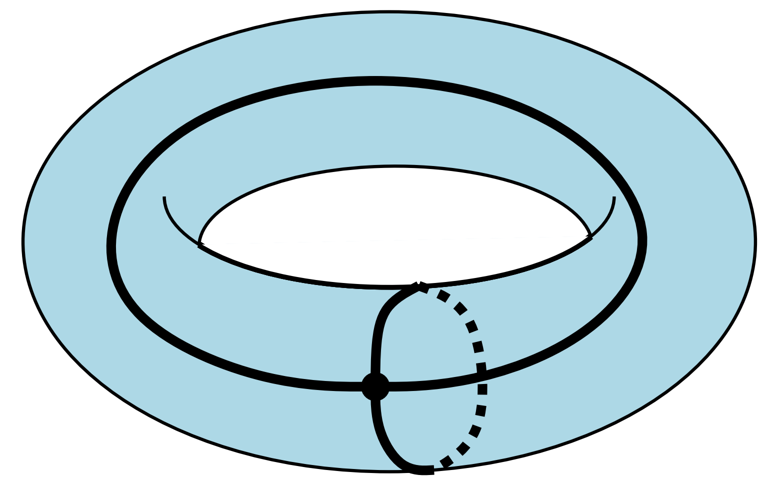

This is a map on the torus.

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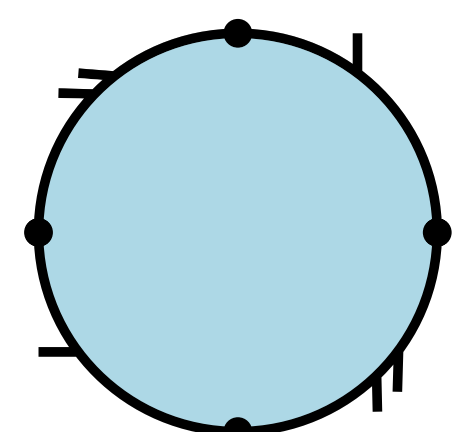

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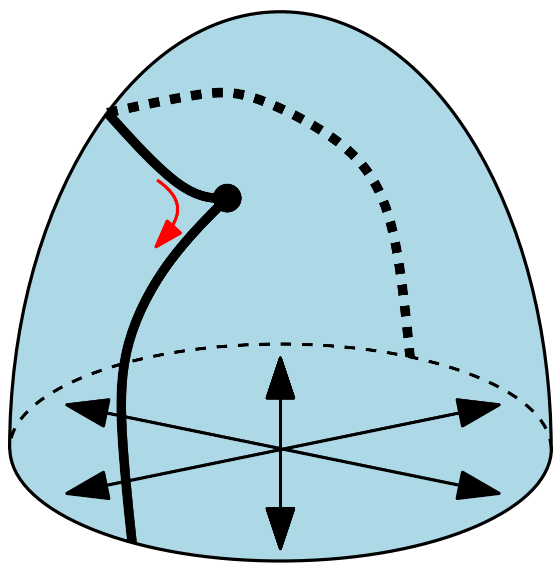

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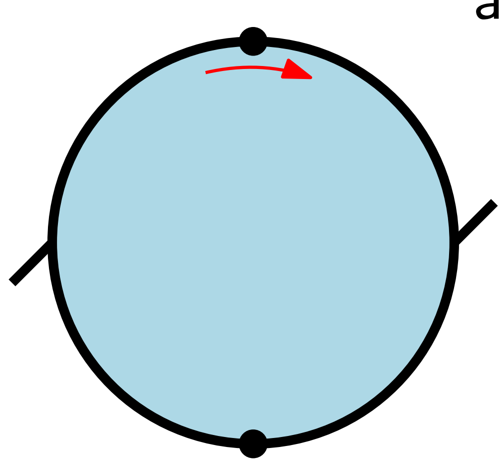
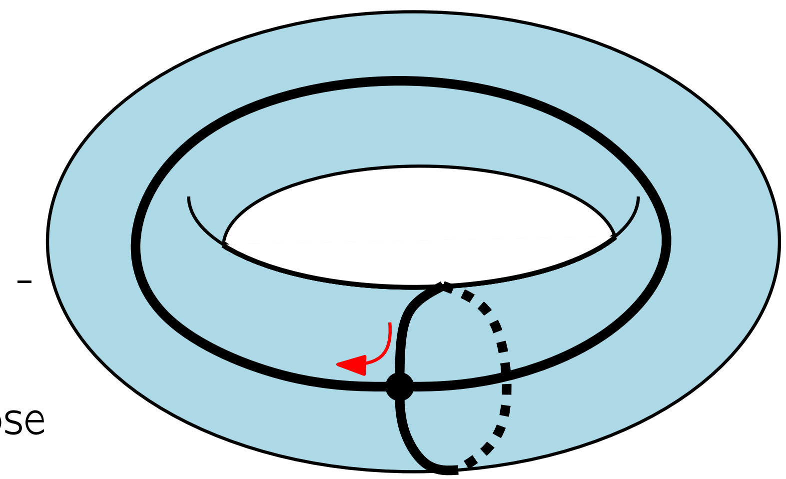


we kill automorphisms - easier to count/decompose

rooted map ≡ map with a distingiuished oriented corner

≡ distingushed oriented edge in the oriented case (warning: not enough in the non-oriented case!)



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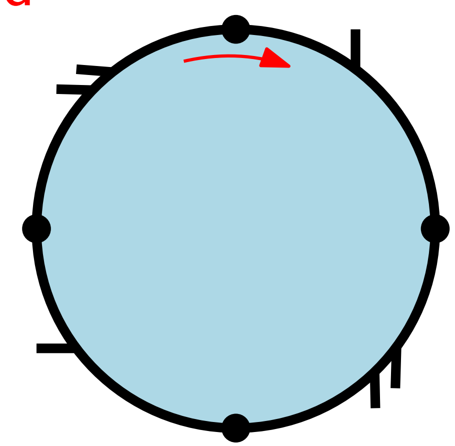

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universality predicted by topological recursion [Checkhov, Eynard–Orantin '06,'07+]: for any reasonable model $\mathscr{M}_{\mathcal{S}}$ on an orientable $\mathcal{S}$

$$m_{\mathscr{M}_{\mathcal{S}}}(n) \sim c(\mathscr{M}_{\mathcal{S}}) \cdot n^{-5/4 \cdot \chi(\mathcal{S})} \cdot \gamma_{\mathscr{M}_{\mathcal{S}}}^n$$

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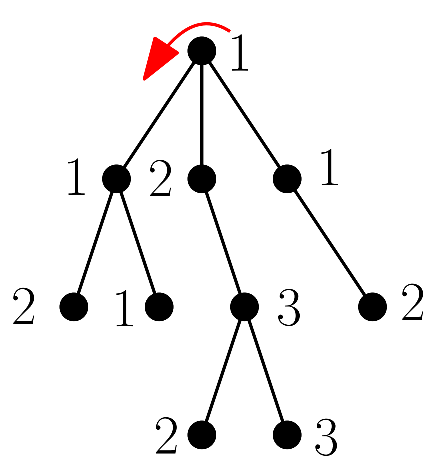
Direct combinatorial explanation:

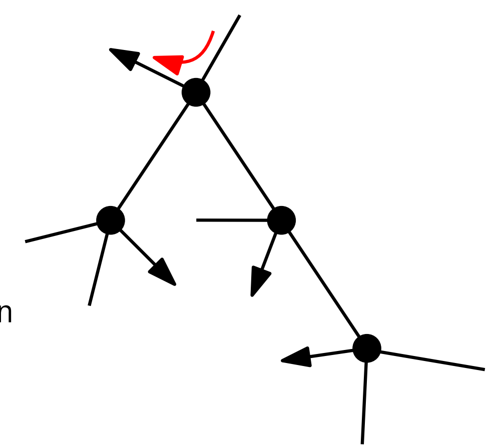
- When $\mathcal{S} = \mathbb{S}^2$: two important bijections with tree-like structures.



- root vertex labeled 1
- positive labels
- difference ≤ 1 along edges

Rooted well-labeled trees
[Cori–Vaquellin '81]
+ [Schaeffer '98]

- binary rooted tree on n vertices
- each vertex has an additional "bud"
- closing operation leaves the root leaf open



Balanced blossoming trees
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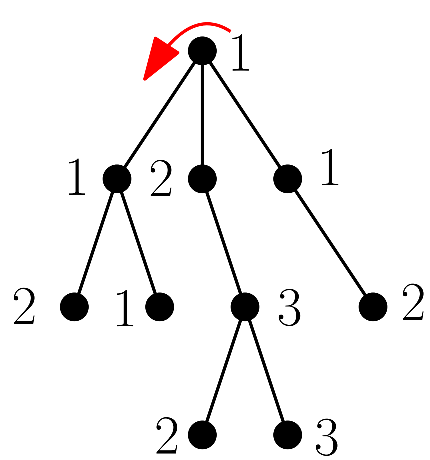

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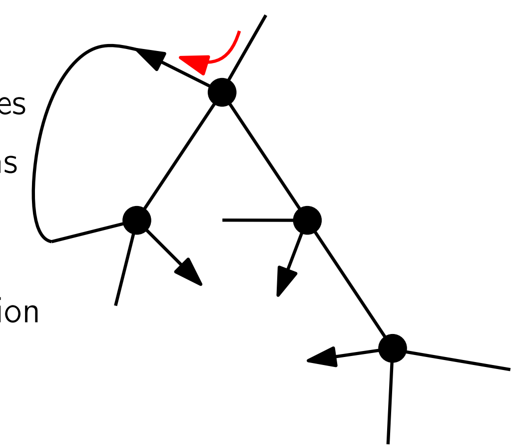

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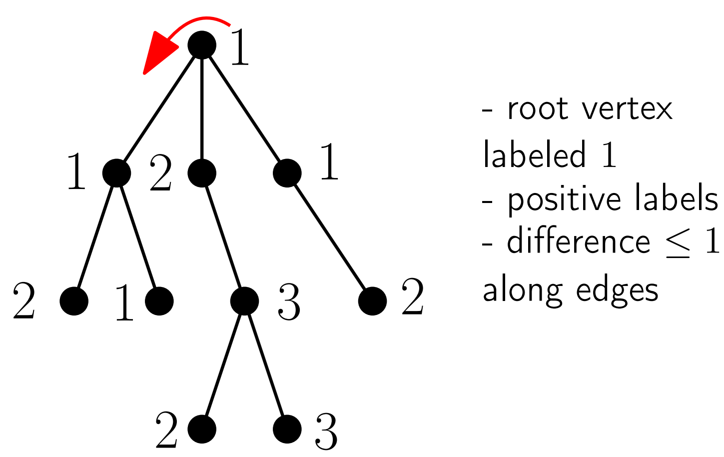

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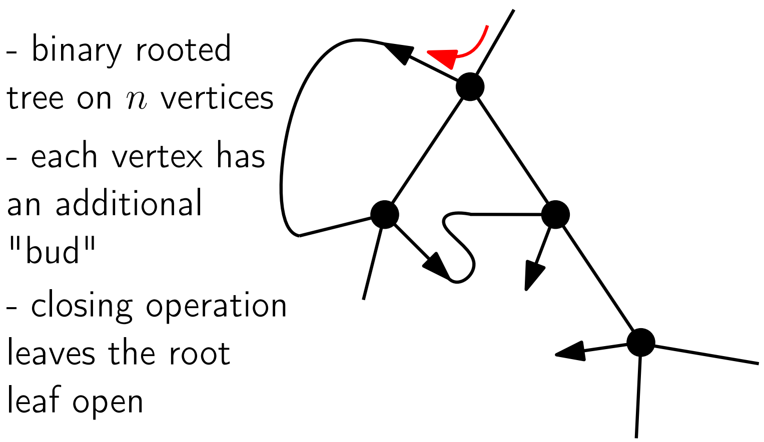

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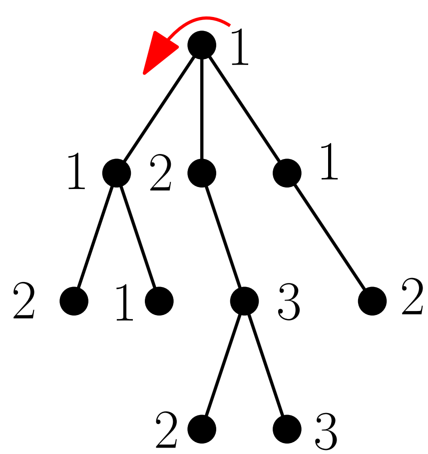

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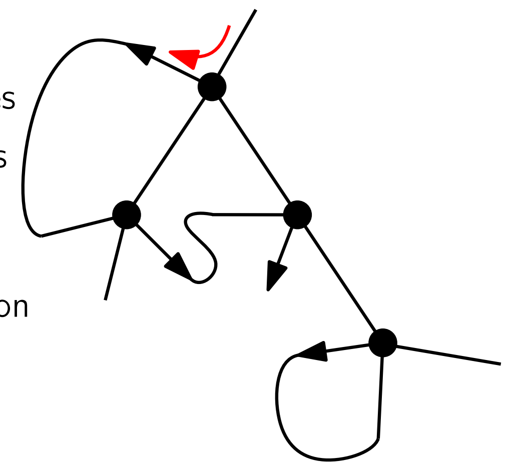

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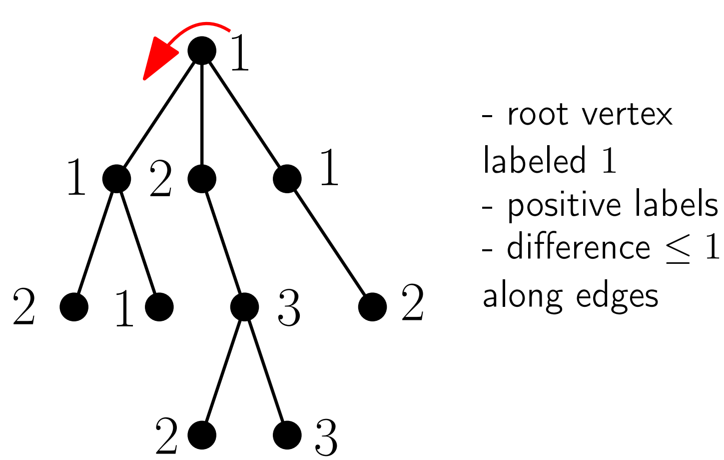

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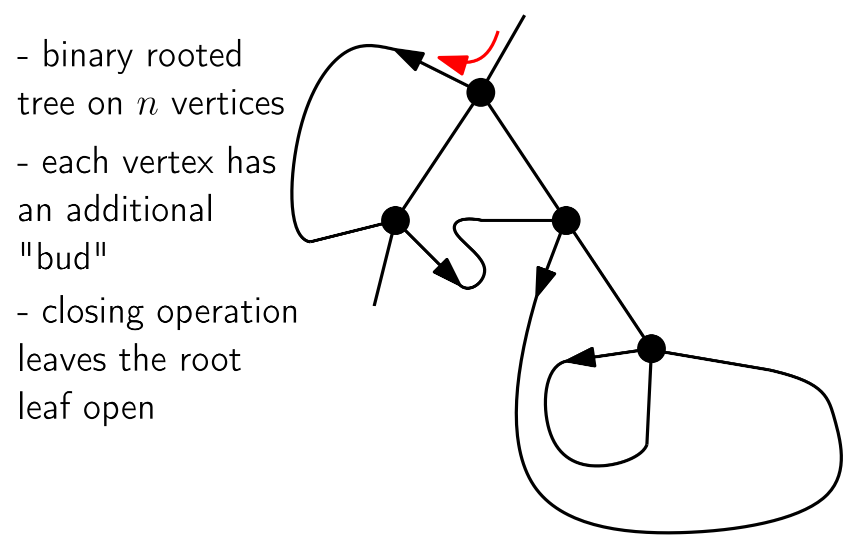

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Initial motivation:

- direct explanation of the simple formula of Tutte,
- better understanding of the structure of planar maps
- good way to generate maps

...or enumeration of bipartite quadrangulations

Map M is bipartite if vertices can be colored by two different colors ($V_\bullet(M)$ - set of black vertices, $V_\circ(M)$ - set of white vertices, the root vertex is black by convention) such that each edge connects two vertices of different colors.

Quadrangulation is a map with all faces of degree 4.

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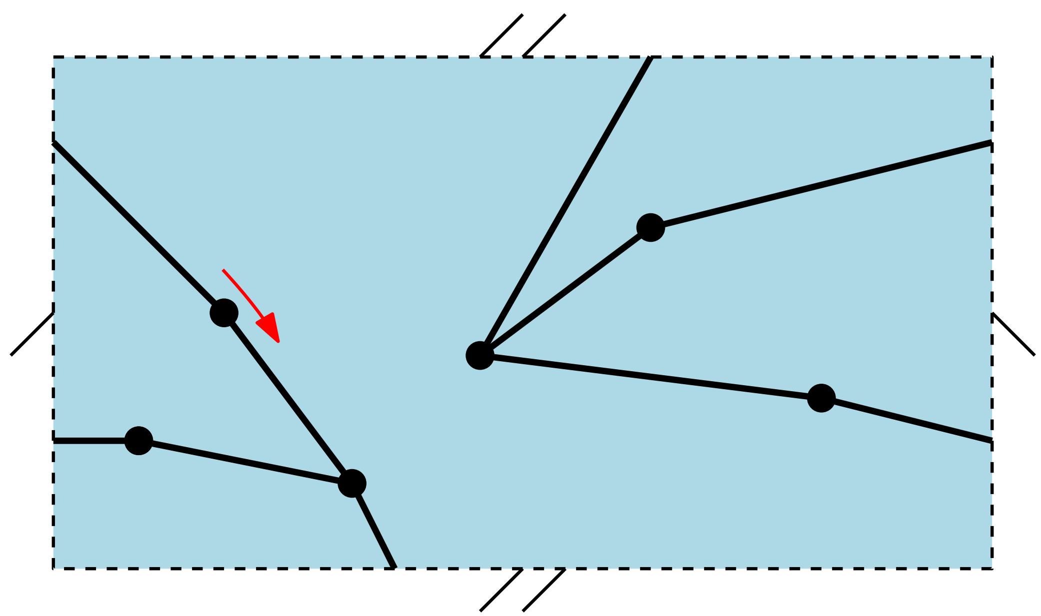

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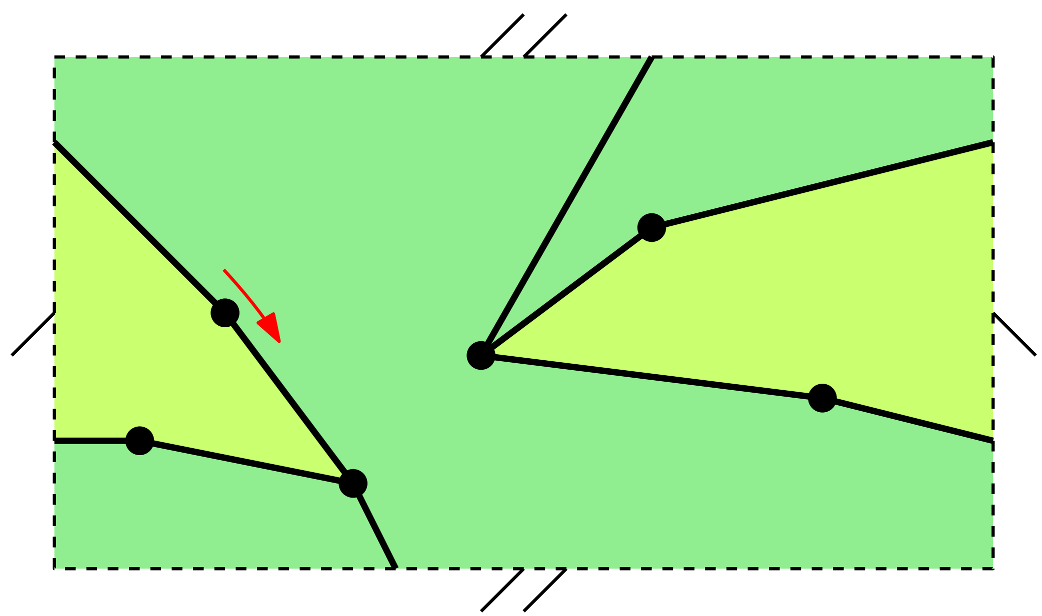

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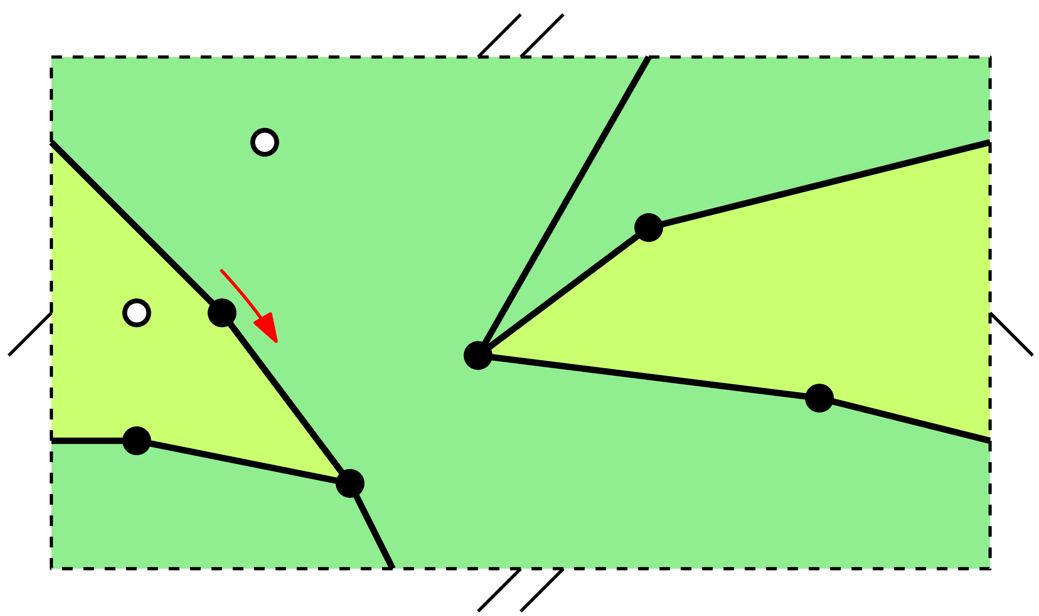

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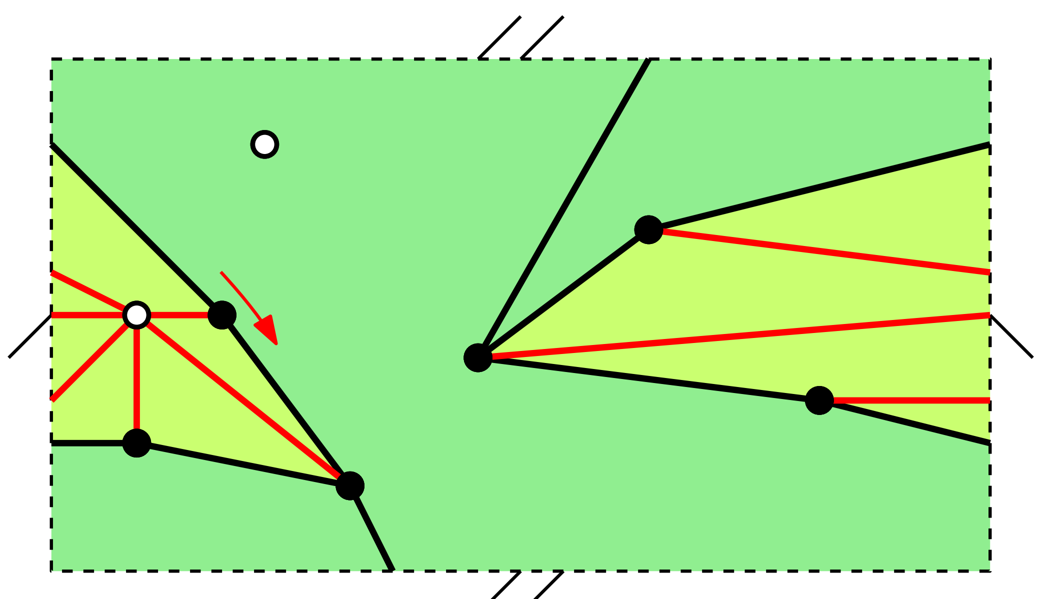

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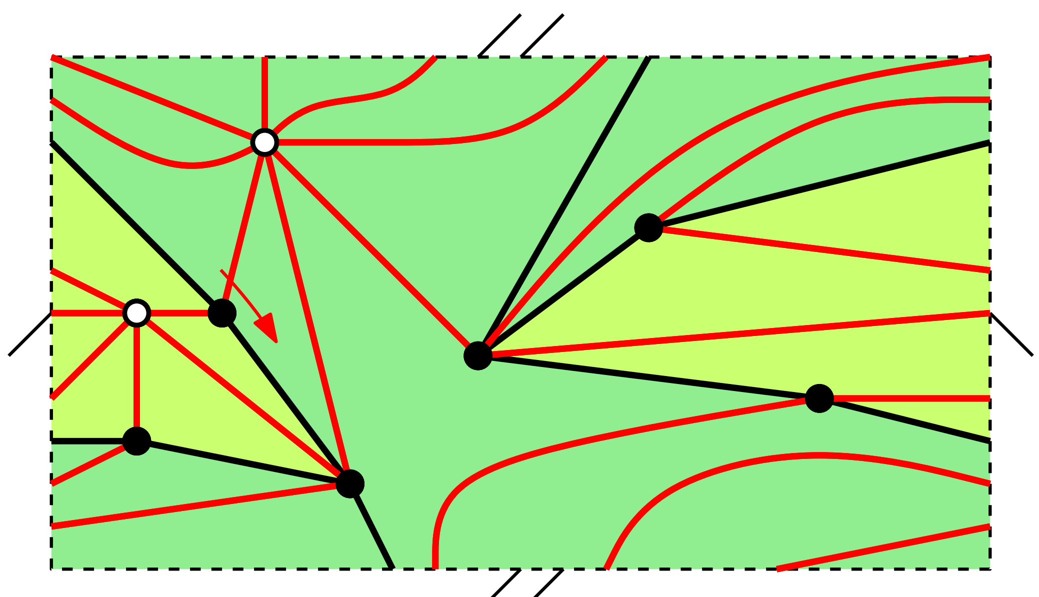

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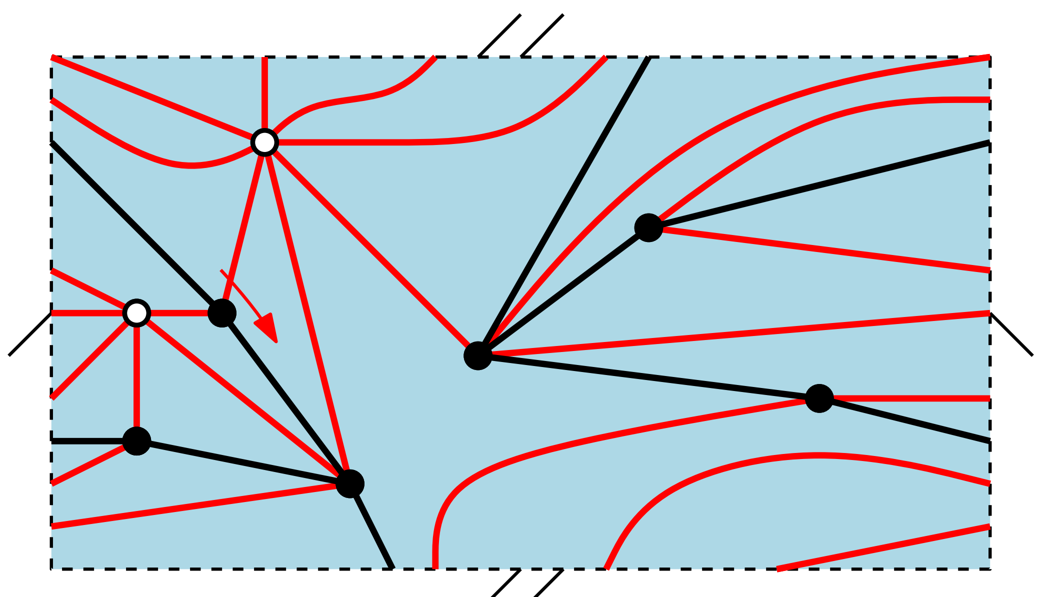

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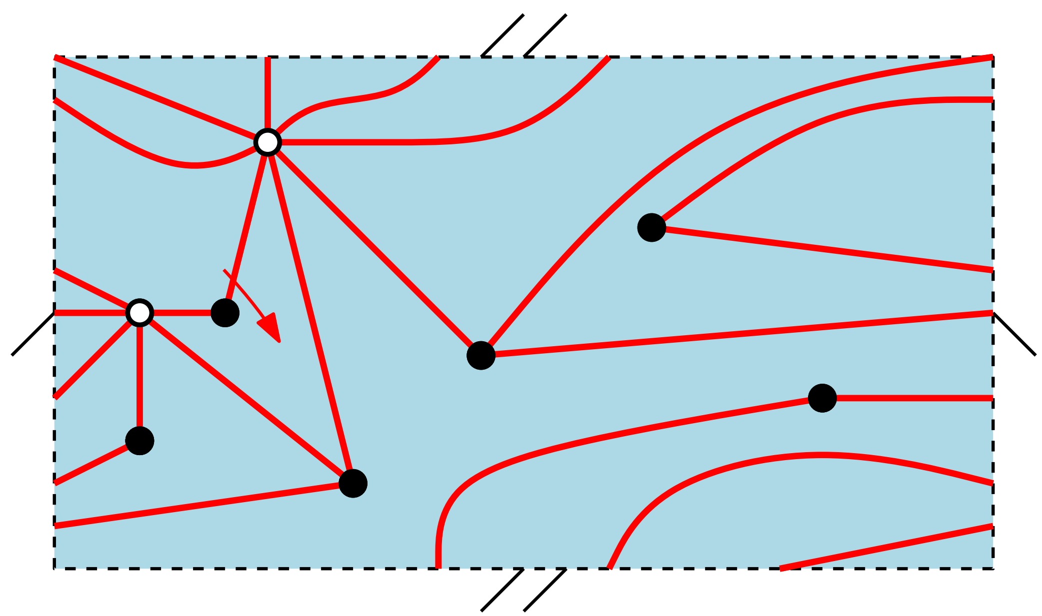

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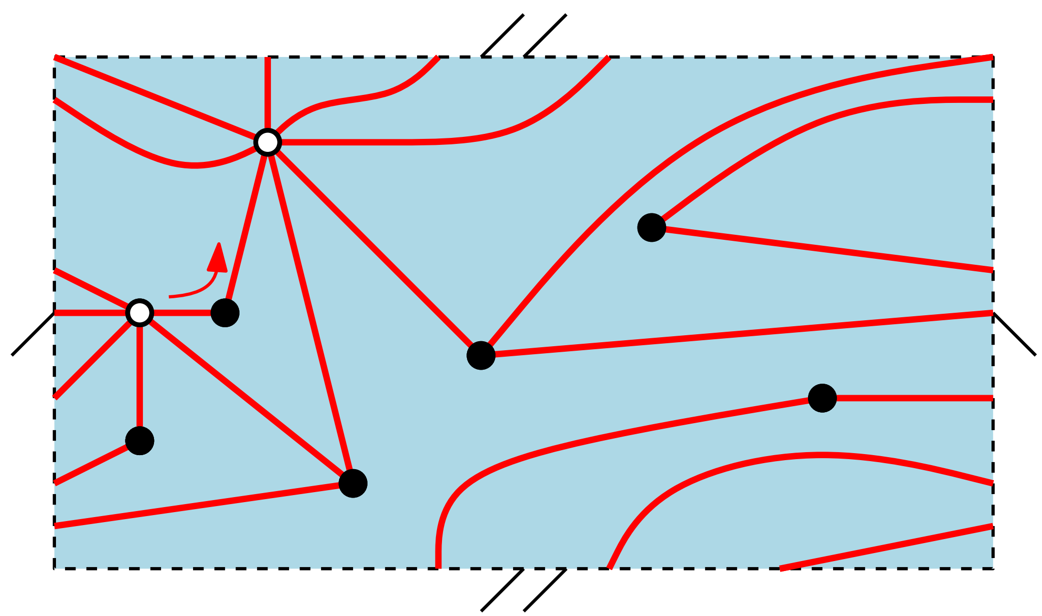

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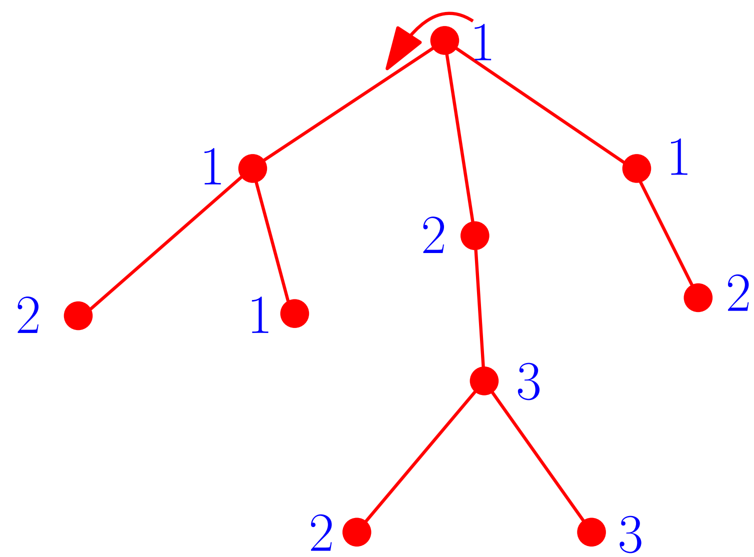

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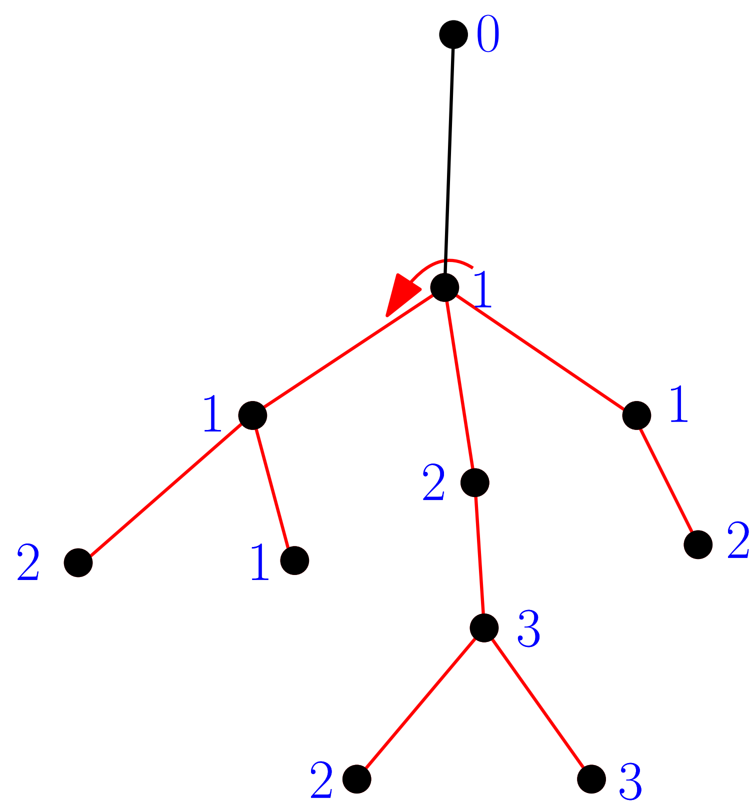

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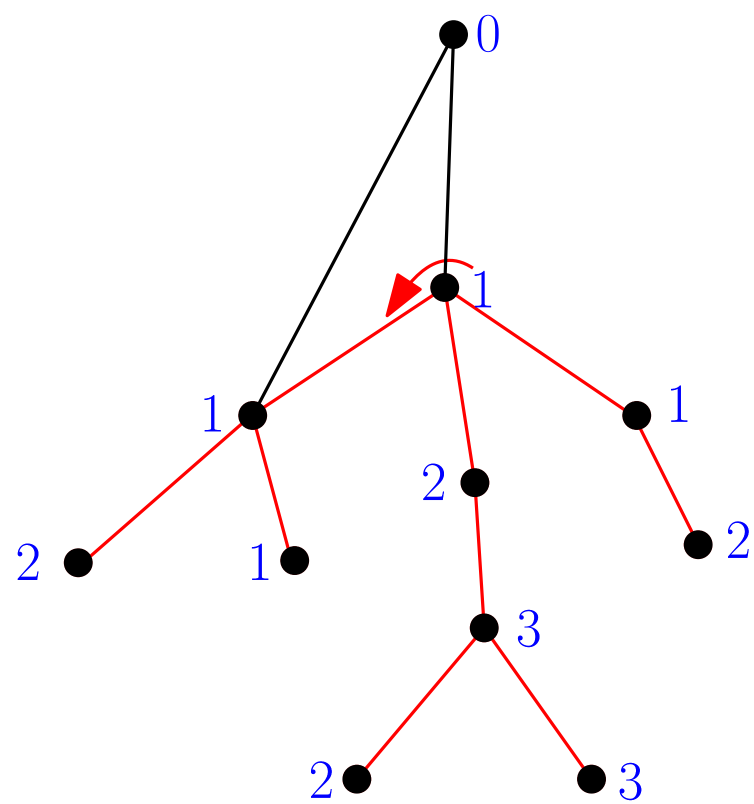

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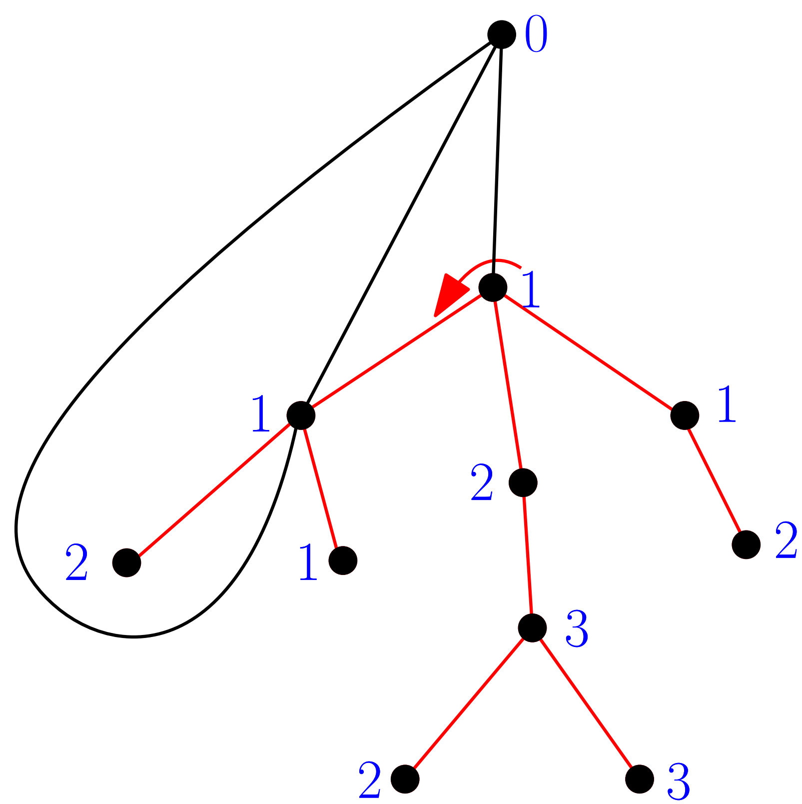

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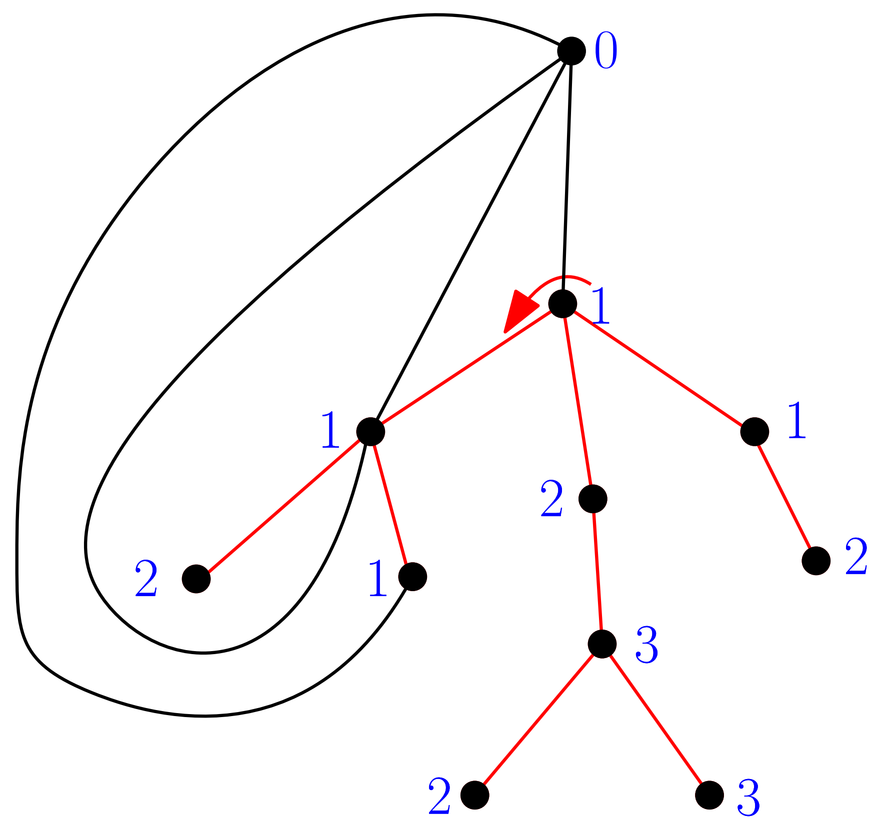

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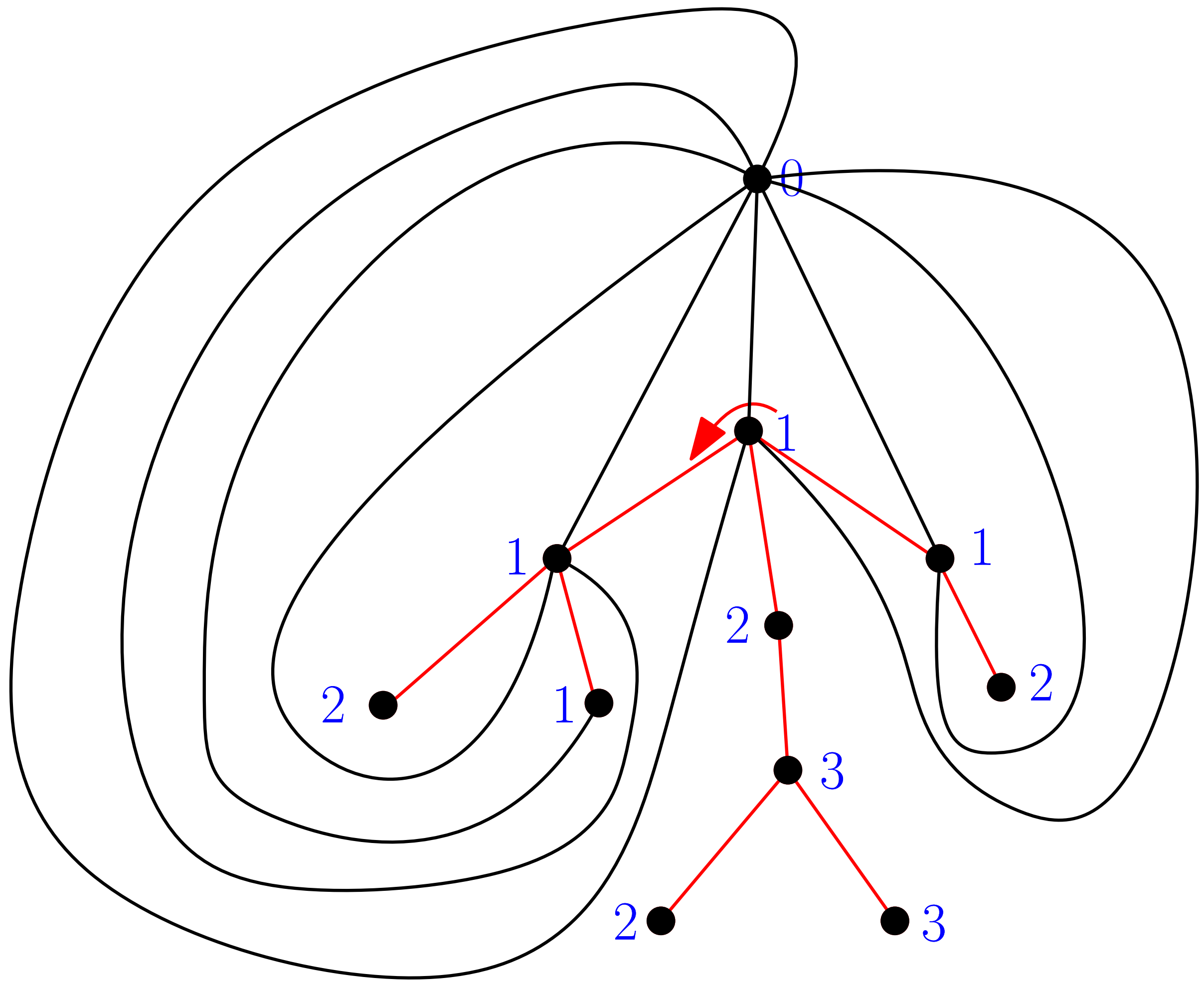

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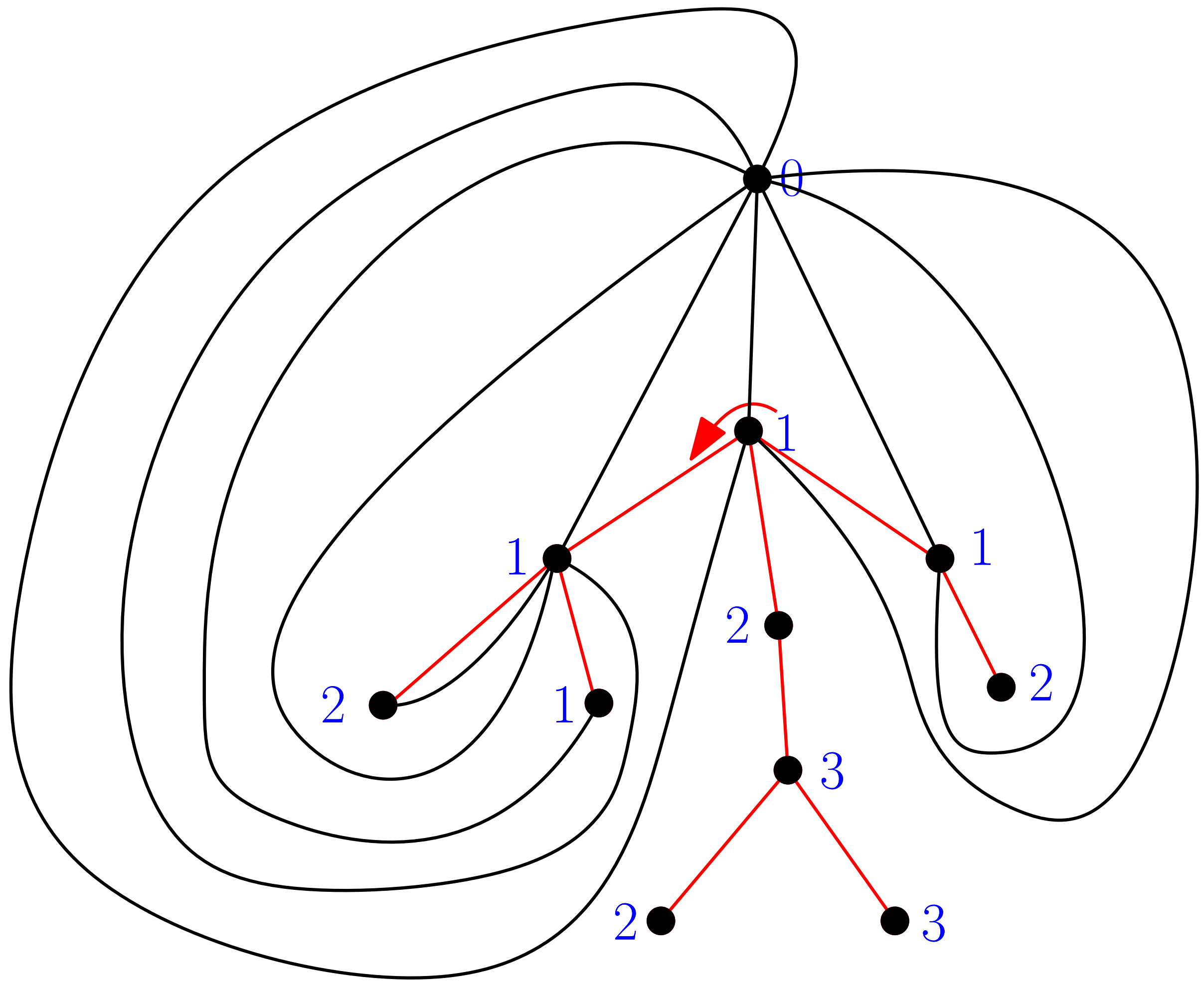

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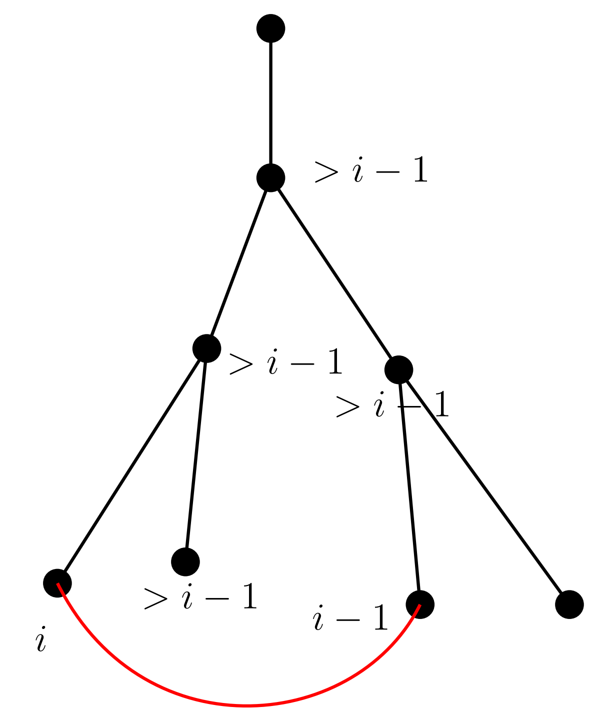

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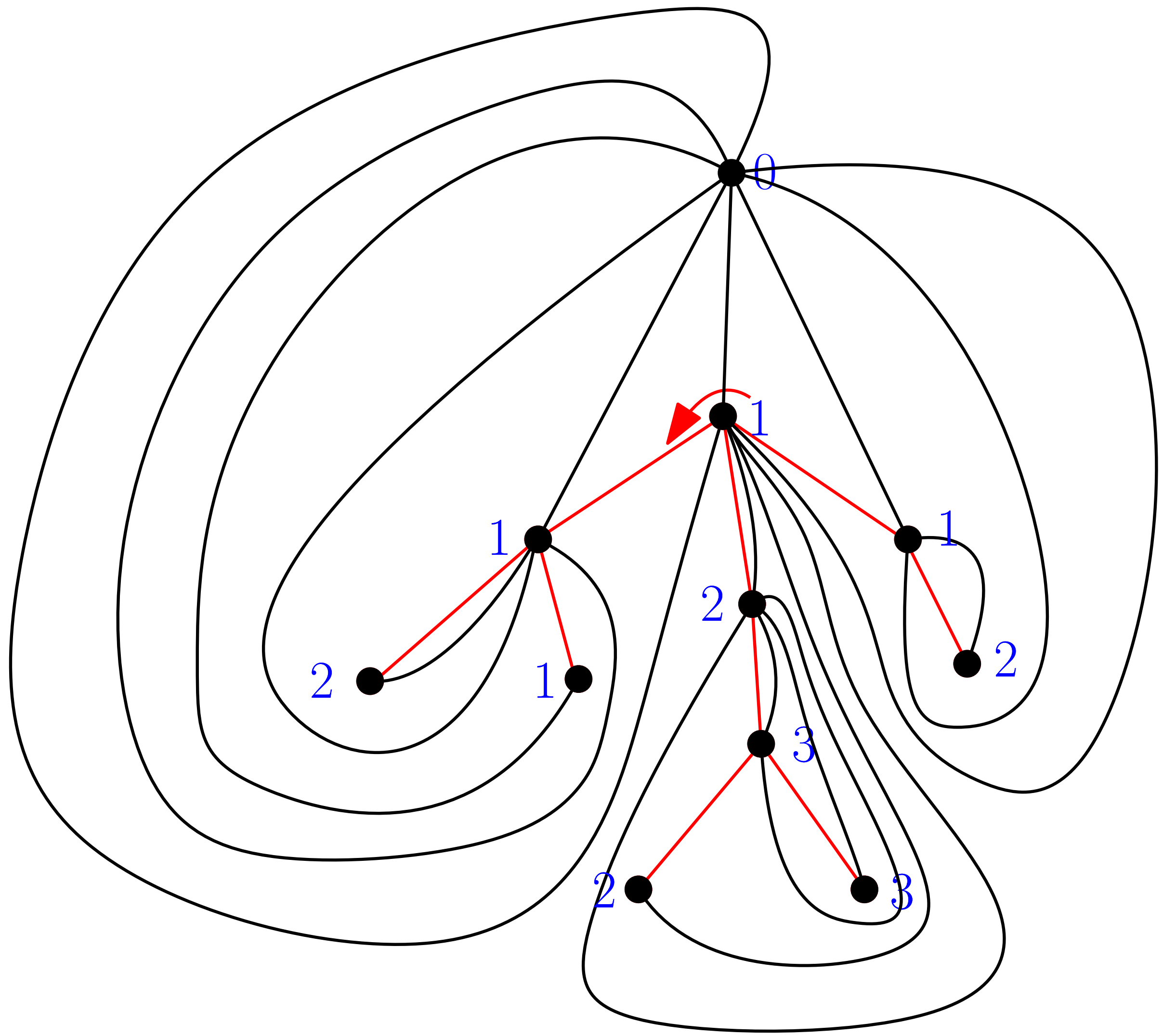

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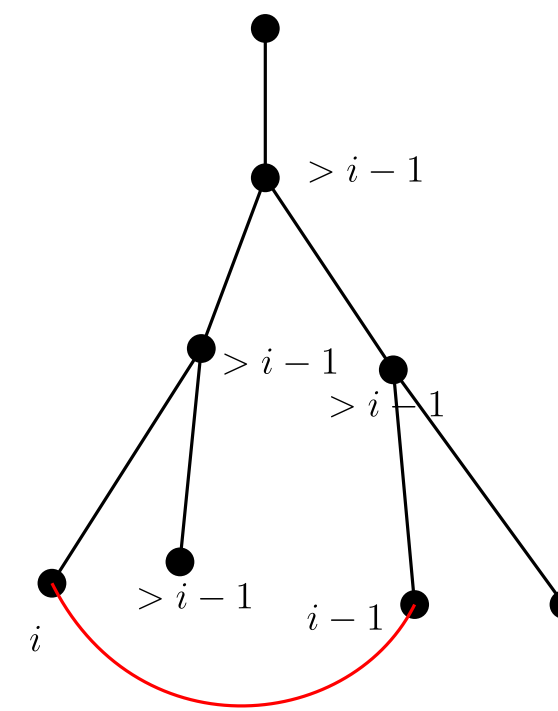

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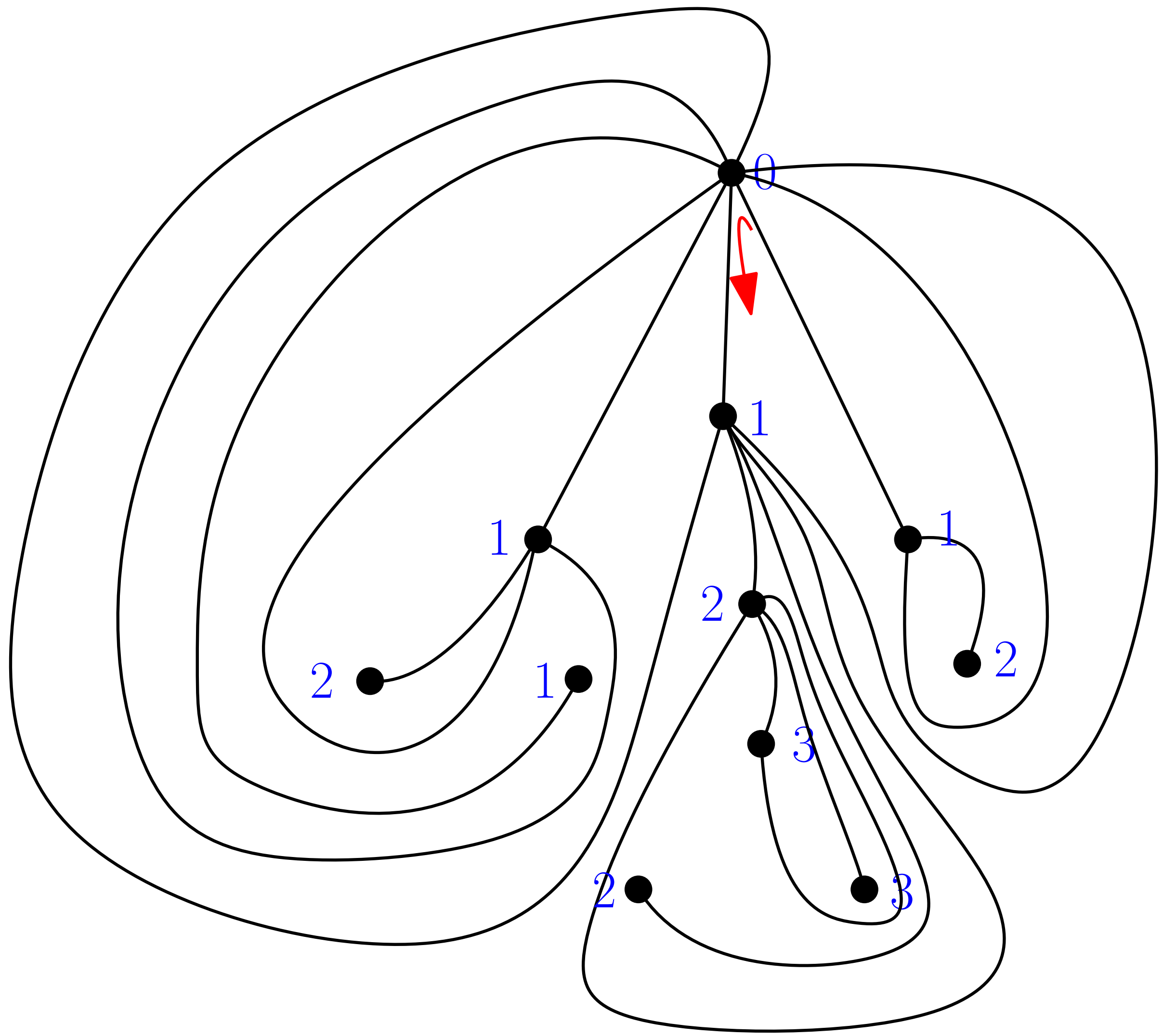

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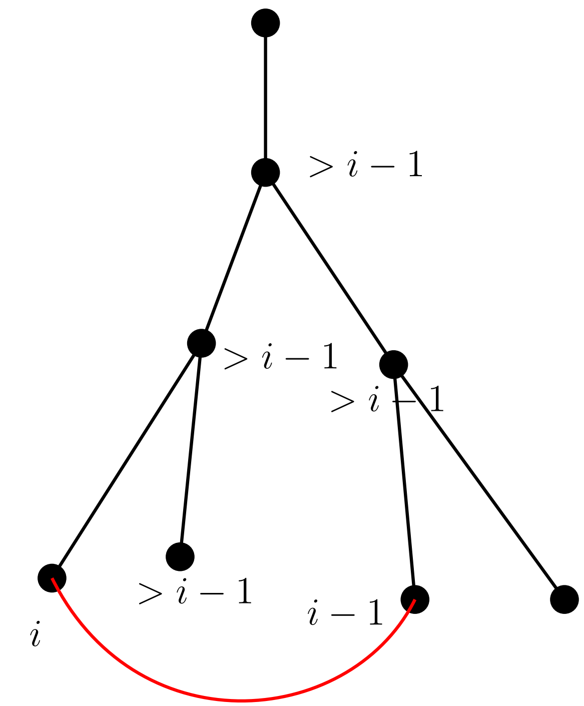
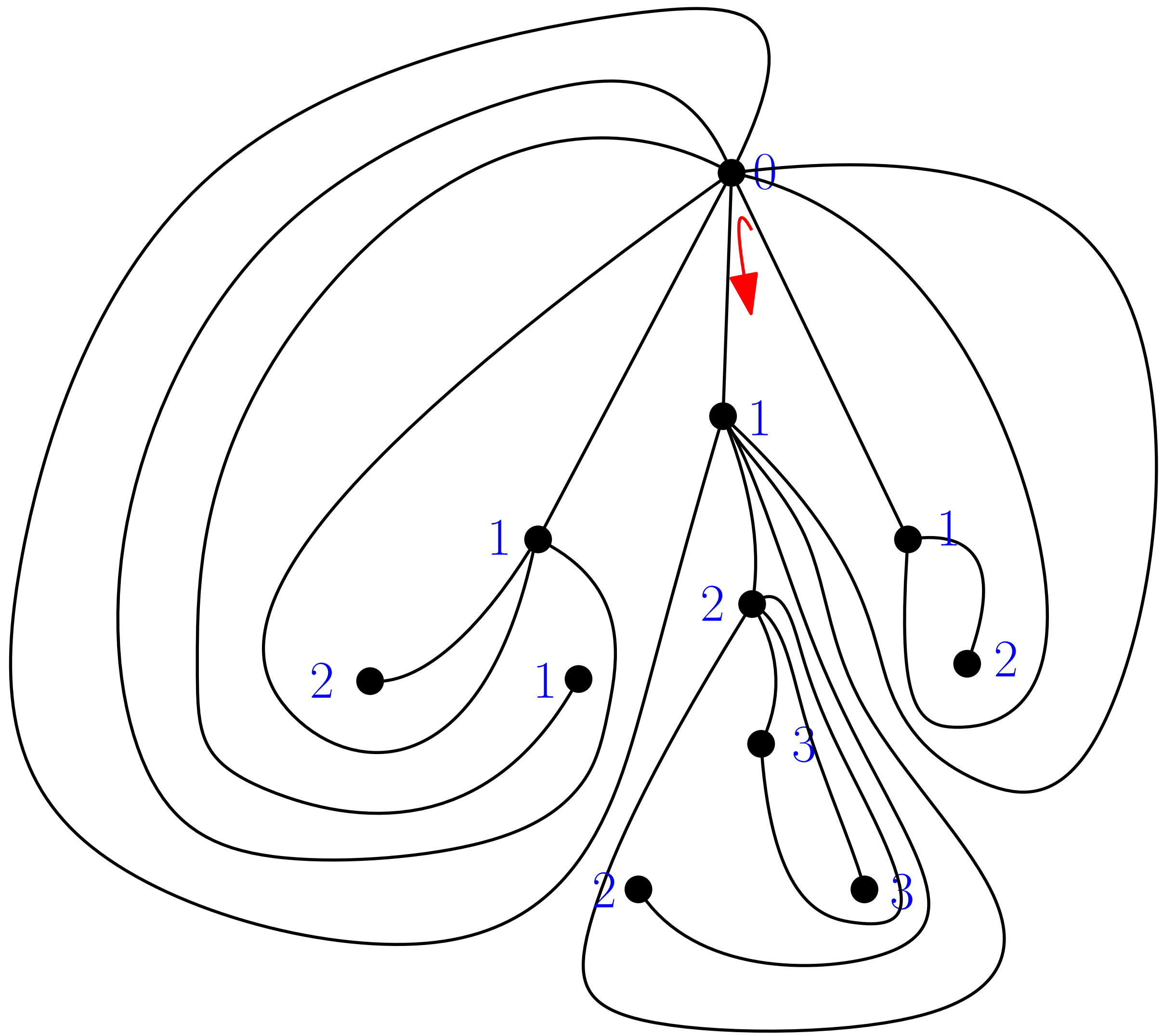


How these bijections work



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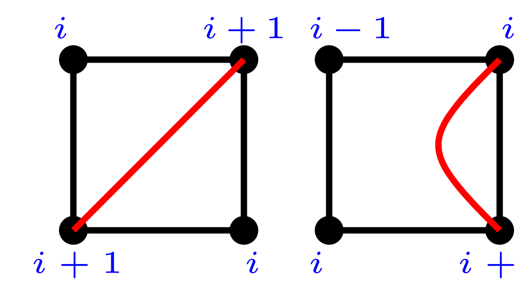

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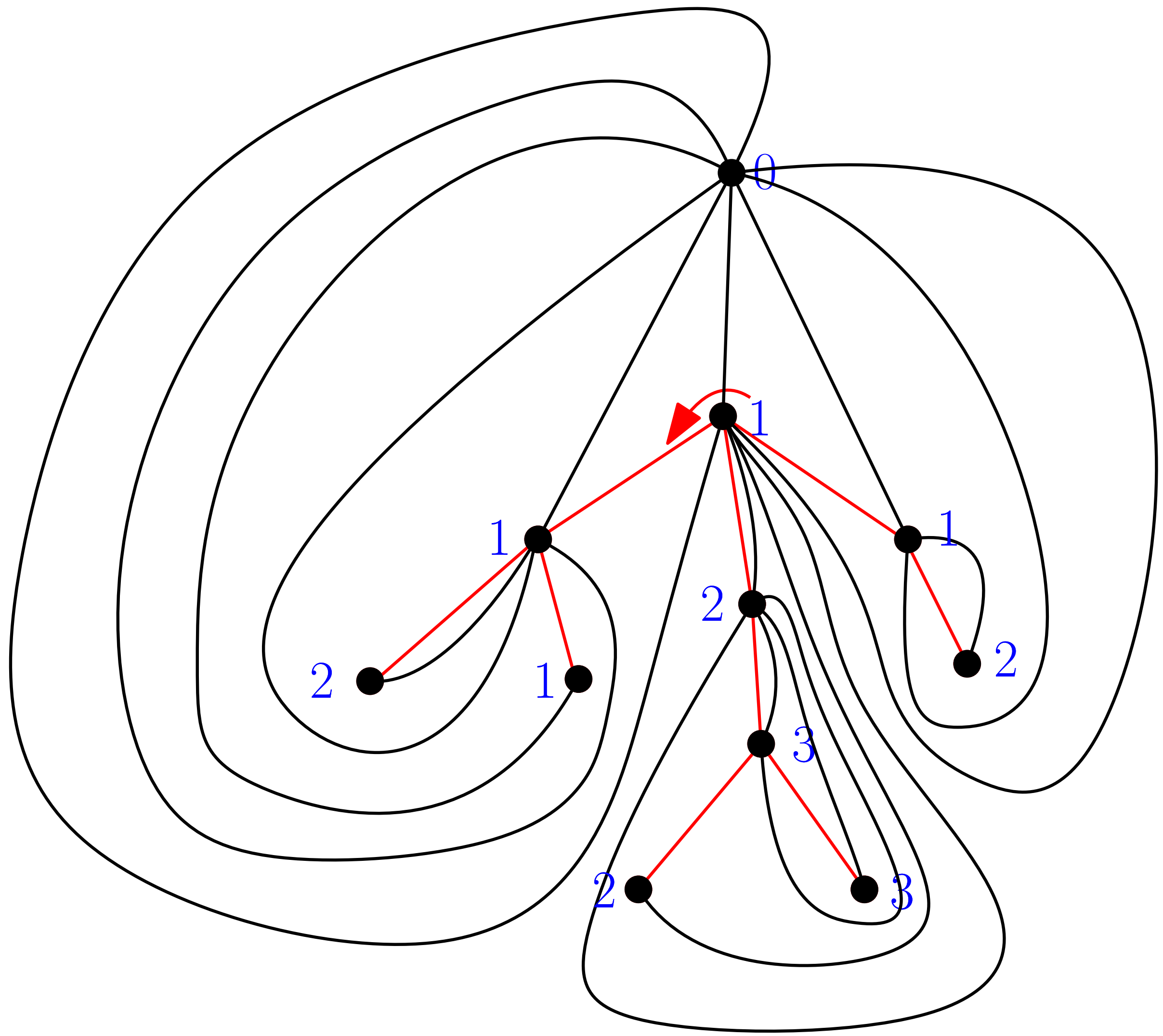

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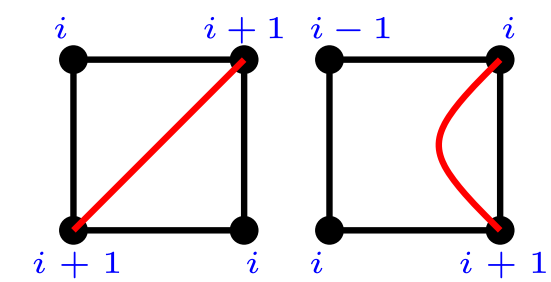

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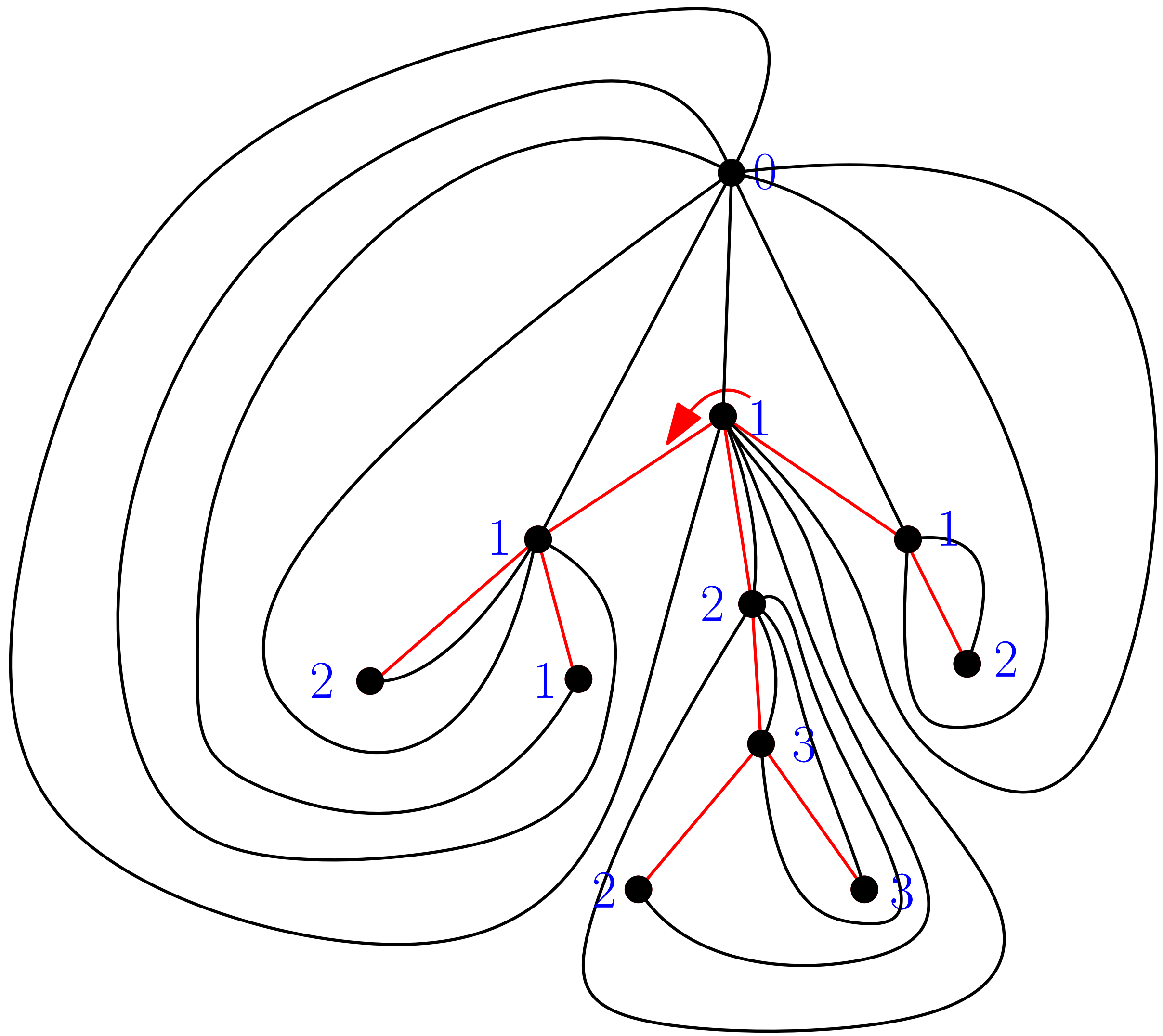

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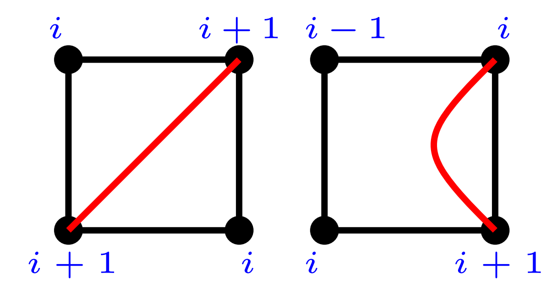

Observation: labels $\equiv$ metric structure of the quadrangulation

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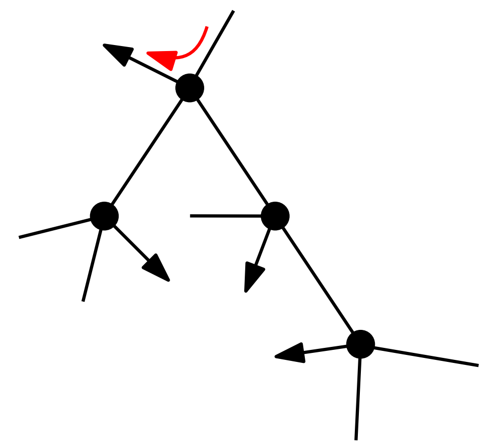

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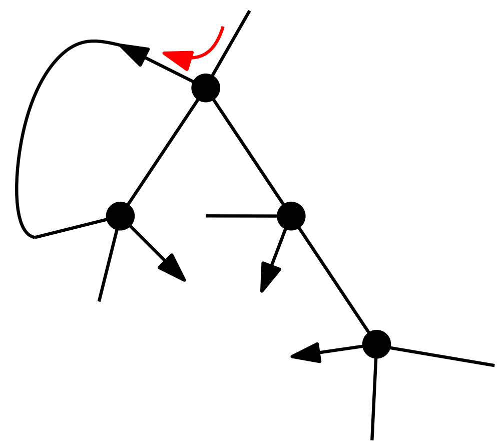

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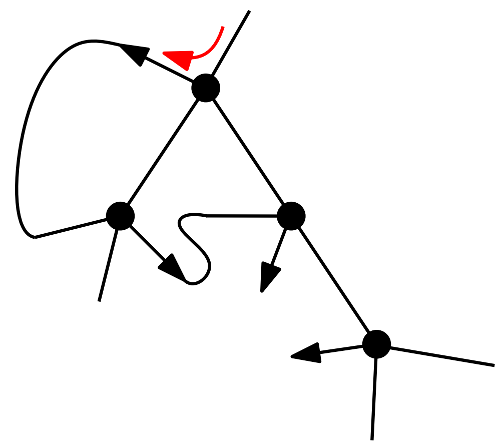

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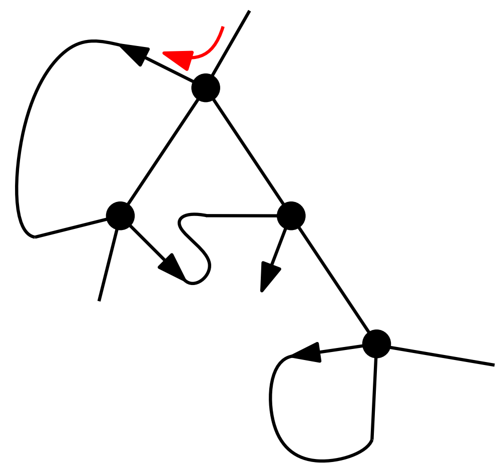

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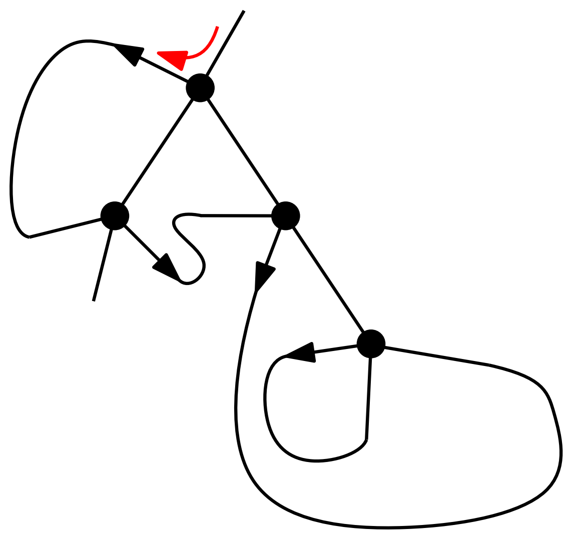

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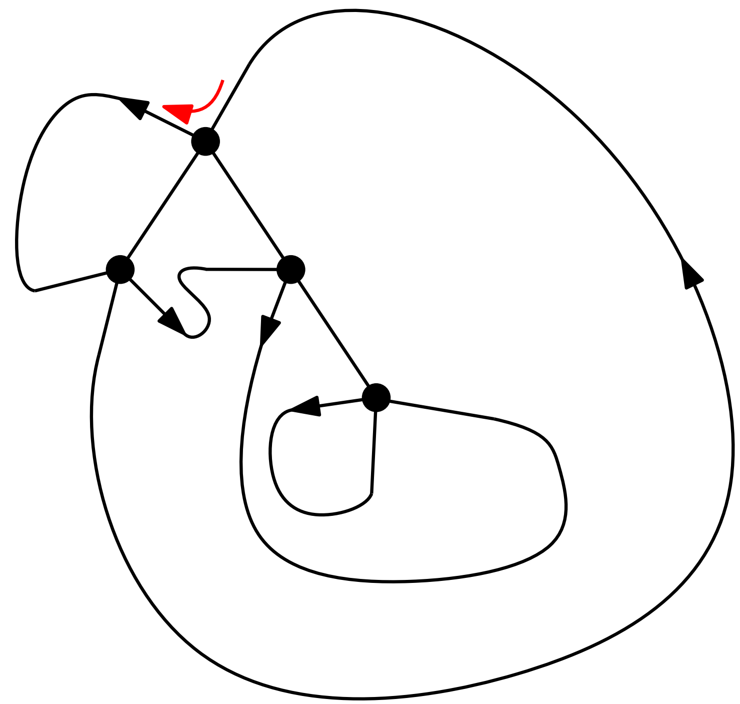

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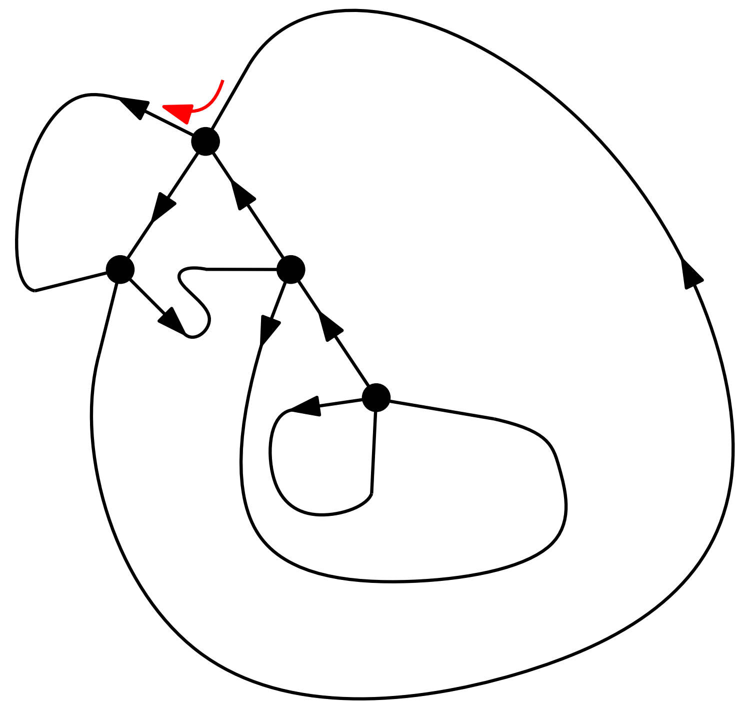
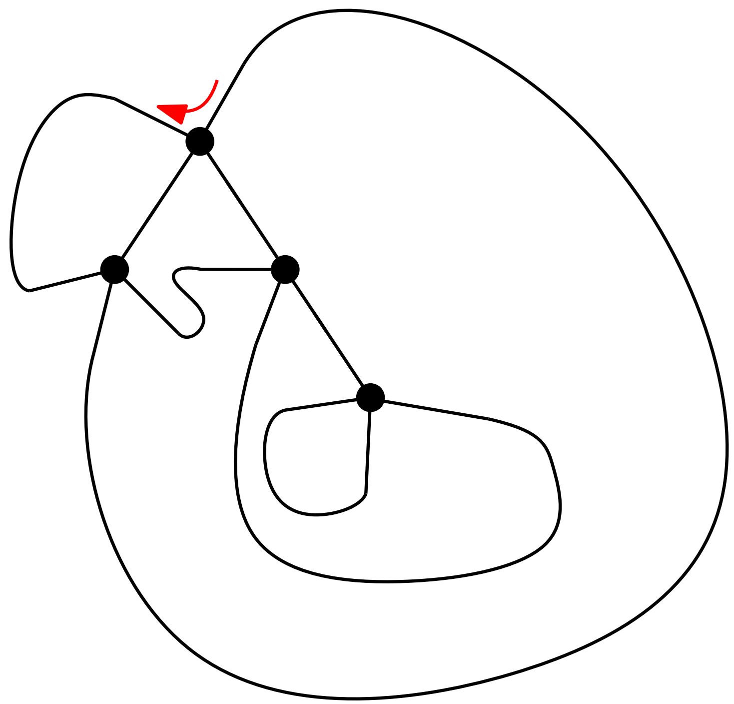


Theorem: [Felsner '04]
There is a unique Eulerian orientation (indegree=outdegree) without clockwise circuit

How these bijections work



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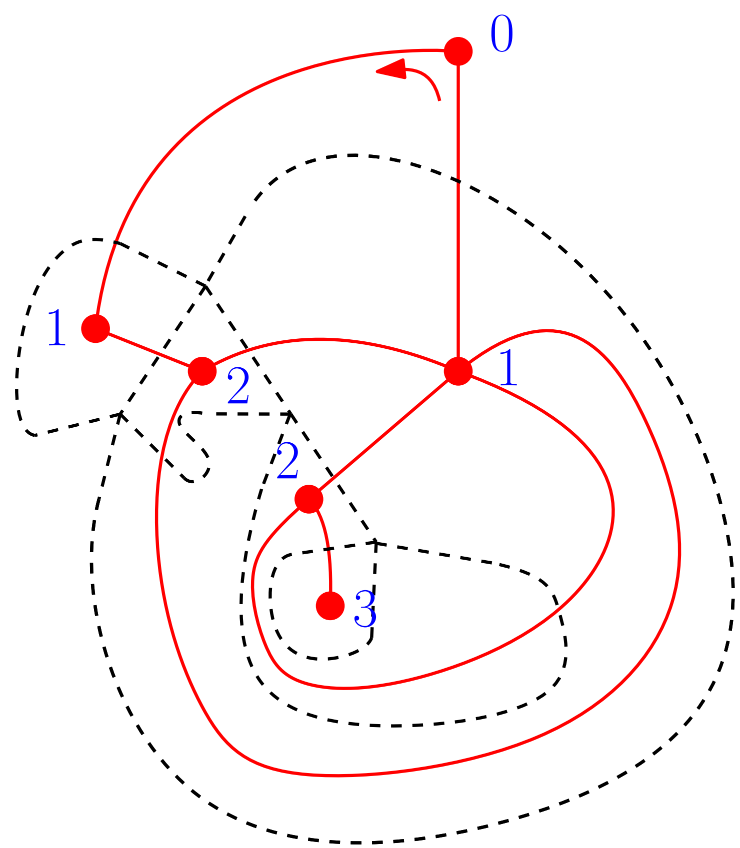

dual map = bipartite quadrangulation

Observation: metric structure in the quadrangulation is again encoded by the blossoming tree!

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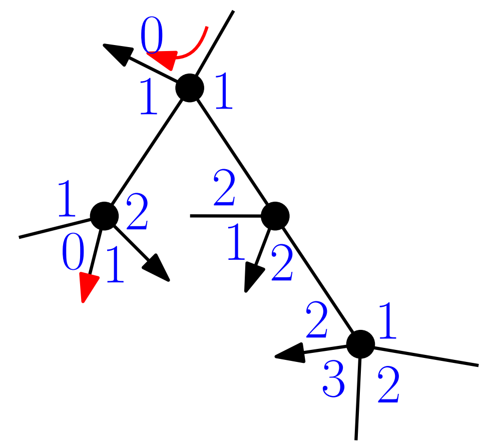

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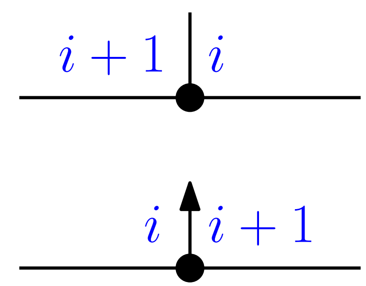

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New motivation

Find a bijection between maps and some objects with a WELL-UNDERSTOOD (tree-like) structure!

Understanding a geometry of a random surface:

- growing maps as a discrete model of a continuous manifold,
- metric geometry of a random surface = metric in a random map, when its size tends to infinity,
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- quadrangulations [Le Gall '11 + Miermont '11]
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• simple triangulations and simple quadrangulations [Addario-Berry–Albenque '13]
• simple maps [Albenque–Bernardi–Collet–Fusy '14]

II. Bijections for bipartite quadrangulations and labeled tree-like structures

Labeled and well-labeled maps

A map is called labeled if its vertices are labeled by integers such that:

- the root vertex has label 1;
- if two vertices are linked by an edge, their labels differ by at most 1.

If in addition we have:

- all the vertex labels are positive,

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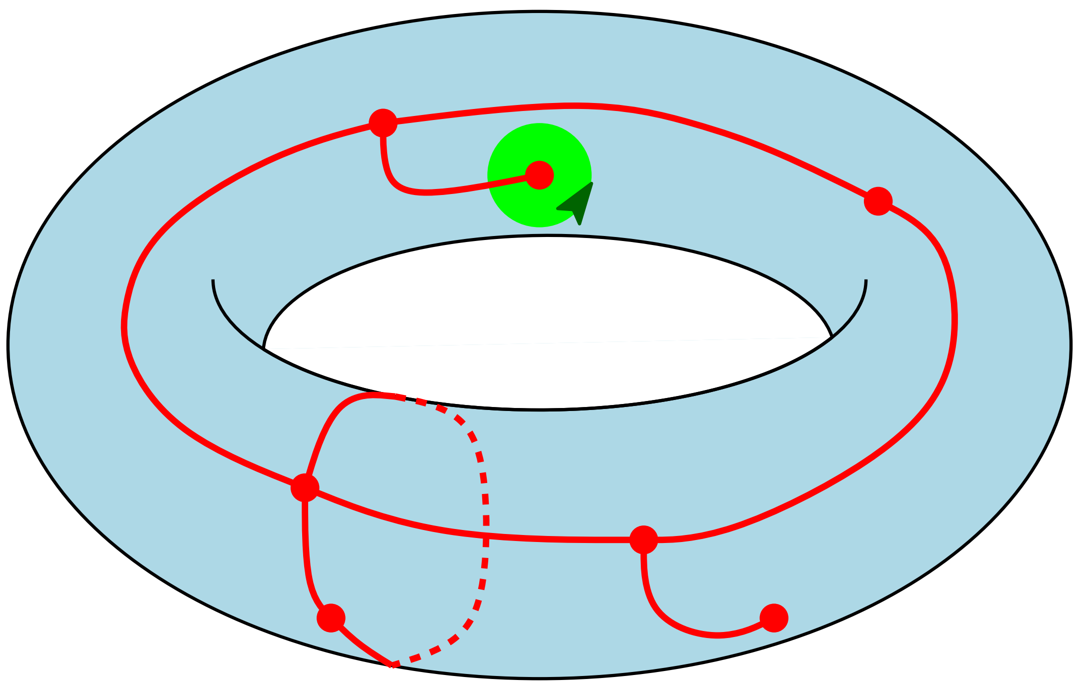

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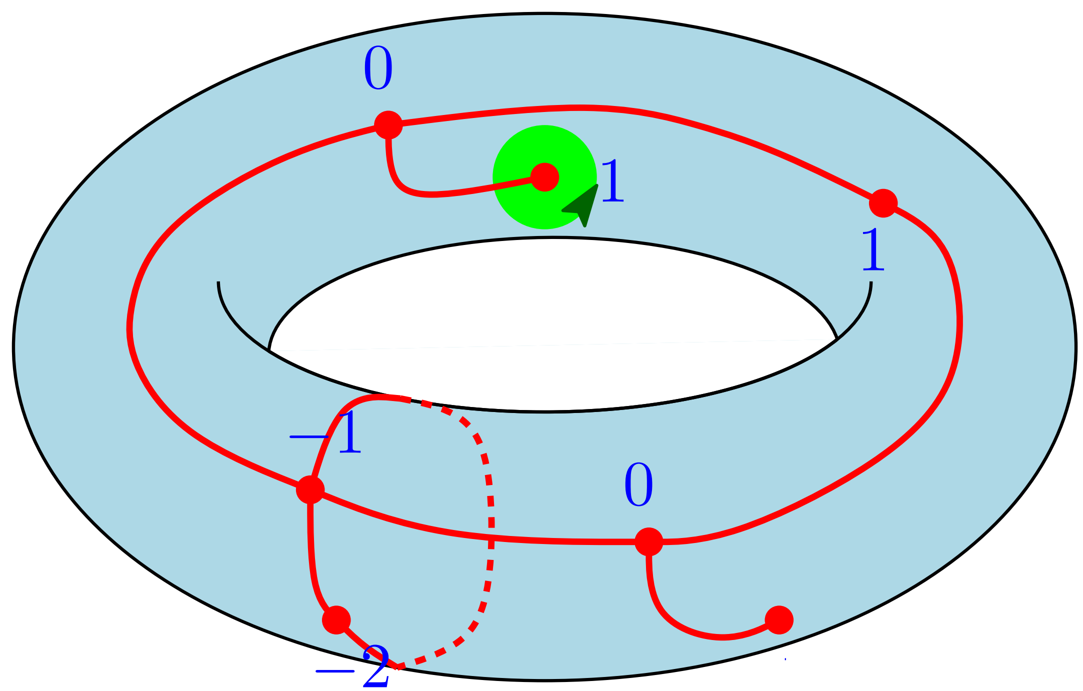

labeled map on the torus

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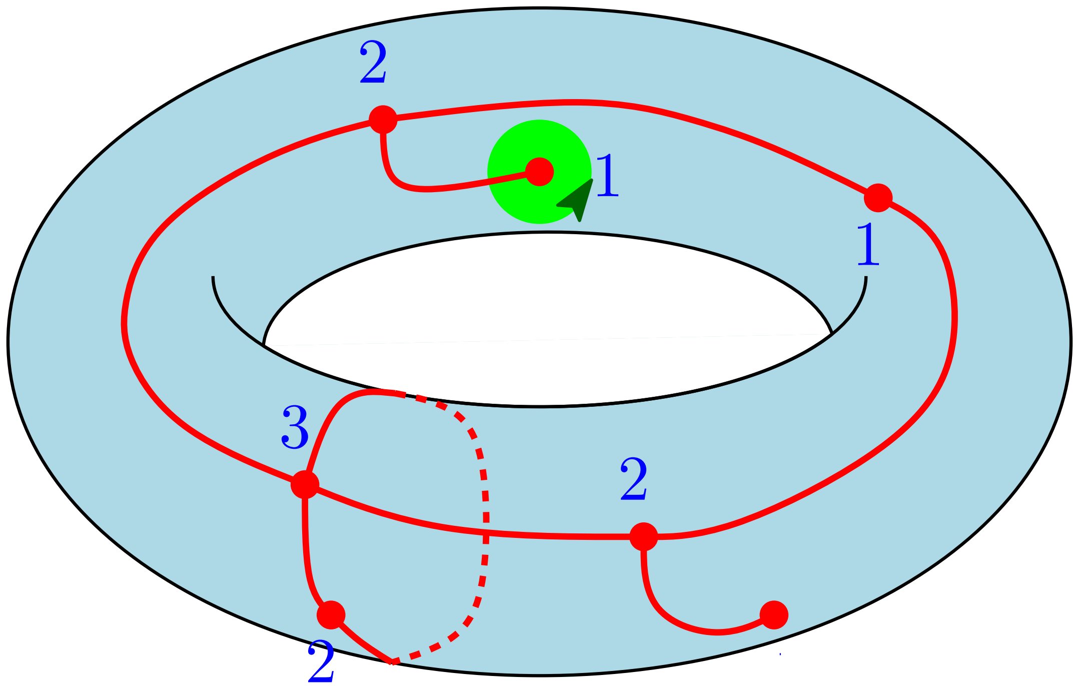

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Orientable case

Theorem [Marcus–Schaeffer '98]
There exists a bijection between:

- rooted, bipartite quadrangulations on ORIENTABLE surface $\mathcal{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)$;
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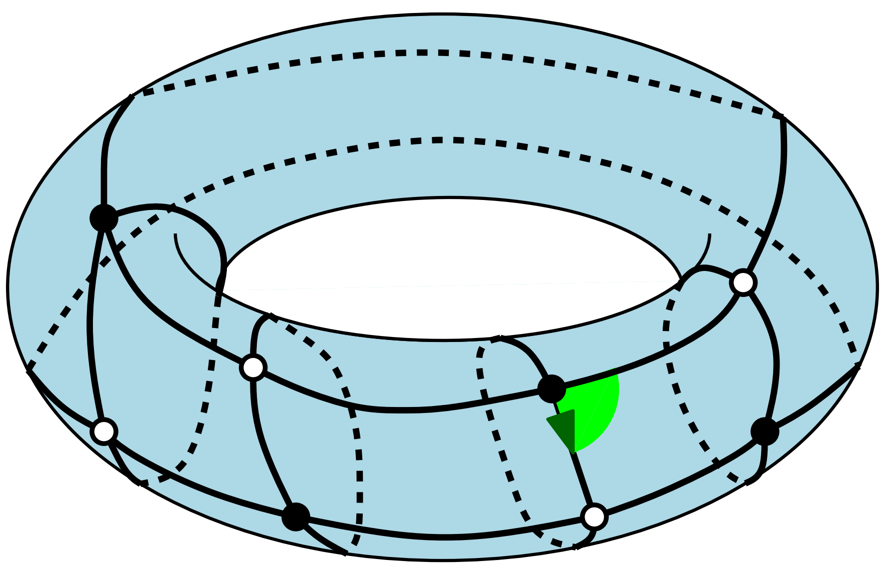

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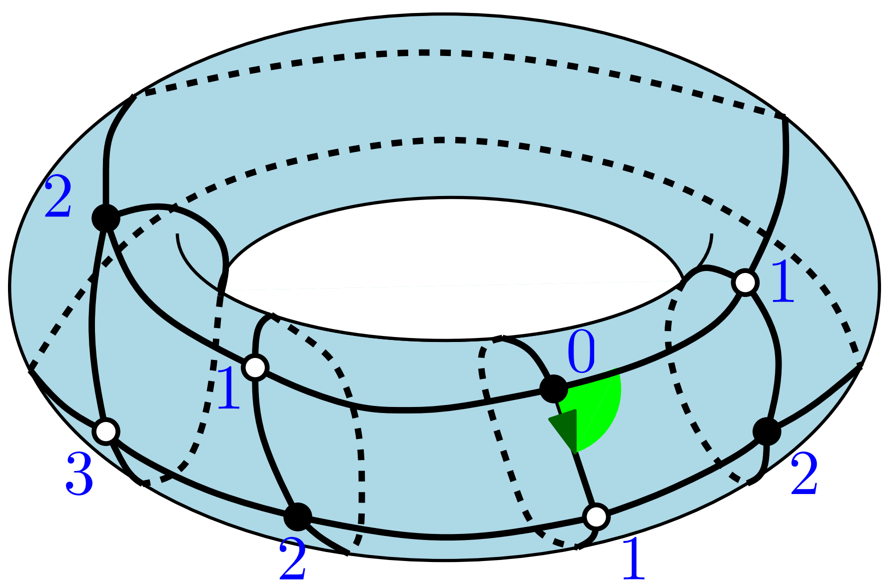

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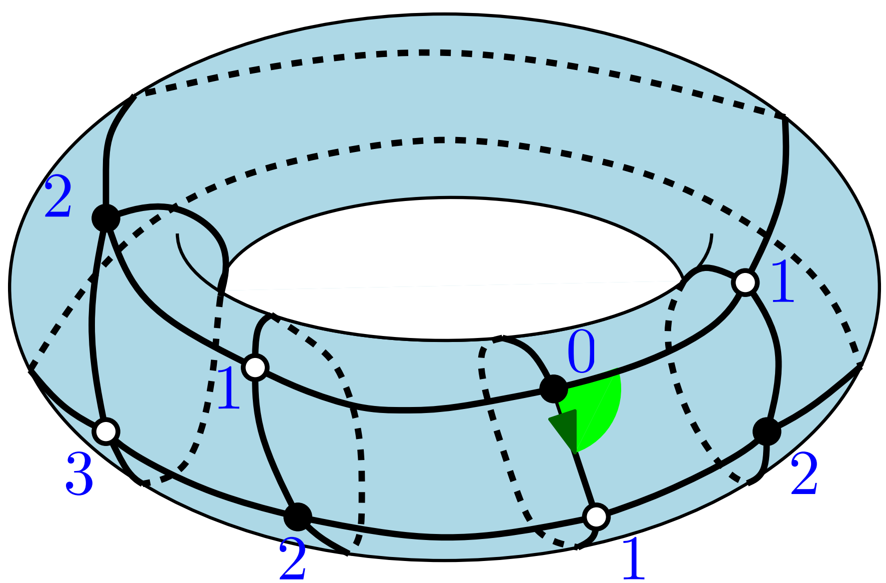

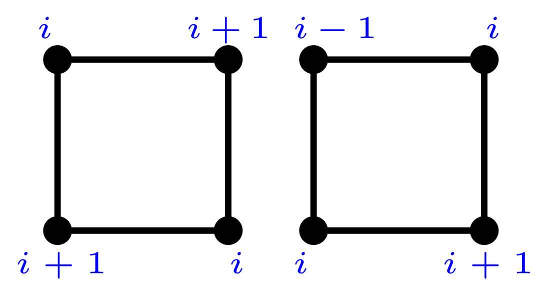

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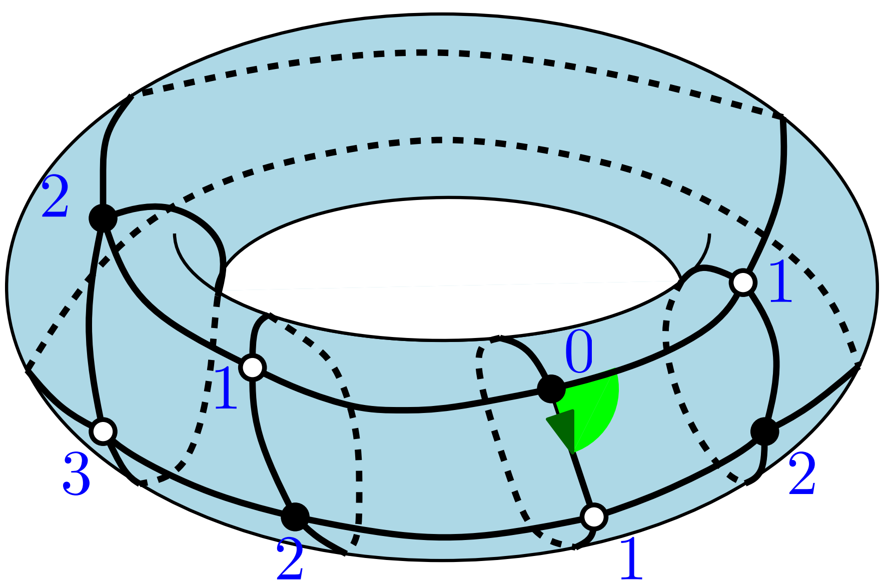

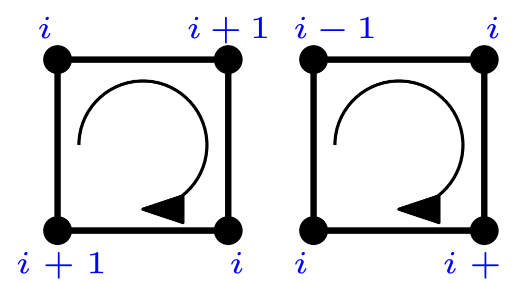

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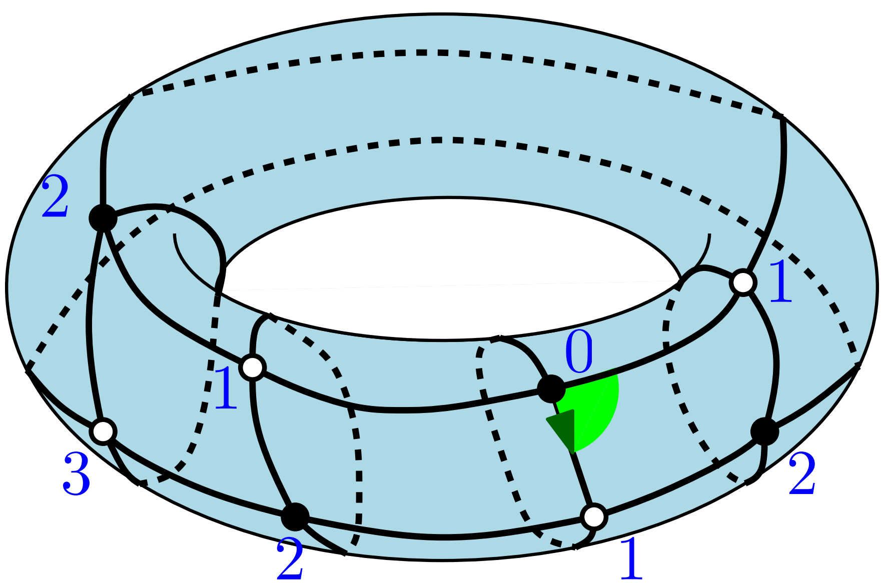

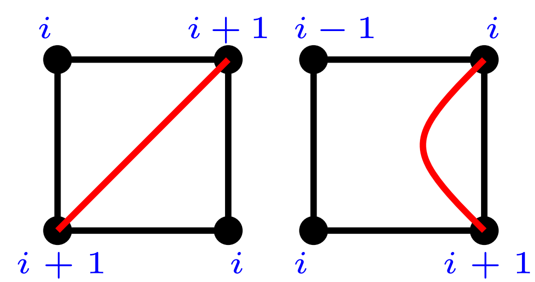

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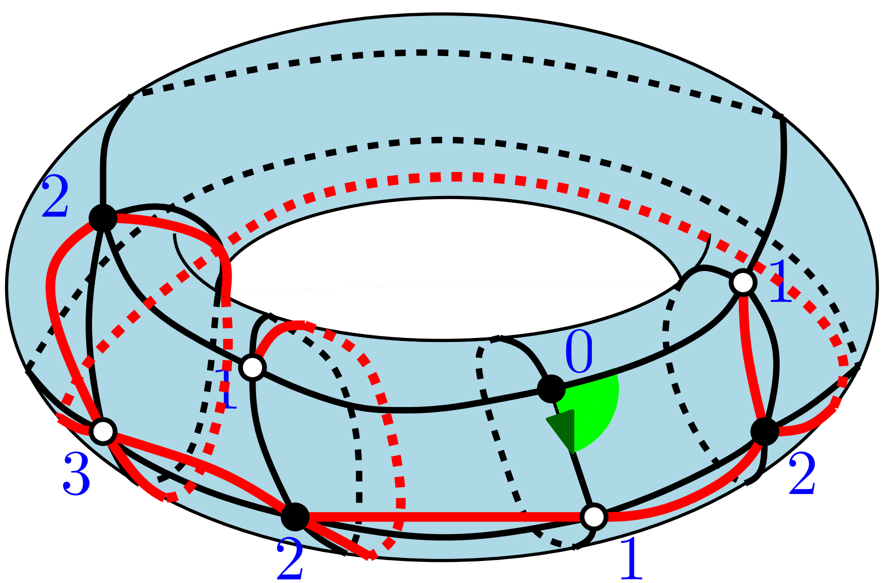

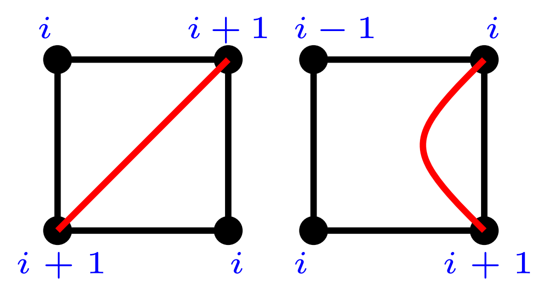

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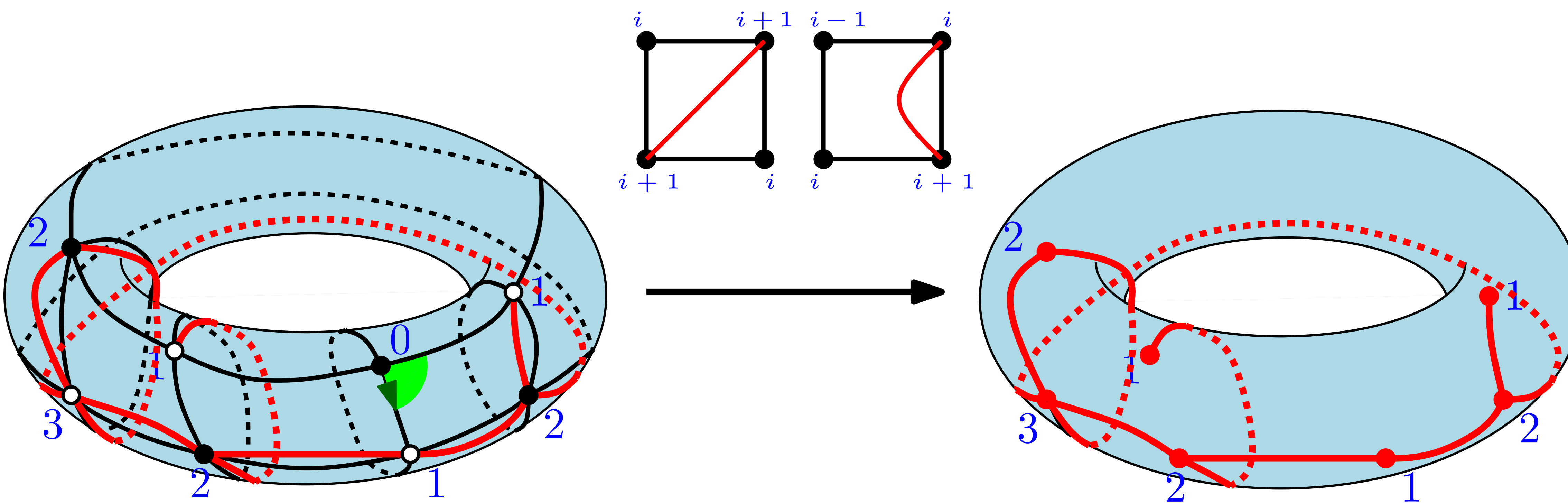

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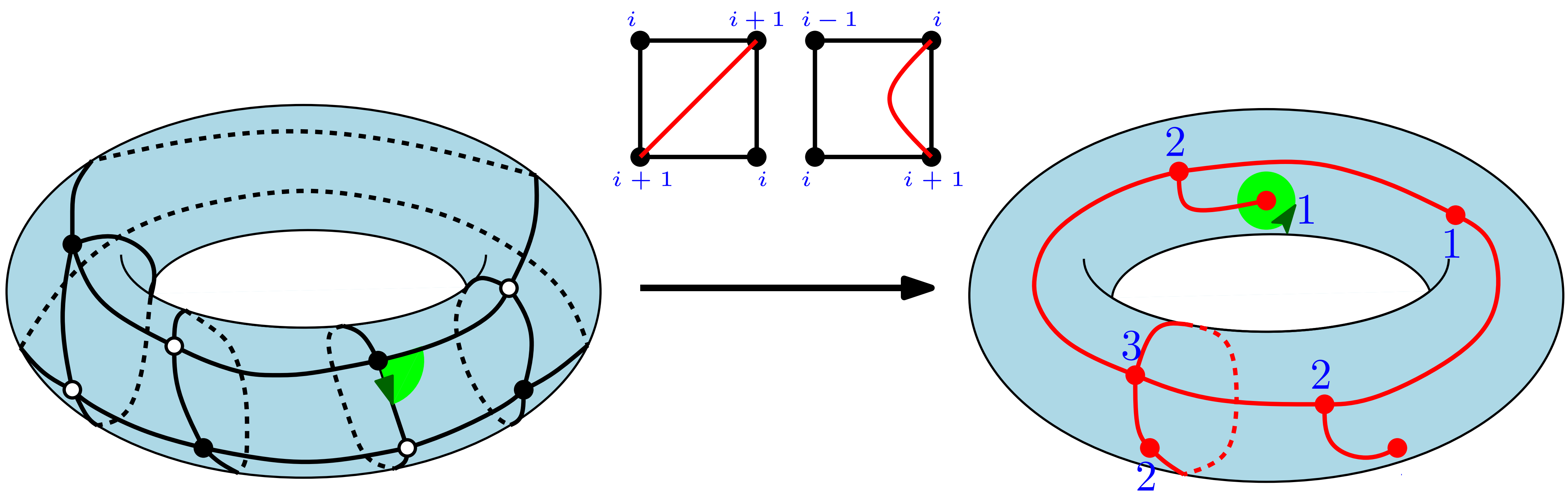

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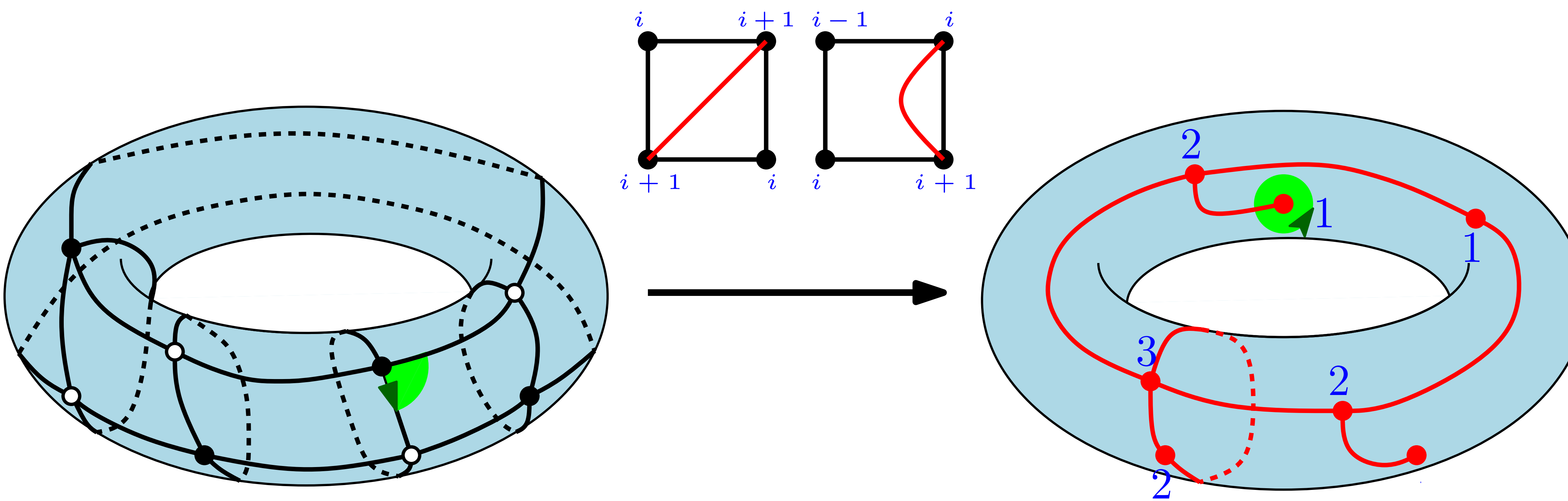

Are **non-orientable** maps different?

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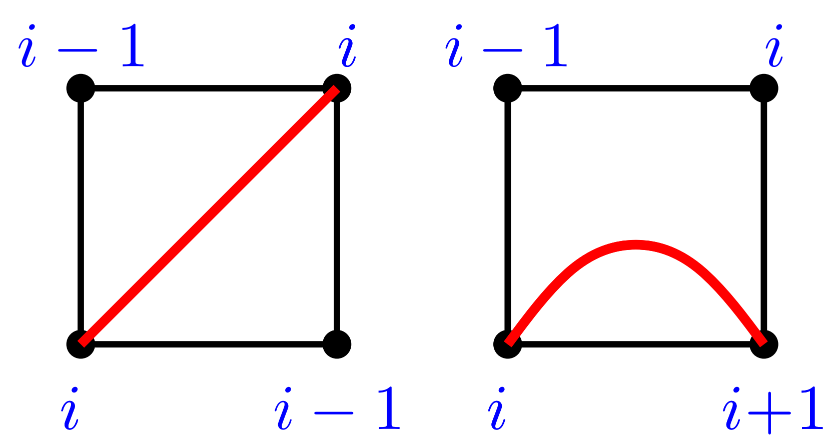

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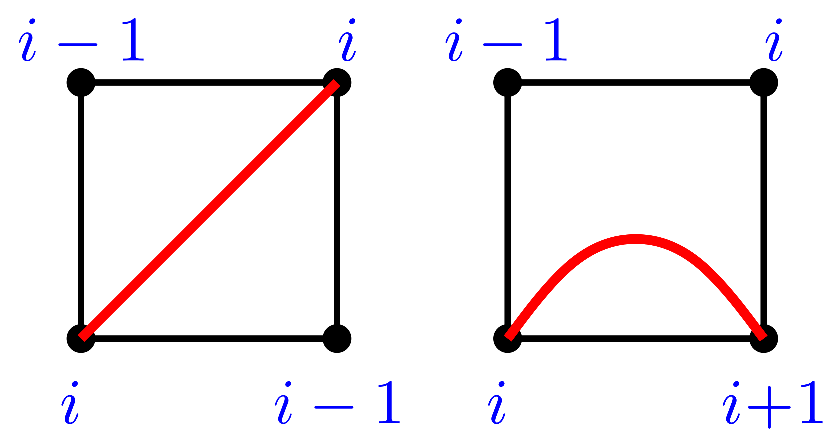

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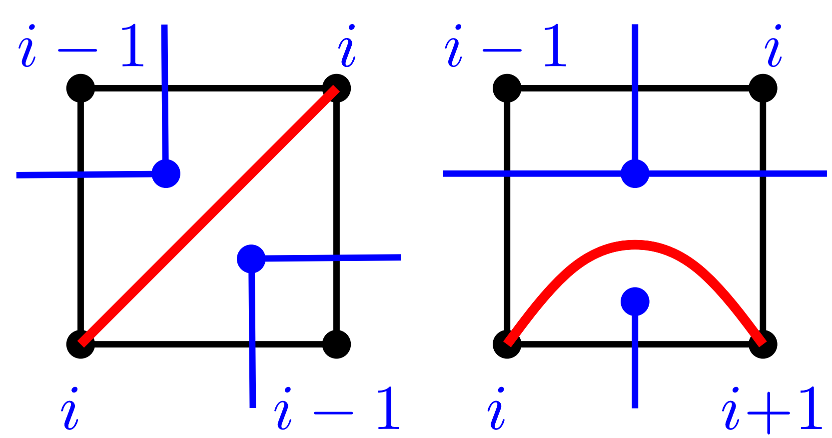

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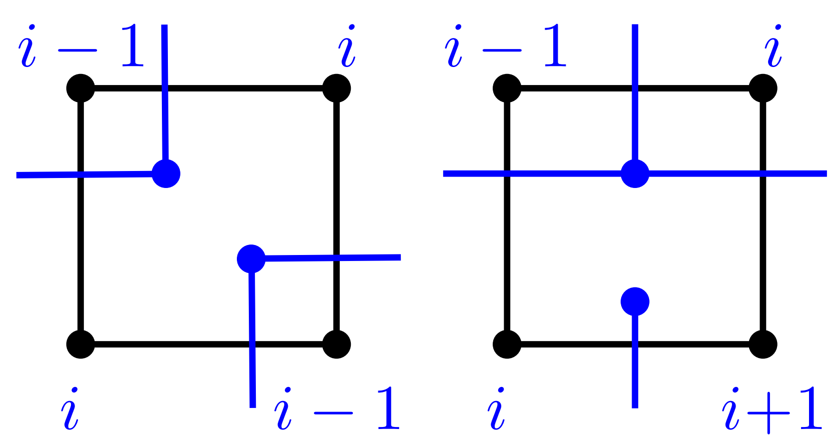

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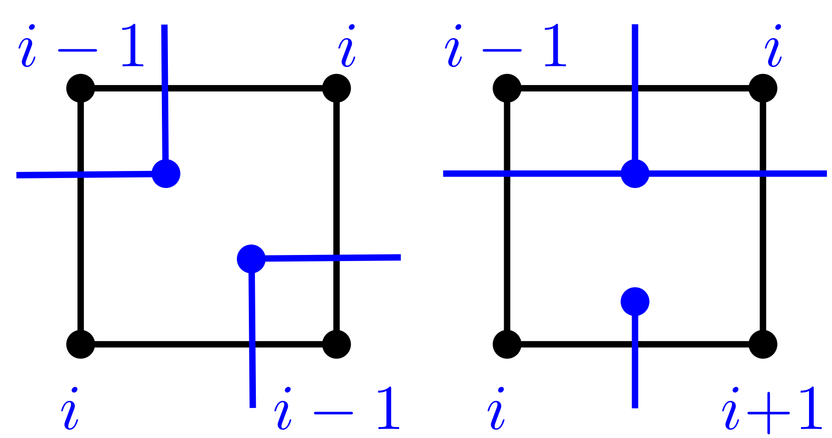

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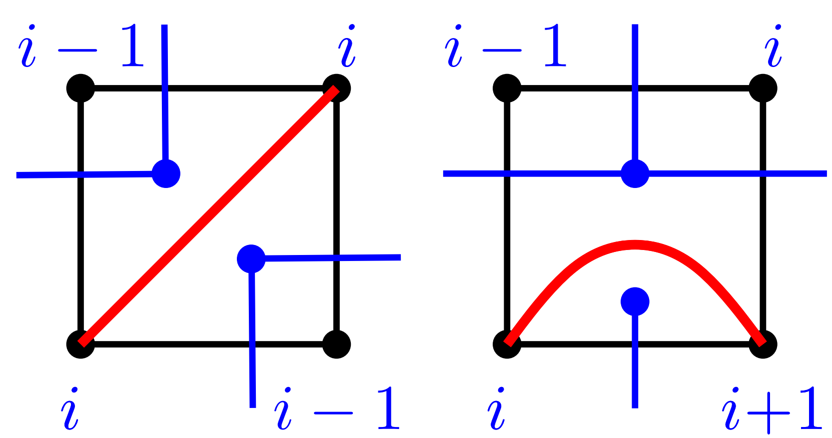

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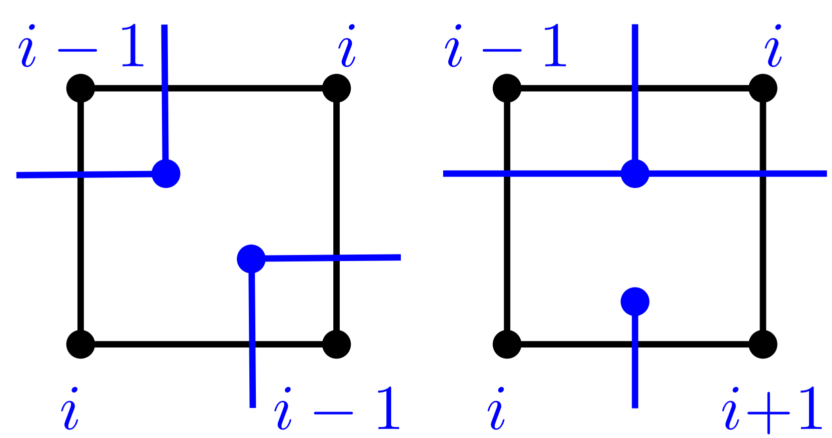
- If the construction of blue graph is local then it is invertible and it leads to a **BIJECTION!**



General case (II)

$\{$rooted, bipartite quadrangulations on $\mathcal{S}$ with n faces and N_i vertices at distance i from the root vertex $(i \geq 1)\}$

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Double rooting trick and Hall's marriage theorem!

Random maps

Let $(\mathcal{M}, v)$ be a map with a distinguished vertex v. We define:

- radius of a map $\mathcal{M}$ centered at v by the quantity
$$R(\mathcal{M}, v) = \max_{u \in V(\mathcal{M})} d_{\mathcal{M}}(v, u);$$
- profile of distances from the distinguished point v (for any $r > 0$) by:
$$I_{(\mathcal{M},v)}(r) = \#\{u \in V(\mathcal{M}) : d_{\mathcal{M}}(v, u) = r\}.$$

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Theorem [Chapuy–D. '15]
Let q_n be uniformly distributed over the set of rooted, bipartite quadrangulations with n faces on $\mathcal{S}$, let v_0 be a root vertex of q_n and let v_* be uniformly chosen vertex of q_n. Then, there exists a continuous, stochastic process $L^{\mathcal{S}} = (L^{\mathcal{S}}_t, 0 \leq t \leq 1)$ such that:

- $\left(\frac{9}{8n}\right)^{1/4} R(q_n, v_*) \to \sup L^{\mathcal{S}} - \inf L^{\mathcal{S}}$;
- $\left(\frac{9}{8n}\right)^{1/4} d_{q_n}(v_0, v_*) \to \sup L^{\mathcal{S}}$;
- $\frac{I_{(q_n,v_*)}\left((8n/9)^{1/4}\cdot\right)}{n+2-2h} \to \mathcal{I}^{\mathcal{S}}$,

where $\mathcal{I}^{\mathcal{S}}$ is defined as follows: for every non-negative, measurable $g : \mathbb{R}_+ \to \mathbb{R}_+$,
$$\langle \mathcal{I}^{\mathcal{S}}, g \rangle = \int_0^1 dt g(L^{\mathcal{S}}_t - \inf L^{\mathcal{S}}).$$

Generalization by Bettinelli

- [Bettinelli '15] rephrased our orientation process of a quadrangulation (given by the Dual Exploration Graph) in terms of level loops.

direct construction of a bijection between pointed quadrangulations and labeled unicellular maps on a non-oriented surface $\mathcal{S}$

extension to arbitrary bipartite (and finally not necessarily bipartite - more technical) maps on a non-oriented surface $\mathcal{S}$. Bijection with so-called well-labeled unicellular mobiles on $\mathcal{S}$.

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Applications: Enumeration of triangulations of any non-oriented surface $\mathcal{S}$.

III Bijections for bipartite maps and blossoming tree-like structures

Idea

• In the planar case the crucial idea was to use the set of **Eulerian** orientations and rely on the fact that it is a lattice. In positive genus: Eulerian maps $\neq$ Bicolorable maps (Bicolorable maps = dual to bipartite maps)

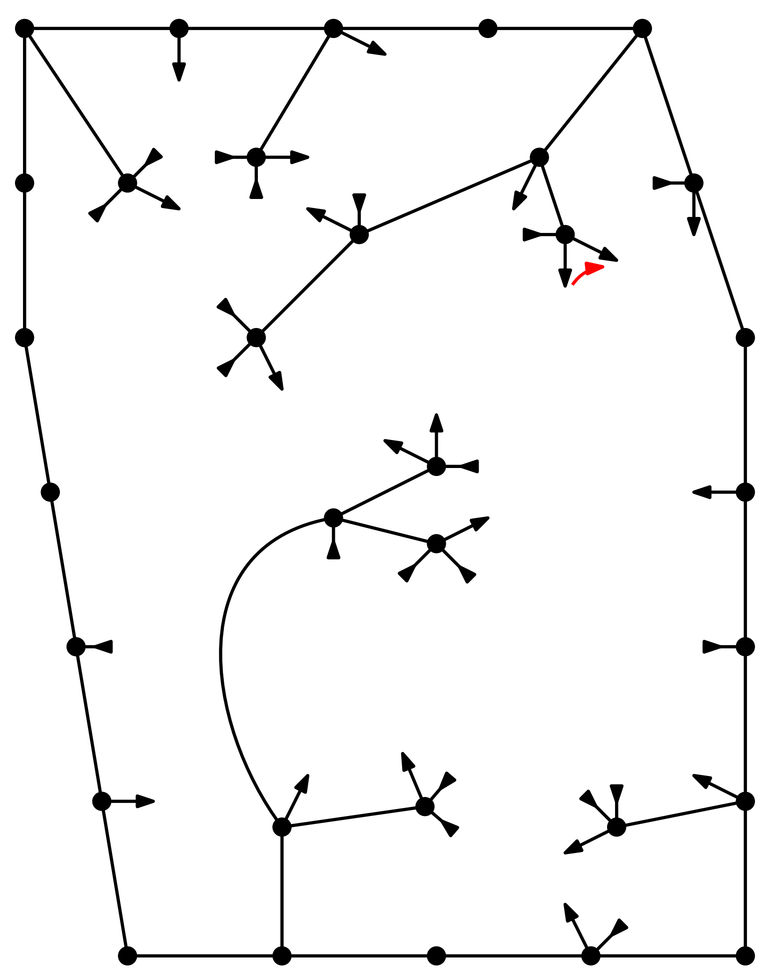
• The set of bicolorable orientations (of a fixed graph) is a lattice [Propp '93]. [Lepoutre '17] used it to extend Schaeffer bijection to all orientable surfaces. Ideas still heavily rely on clockwise/counterclockwise circuits. New ideas:

- try to cut your map using a canonical spanning tree
- redefine blossoming maps

Blossoming and well-blossoming maps

A map is called blossoming if it has additional half-edges (stems):

- buds
- leafs



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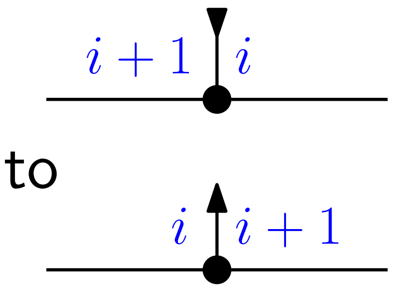

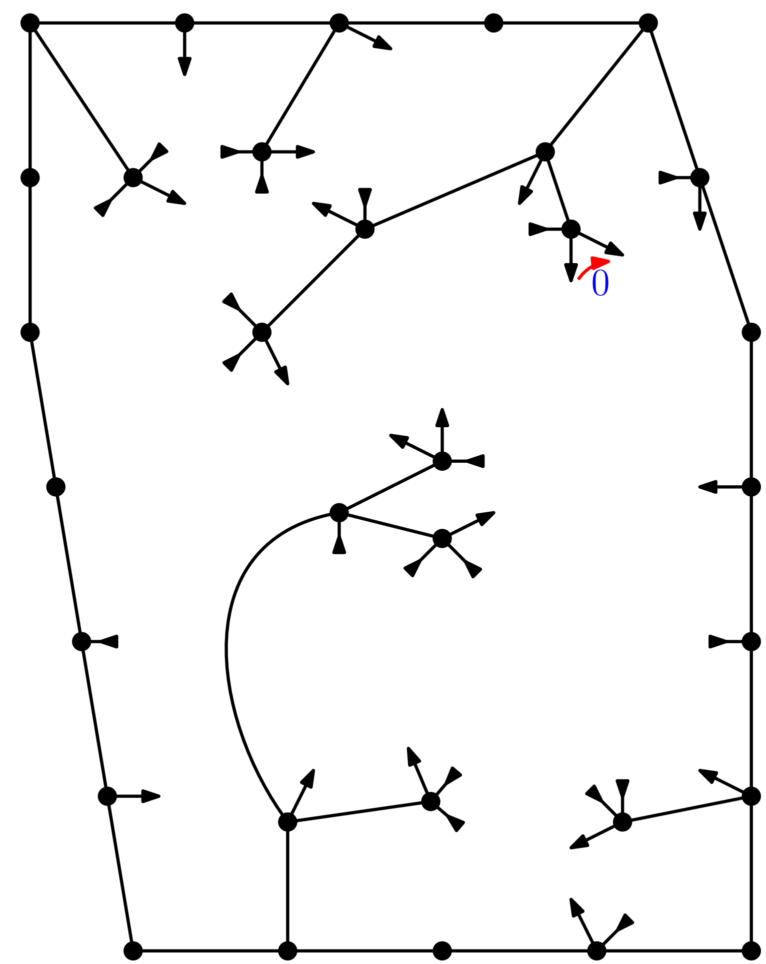

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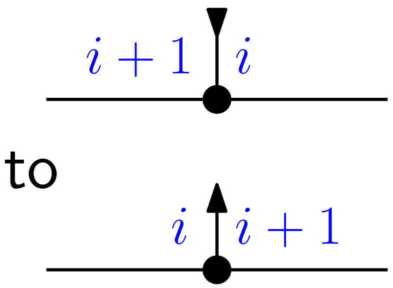

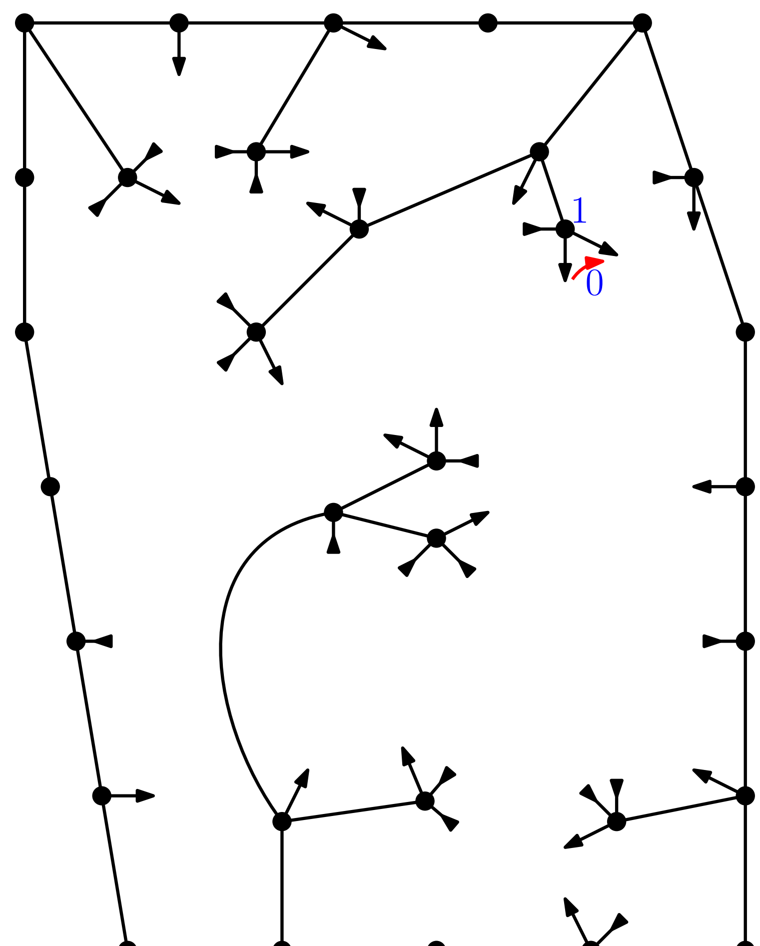

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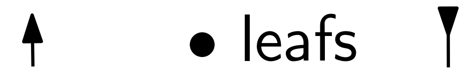

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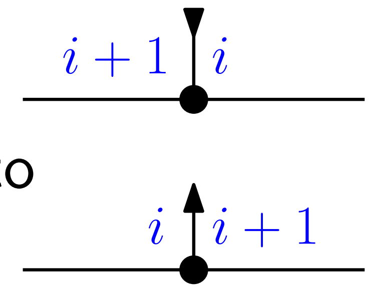

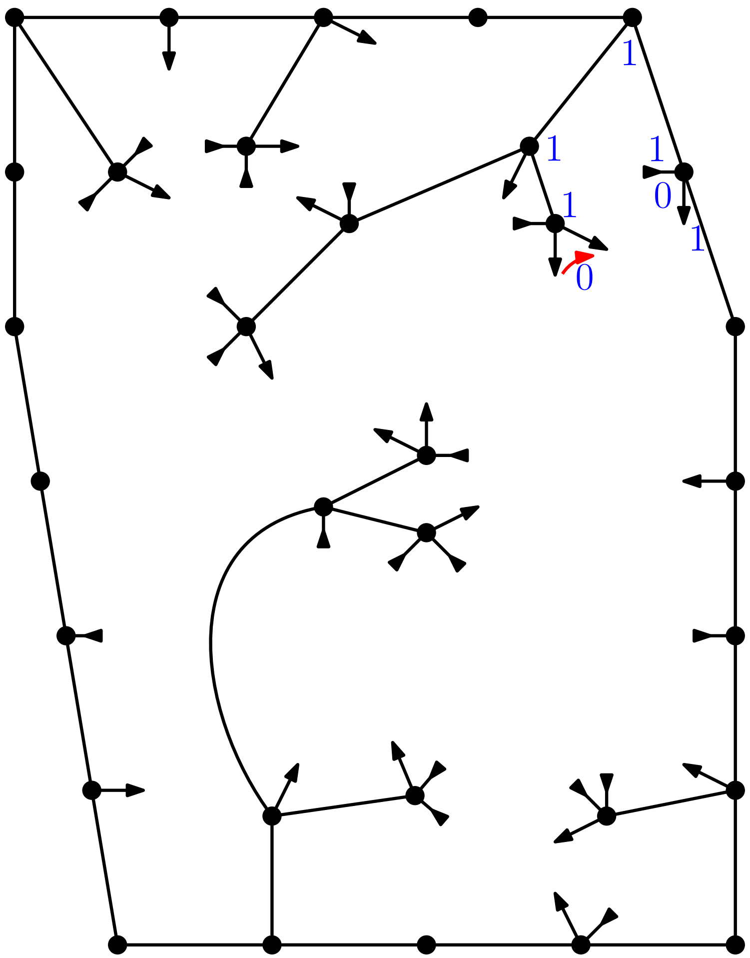

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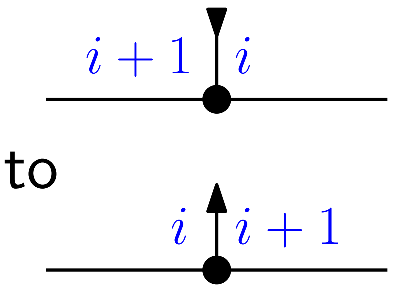

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- it is bud-rooted
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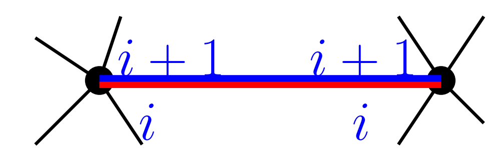

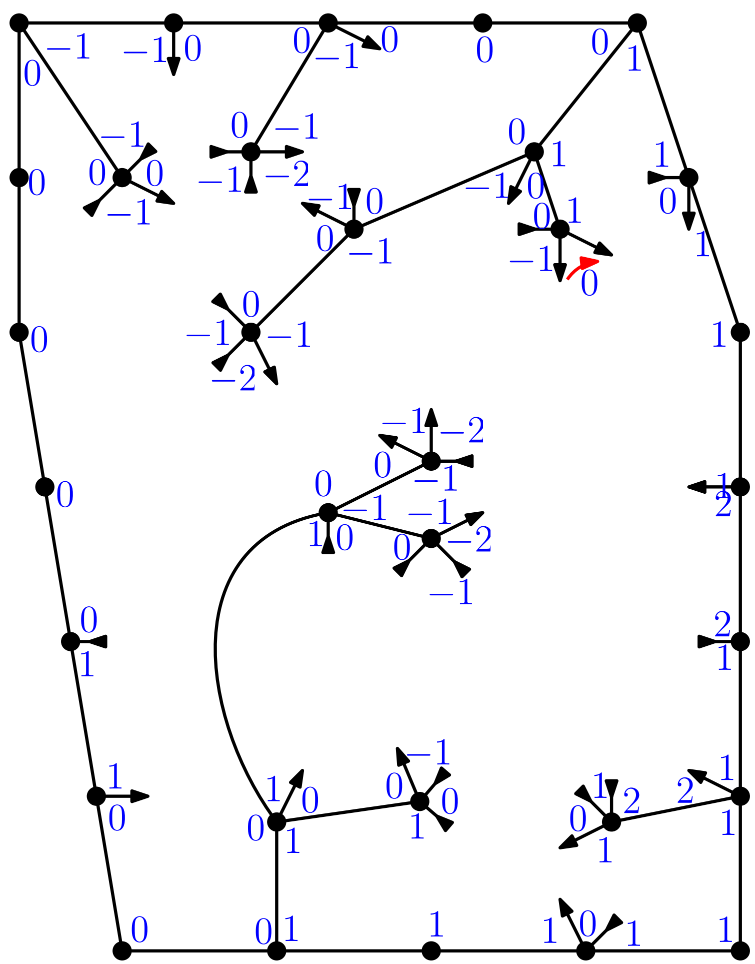

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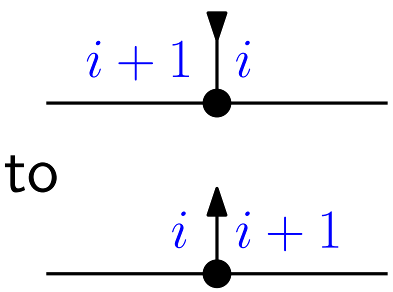

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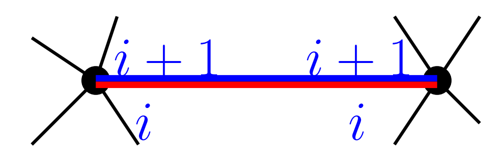

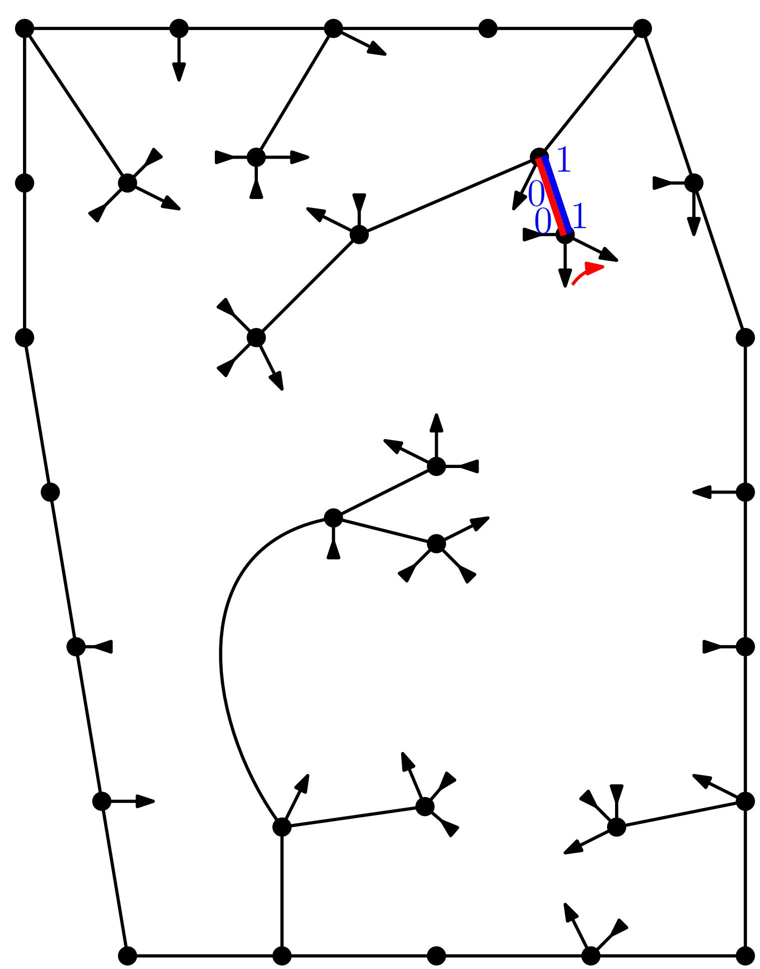

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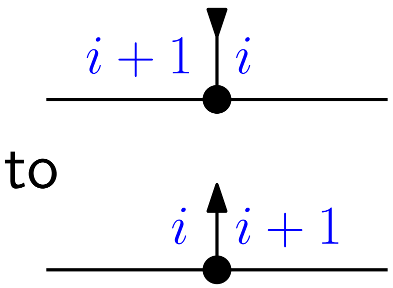

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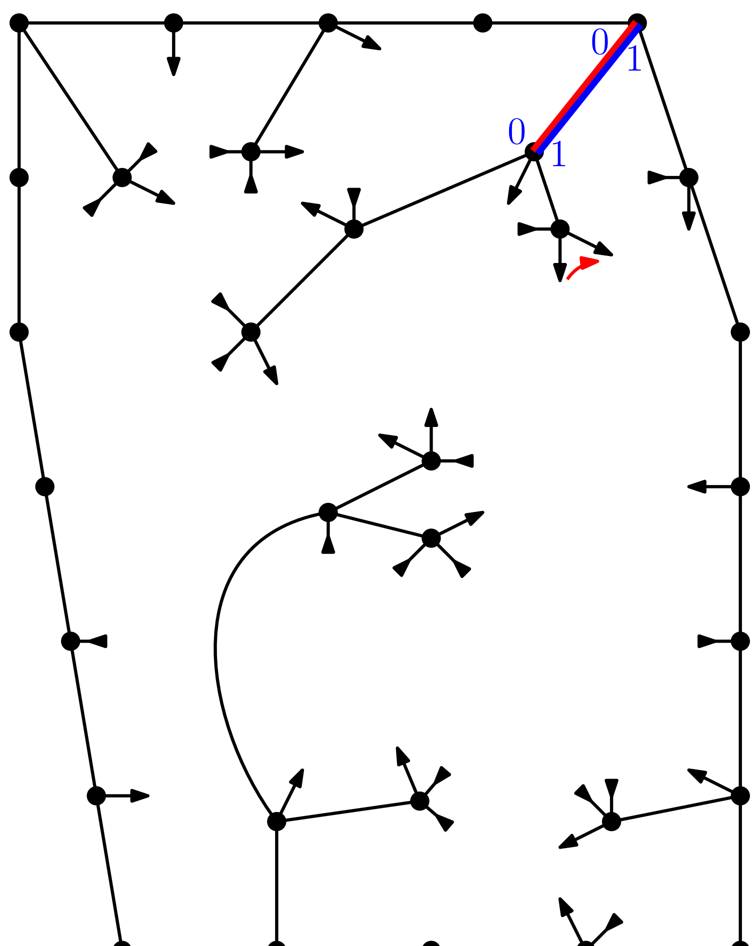

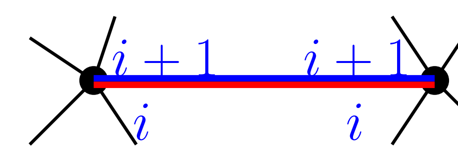

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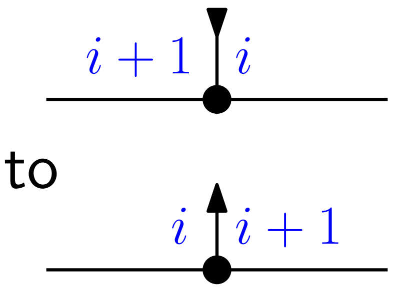

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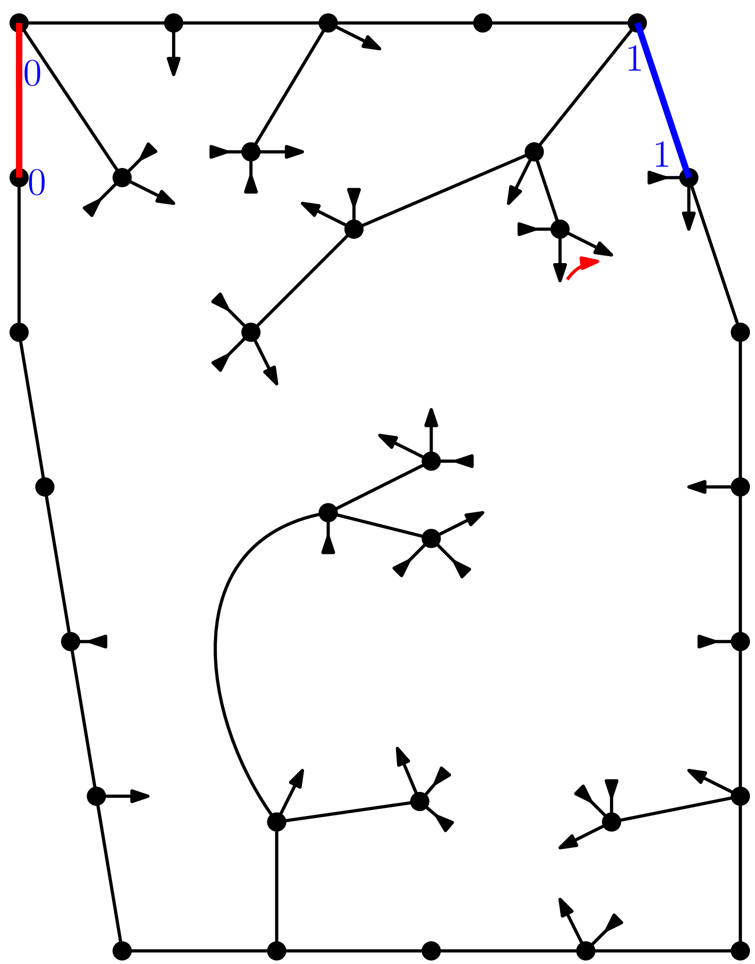
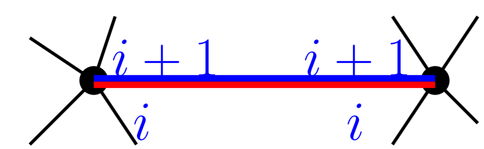


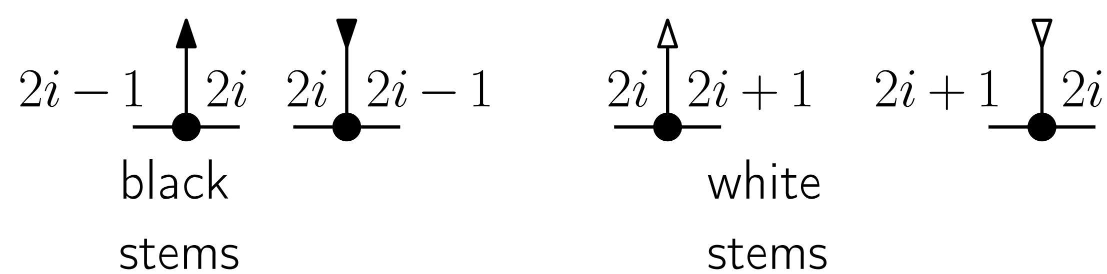


Bijection

Theorem [D.–Lepoutre '20]
There exists a bijection between:

- rooted, bipartite, pointed maps on ANY NON-ORIENTED surface $\mathcal{S}$ with $n_\bullet$ black vertices, $n_\circ$ white vertices, and n_k faces of degree $2k$ $(k \geq 1)$;
- well-blossoming maps on ANY NON-ORIENTED surface $\mathcal{S}$ with $n_\bullet - 1$ black buds, $n_\circ$ white buds and and n_k vertices of degree $2k$ $(k \geq 1)$;

Additionally, distances from the distinguished point correspond to the corner labeling.



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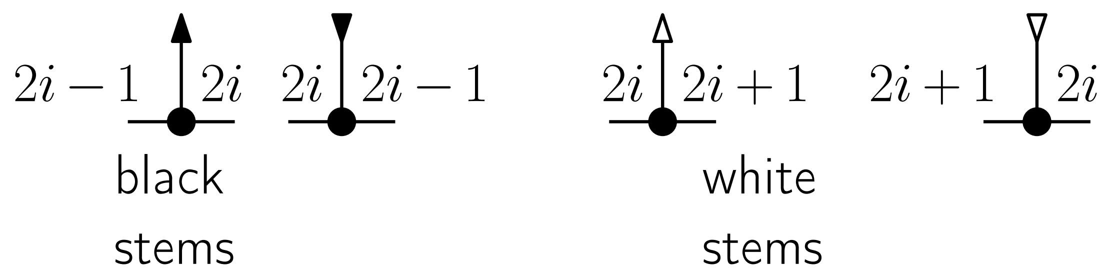
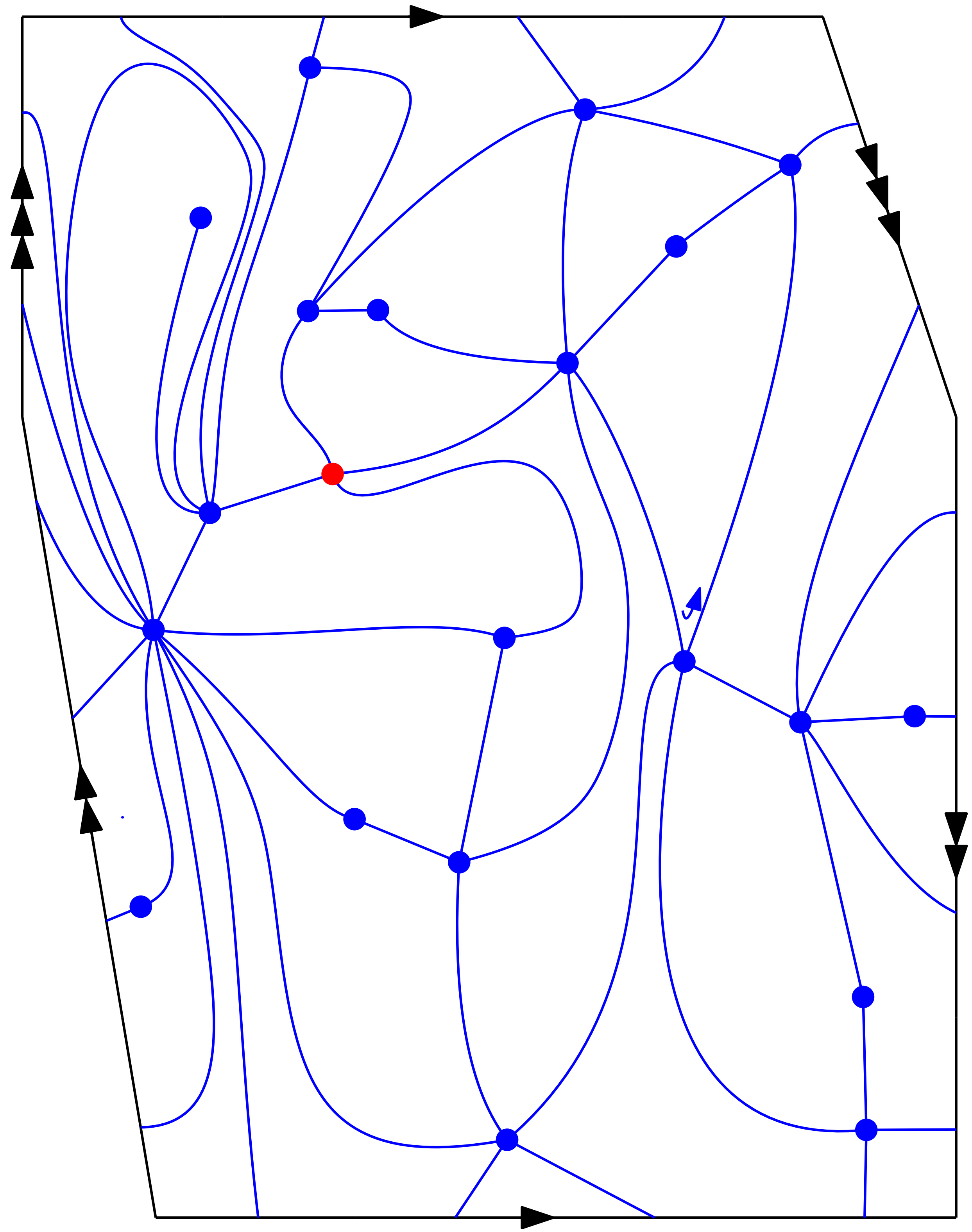


How does it work?

Bijection (II)



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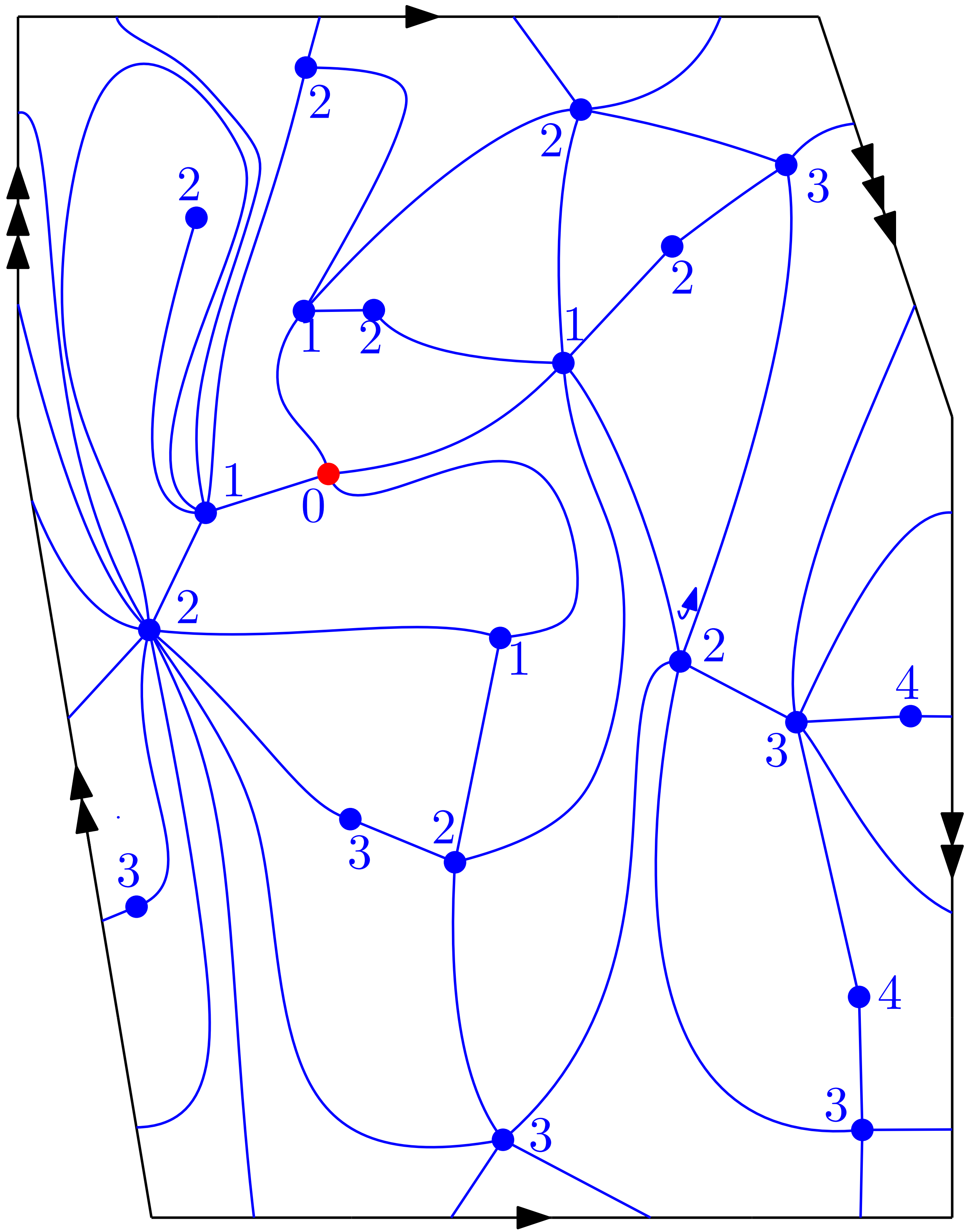

- label the distances from the distiguished point

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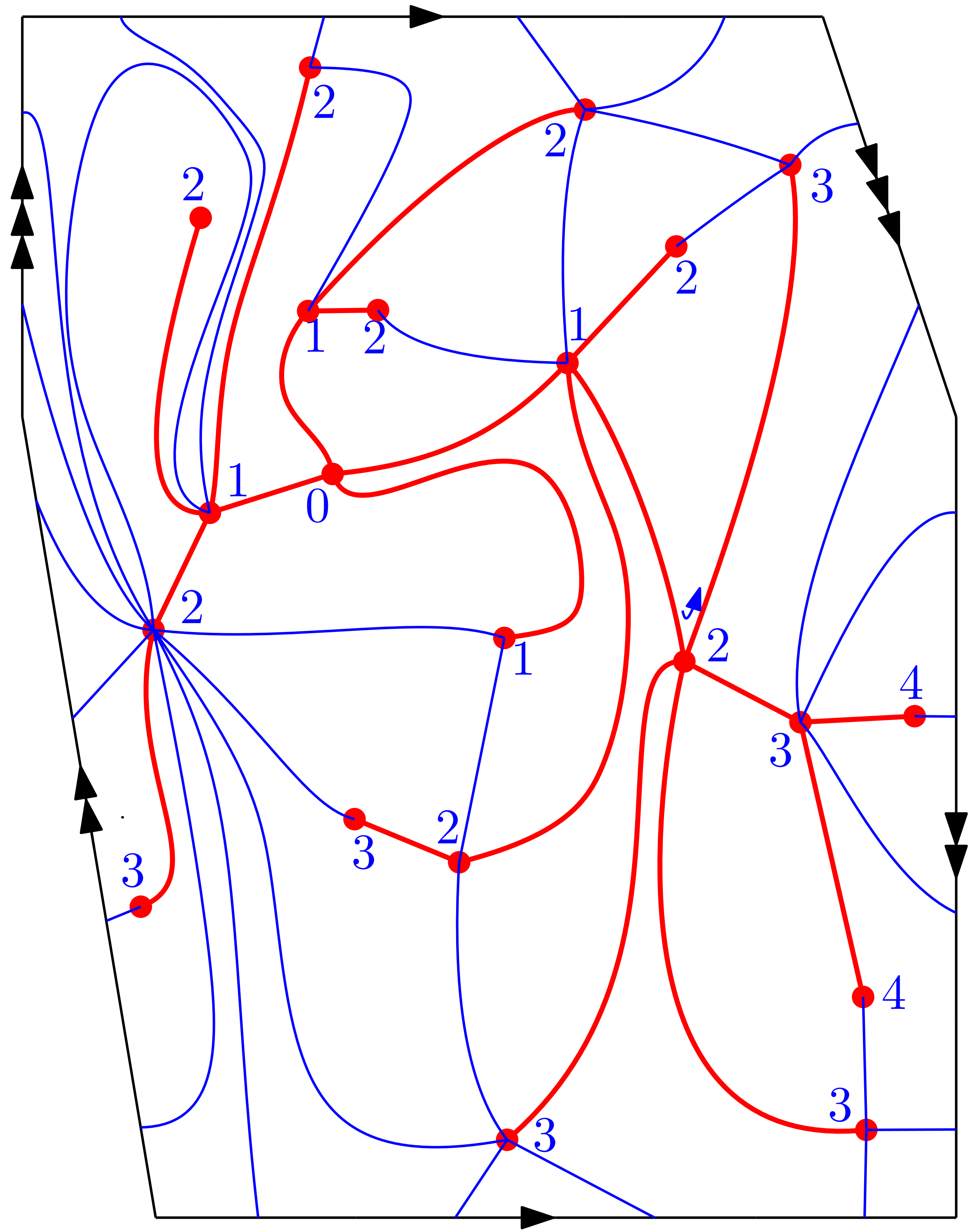

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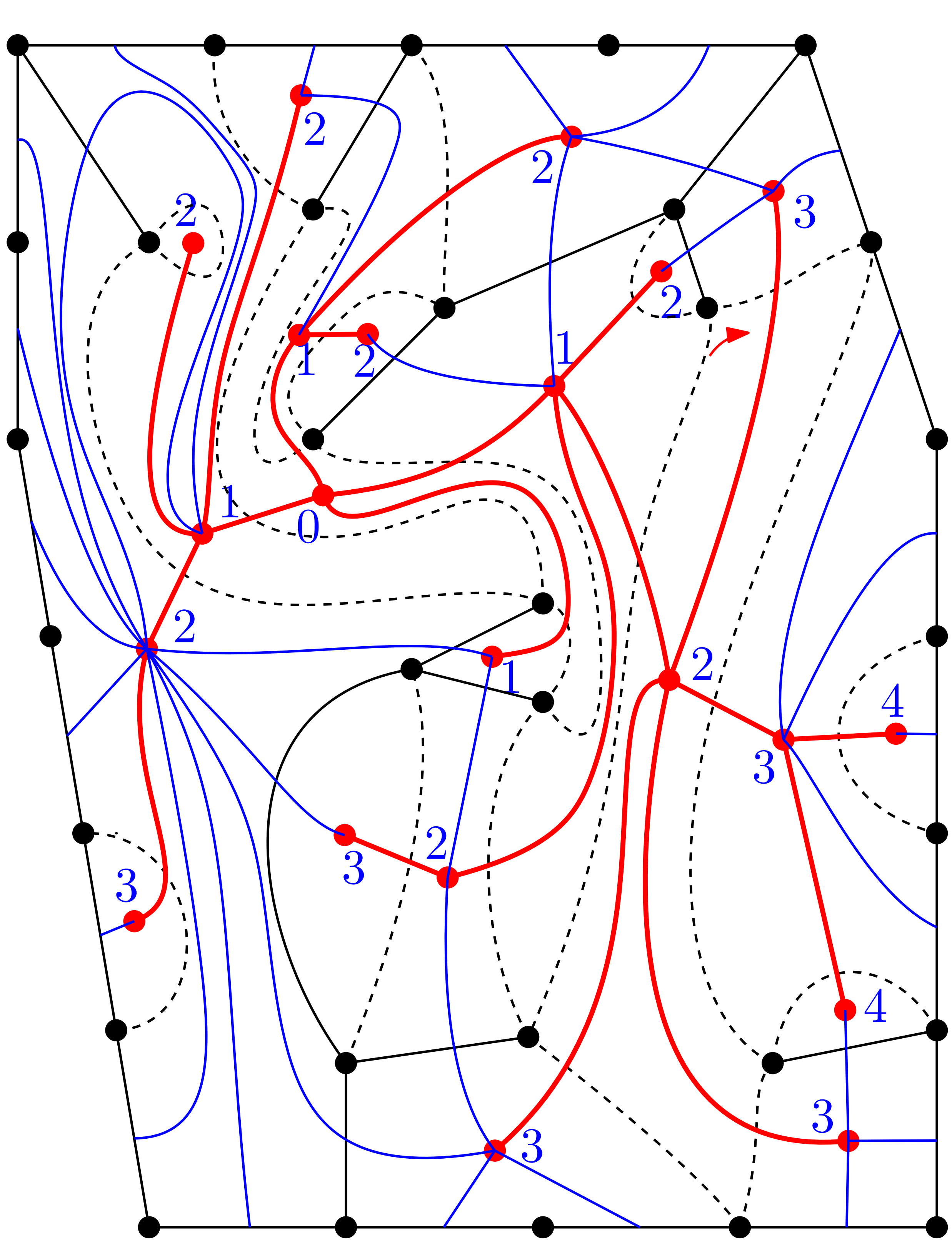
Algorithm: A variant of breadth first search.

Bijection (II)

- draw the dual map



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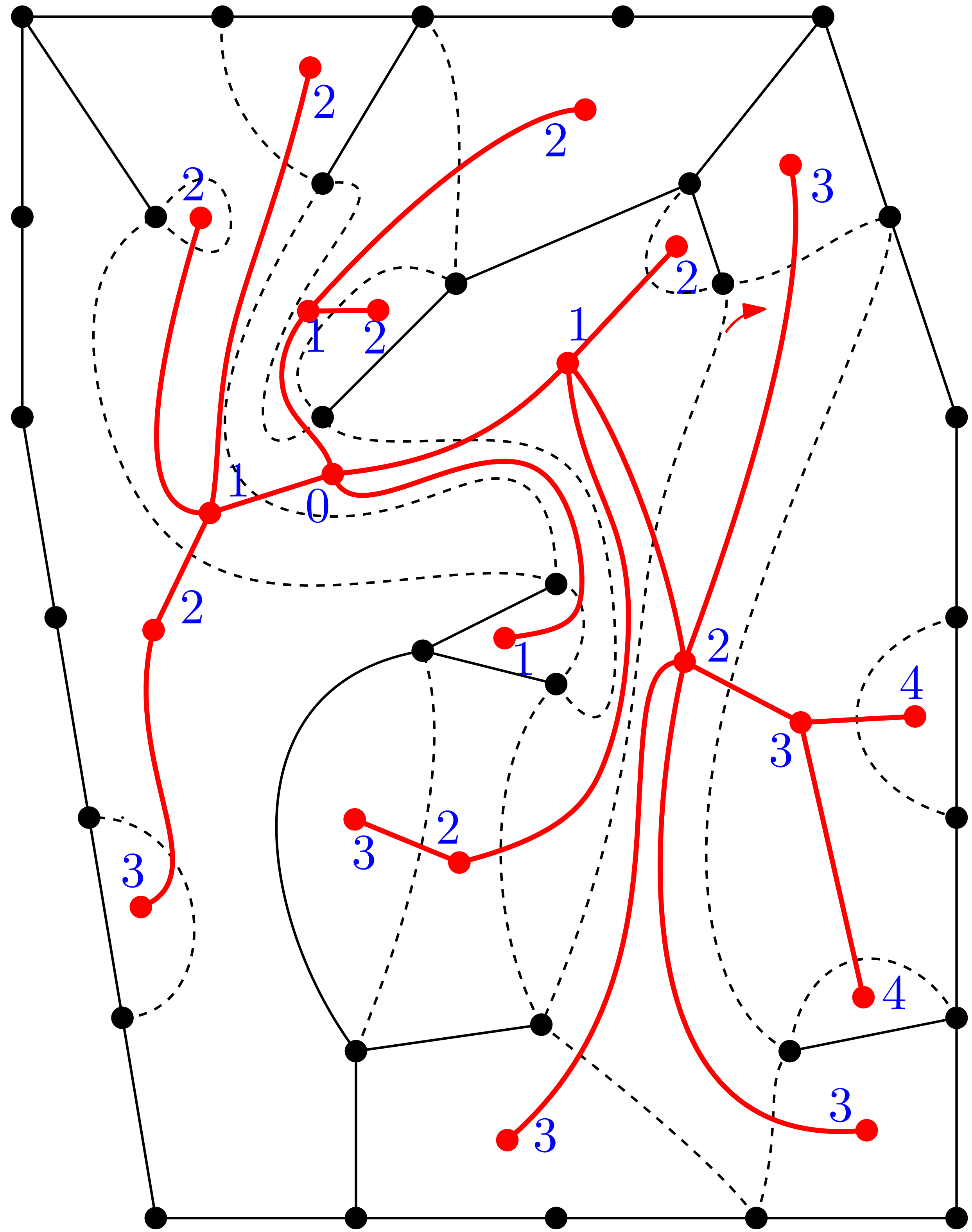

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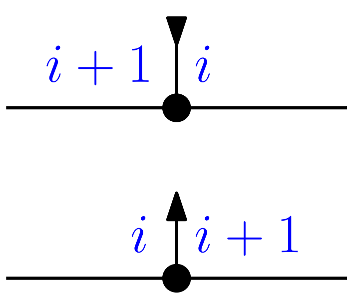

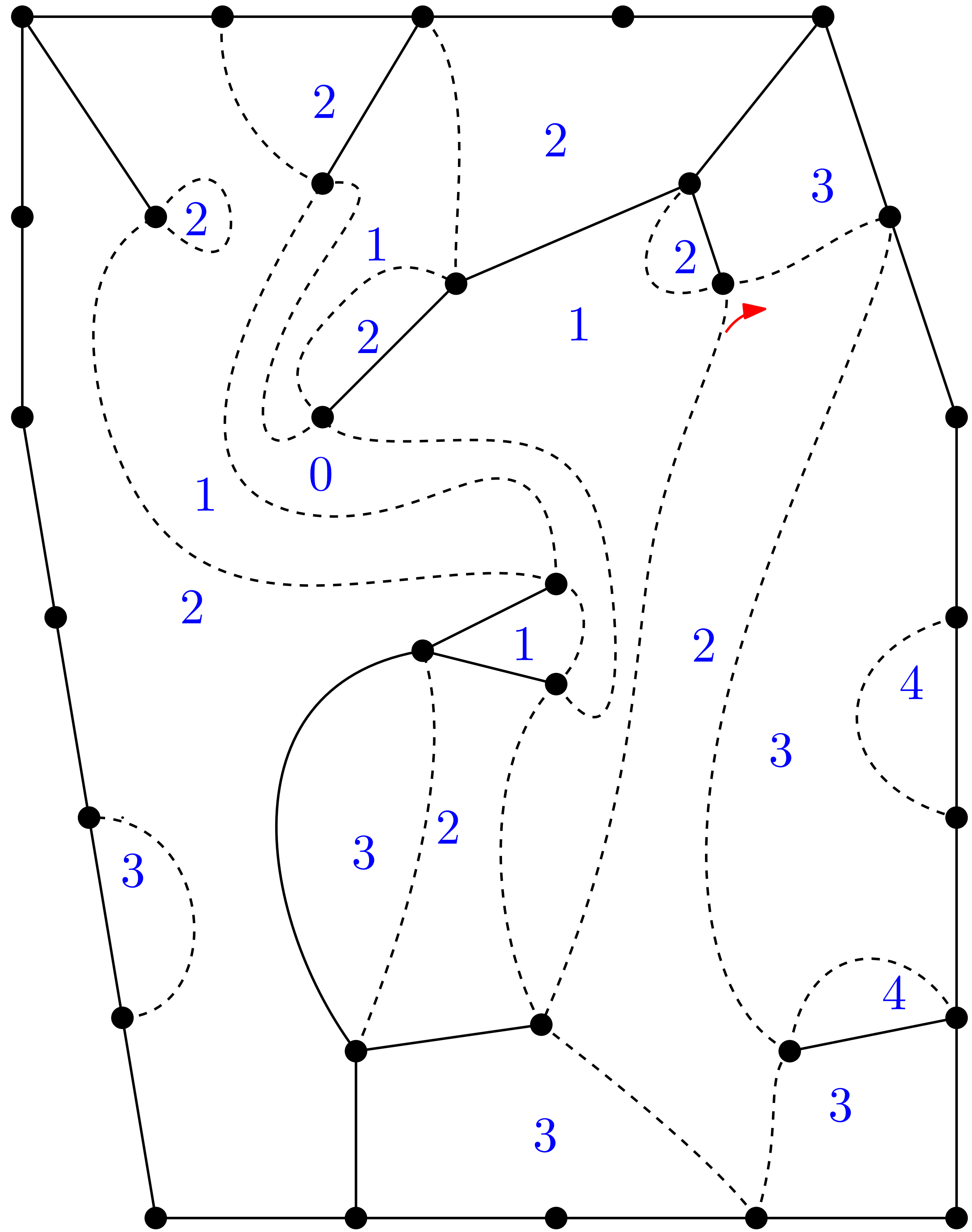

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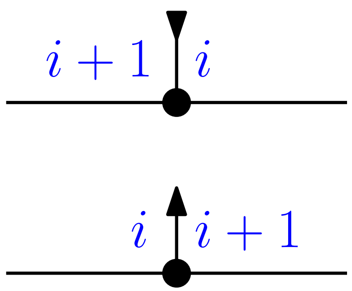

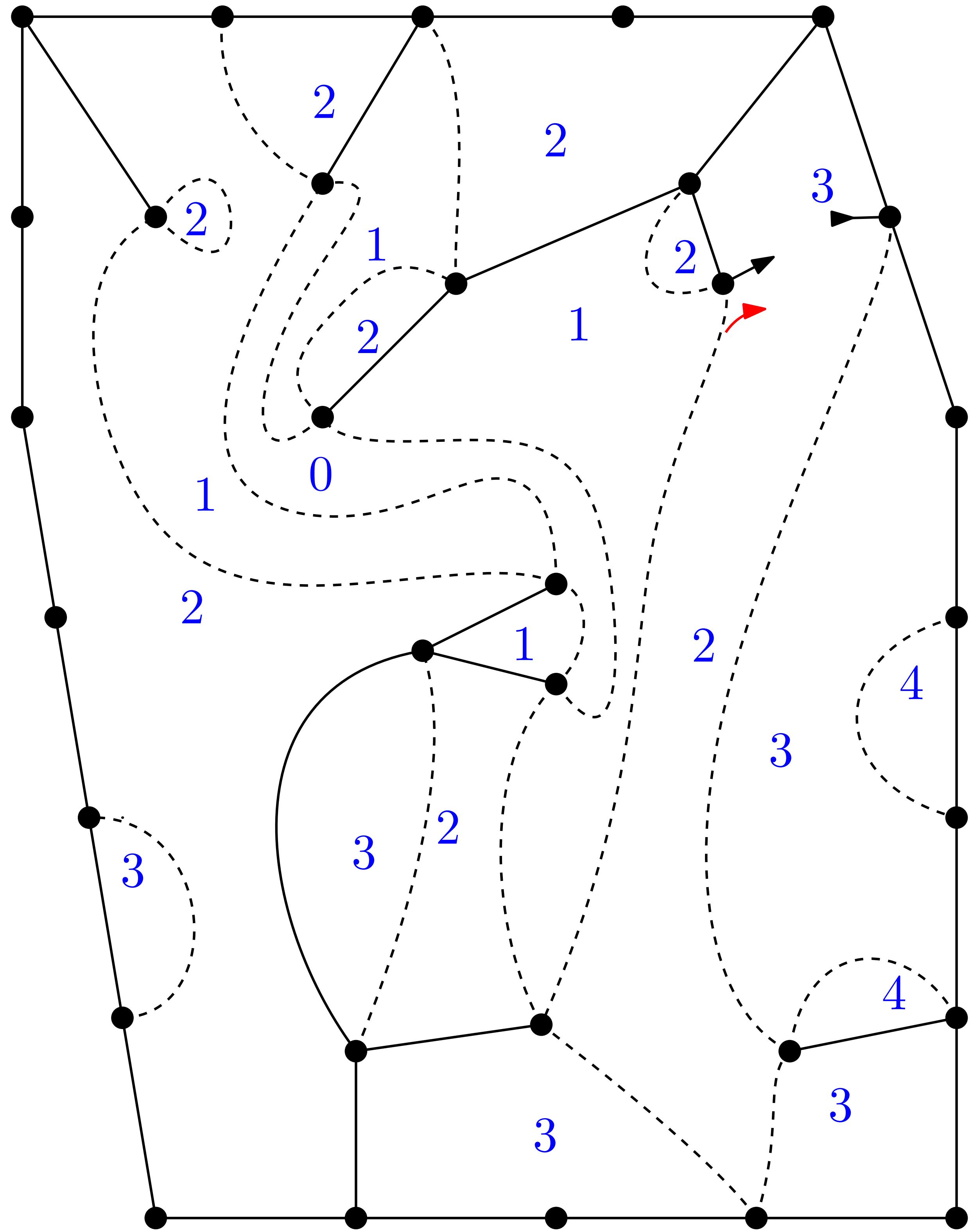

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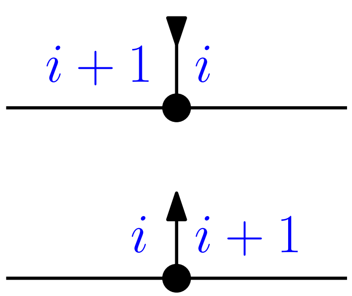

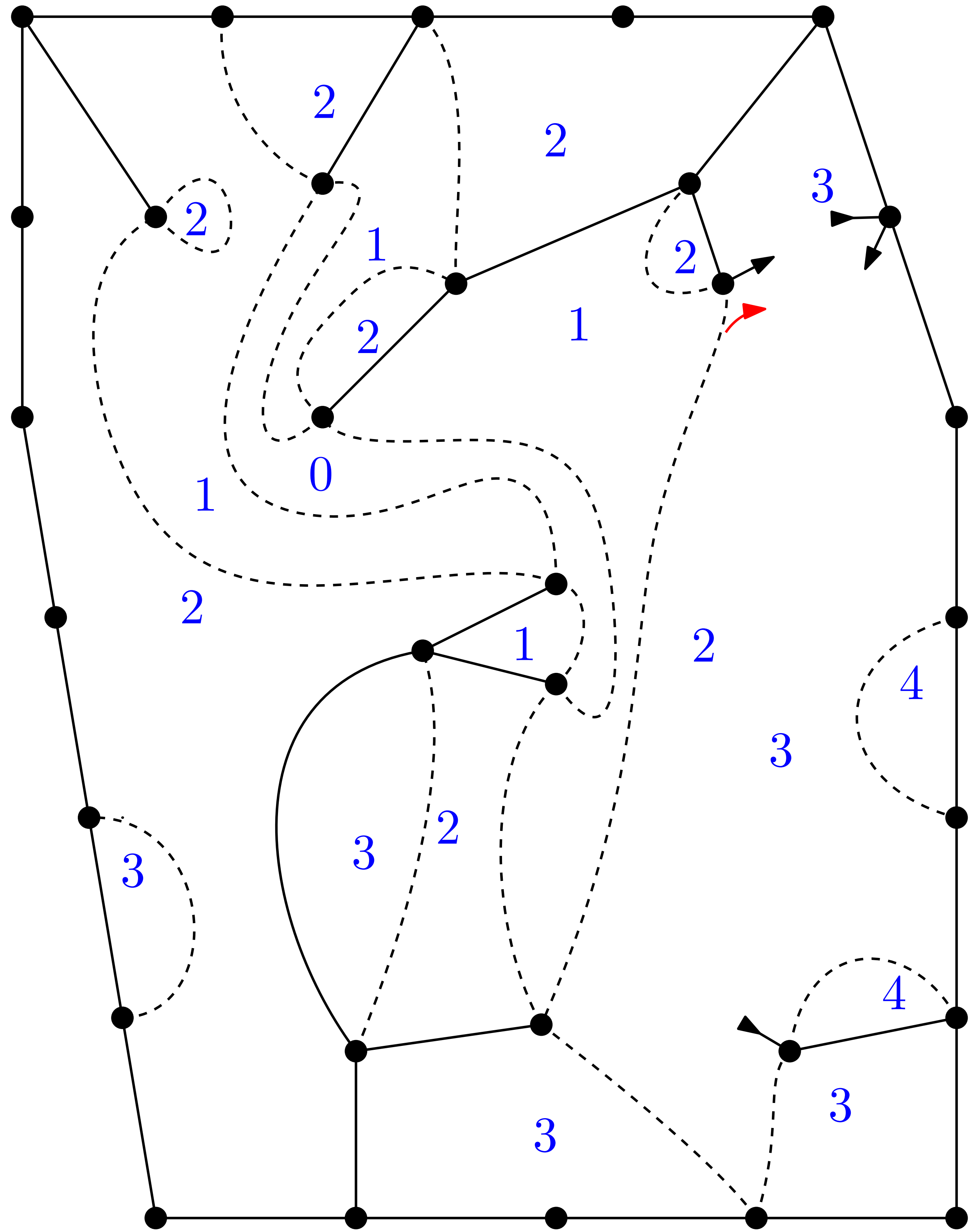

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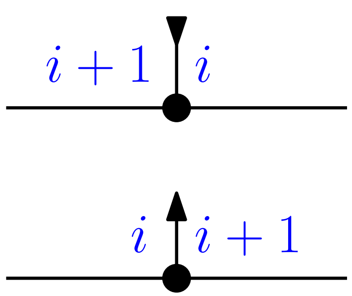

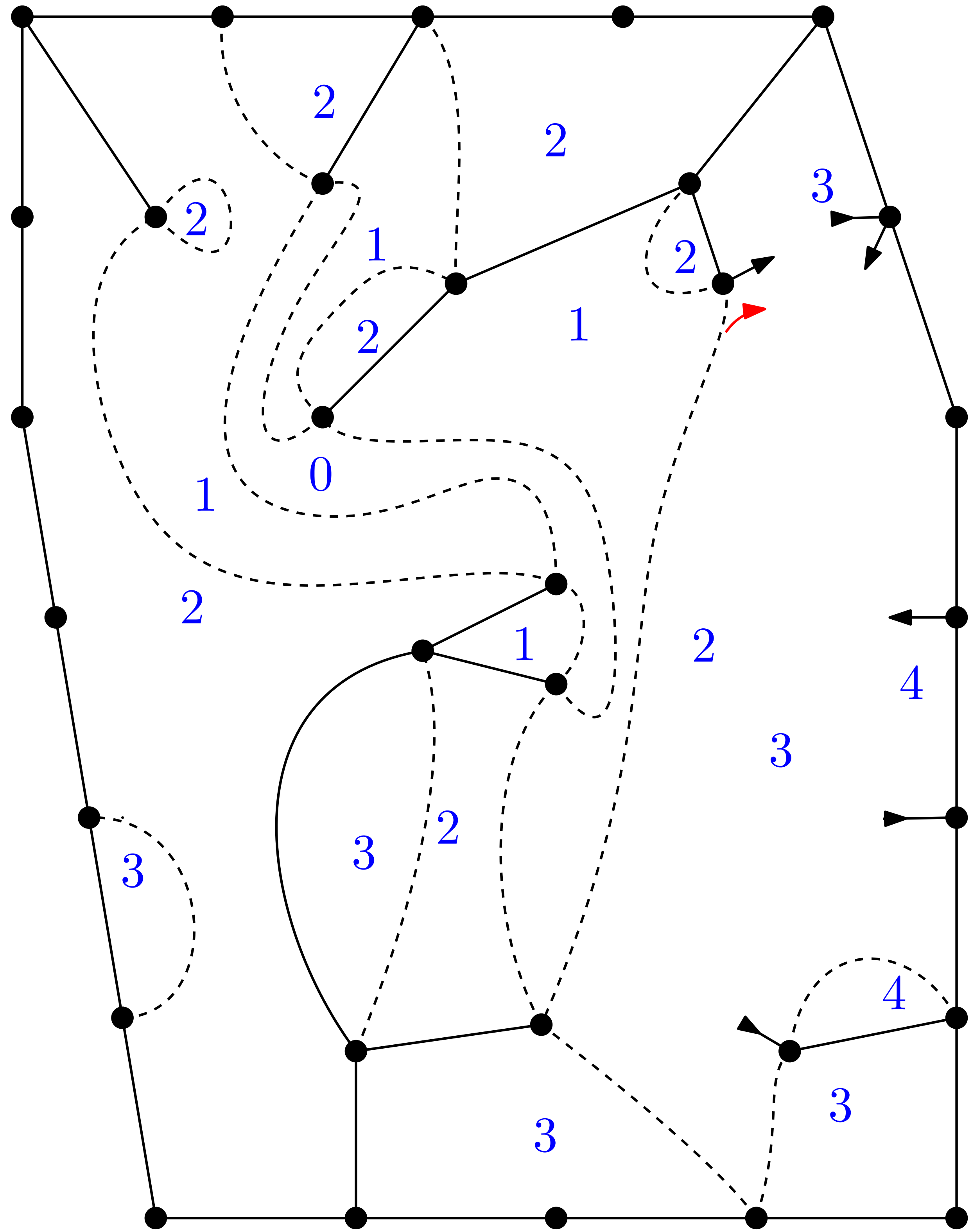

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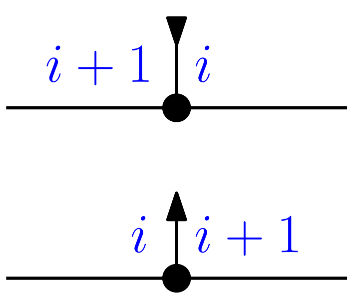

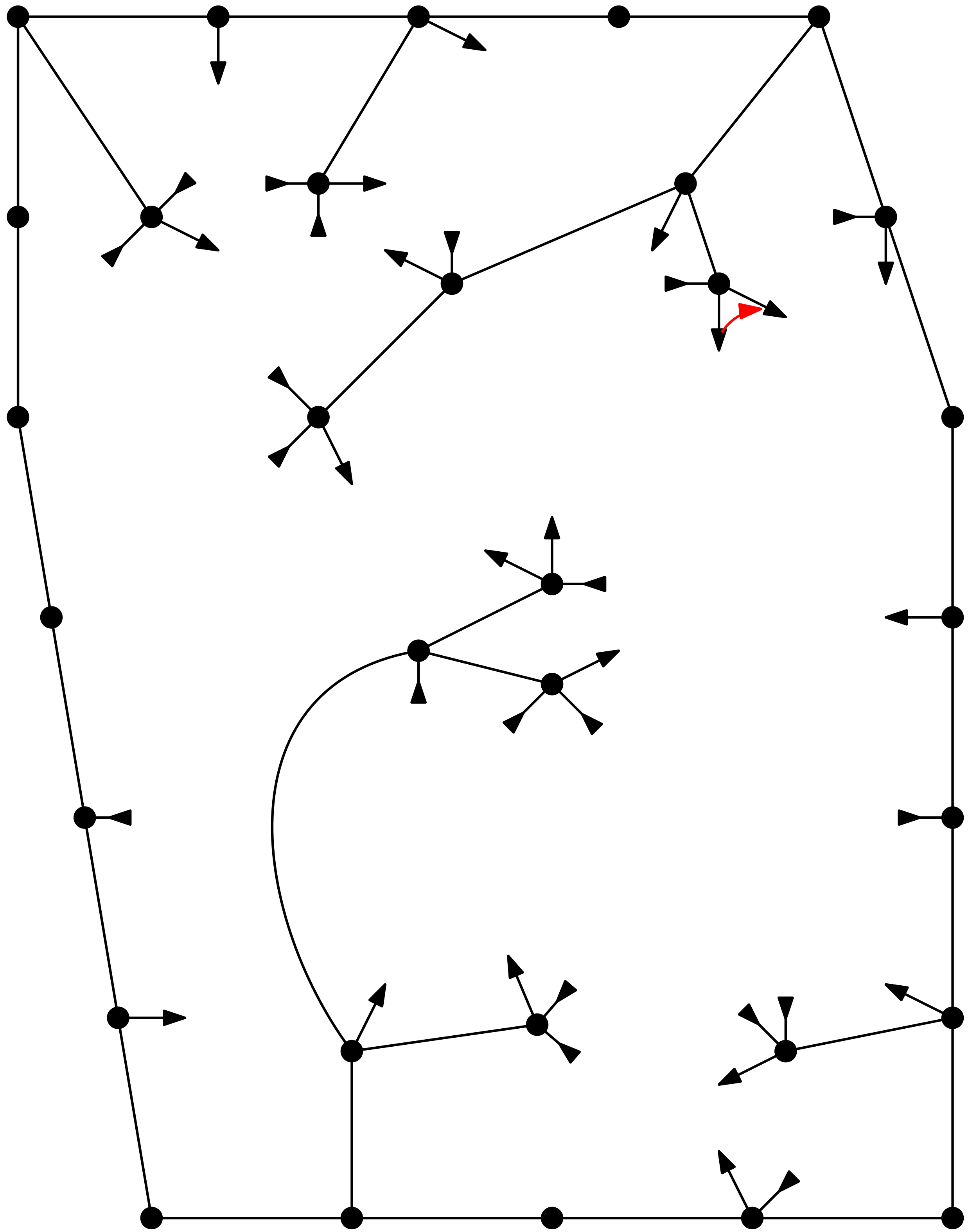

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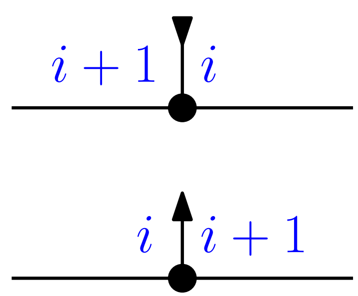

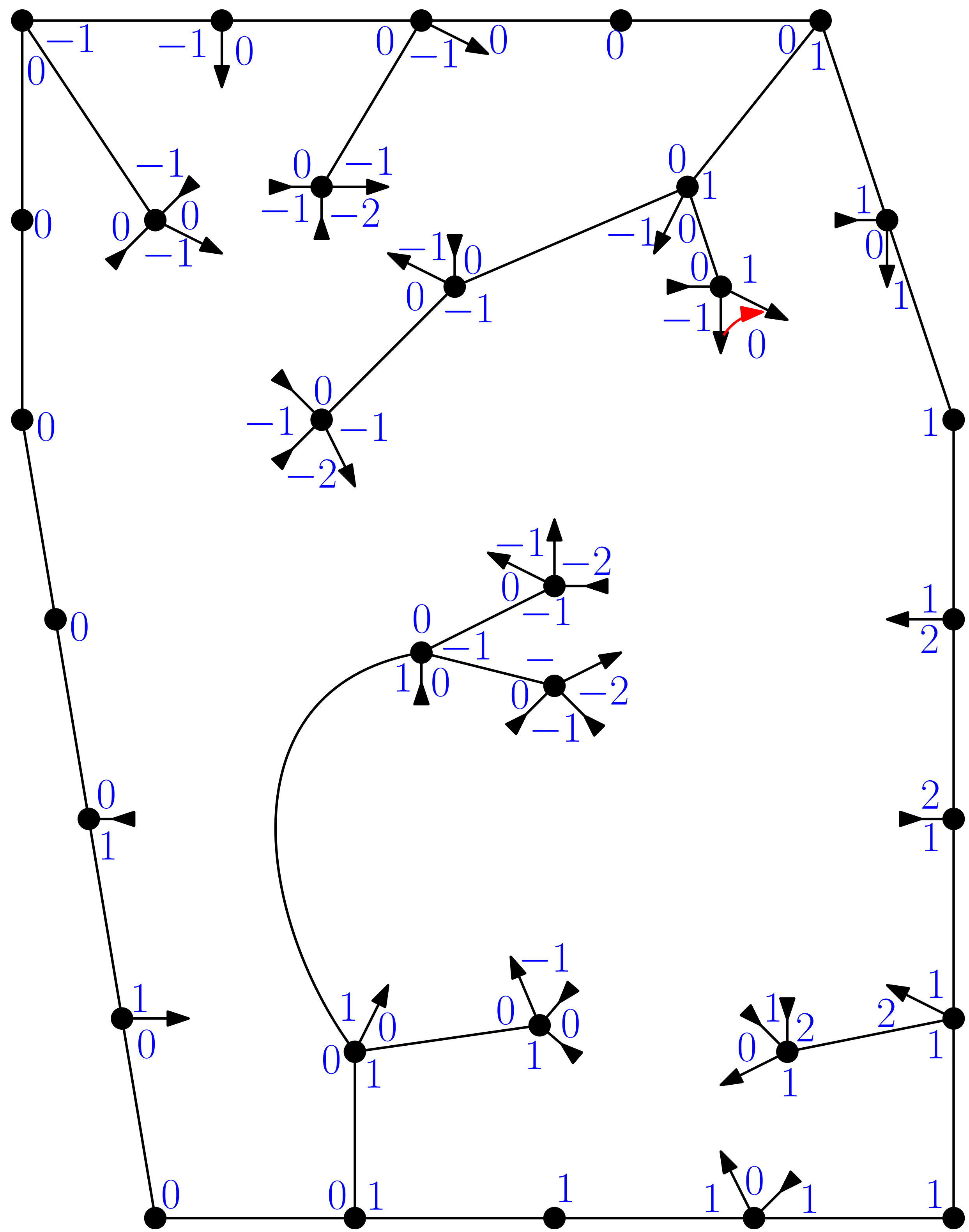

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Consequences

Theorem [Bender–Canfield '86]
Let

$$BQ_{\mathcal{S}}(t) := \sum_{M \in \mathcal{BQ}_{\mathcal{S}}} t^{\chi(\mathcal{S})+\text{number of faces of M}}$$

be the univariate generating function of rooted bipartite quadrangulations of $\mathcal{S}$. Moreover let $U \equiv U(t)$ and $T \equiv T(t)$ be the two formal power series defined by: $T = 1 + 3tT^2, \qquad U = tT^2(1 + U + U^2)$. Then $BQ_{\mathcal{S}}(t)$ is a rational function in U.

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a consequence of our labeled bijection [Chapuy–D. '15]

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be the univariate generating function of rooted bipartite quadrangulations of an orientable surface $\mathcal{S}$. Then $BQ_{\mathcal{S}}(t)$ is a rational function in $\sqrt{1-12t}$.

a consequence of the blossoming bijection [Lepoutre '17]

also a consequence of the topological recursion [Eynard–Orantin '07]

Consequences

Theorem [Bender–Canfield–Richmond '93 (orientable) Arques–Giorgetti '00 (non-oriented)]
Let
$$BQ_{\mathcal{S}}(x,y) := \sum_{M \in \mathcal{BQ}_{\mathcal{S}}} x^{n_\bullet(M)} y^{n_\circ(M)}$$
be the bivariate generating function of rooted bipartite quadrangulations of a surface $\mathcal{S}$. Let
$$t_\bullet = x + 2t_\bullet t_\circ + t_\bullet^2$$
$$t_\circ = y + 2t_\bullet t_\circ + t_\circ^2$$
$$a = \sqrt{(1 - 2(t_\bullet + t_\circ))^2 - 4t_\bullet t_\circ}.$$
Then there exists a polynomial $P_{\mathcal{S}}(t_\bullet, t_\circ, a)$ of degree $\leq 3 - 3\chi(\mathcal{S})$ such that
$$BQ_{\mathcal{S}}(x,y) = \frac{P_{\mathcal{S}}(t_\bullet, t_\circ, a)}{a^{4-5\chi(\mathcal{S})}}.$$
Moreover $\deg_a(P_{\mathcal{S}}) = 0$ when $\mathcal{S}$ is orientable.

a consequence of the blossoming bijection [D.–Lepoutre '20] (orientable case worked out by [Albenque–Lepoutre '20])

THANK
YOU!

References:
arXiv:1501.06942
arXiv:1512.02208
arXiv:2002.07238