Gaussian fluctuations of Jack-deformed random Young diagrams

(joint work with Piotr Śniady)

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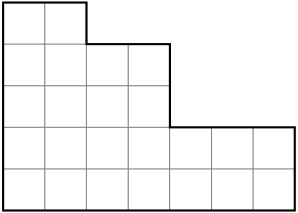
Problem

Definition

- A partition π of the integer n ($\pi \vdash n$, or $\pi \in \mathcal{P}_n$): a finite non-increasing sequence of positive integers $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_k$, such that $|\pi| := \sum_i \pi_i = n$;
- Graphical representation by a Young diagram of size n.



Problem

β-ensembles: the probability distributions on $\mathbb{R}^n$ with the density of the form

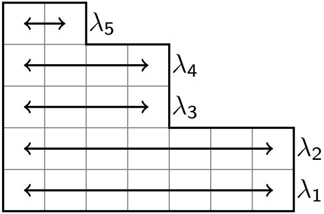
$$p(x_1, \dots, x_n) = \frac{1}{Z} e^{V(x_1)+\cdots+V(x_n)} \prod_{i<j} |x_i - x_j|^{\beta},$$

where V is some real-valued function and Z is the normalization constant. What is the discrete counterpart of β-ensambles?

Problem

Example

- $\pi = (7, 7, 4, 4, 2) \vdash 24$,
- Representem by a Young diagram λ with $\ell(\lambda) = 5$ rows.



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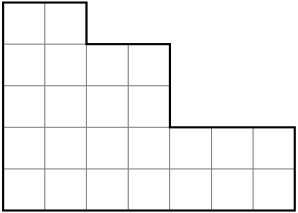

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Solution

- There seems to be no obvious unique way of defining the discrete counterpart of β-ensambles.
- Several alternative approaches are available, see for example approaches recently proposed by Borodin, Gorin and Guionnet, or by Moll.
- We propose different approach which produces measures with many desirable asymptotic properties and allows to study the double-scaling limit.

Examples and the representation theory I

ρ_n - a representation of the symmetric group $\mathfrak{S}_n$ defines a probability measure $\mathbb{P}_n$ on the set of Young diagrams $\mathbb{Y}_n$ in the following way:

$$\chi_n(\pi) := \frac{\text{Tr } \rho_n(\pi)}{\text{Tr } \rho_n(\text{id})} = \sum_{\lambda \in \mathbb{Y}_n} \mathbb{P}_n(\lambda)\chi_\lambda(\pi)$$

for each $\pi \in \mathfrak{S}_n$, where χ_λ is an irreducible character, i.e.

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More generally, we call $\chi : \mathcal{P}_n \to \mathbb{R}$ a reducible character, if it is a convex combination of irreducible characters.

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Examples and the representation theory II

Example

- Plancherel measure

$$\chi(\pi) := \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases} \quad \leftrightarrow \quad \mathbb{P}_\chi(\lambda) := \frac{(\dim \rho_\lambda)^2}{n!}$$

- Schur-Weyl measure

$$\chi(\pi) := N^{\ell(\pi)-|\pi|} \quad \leftrightarrow \quad \mathbb{P}_\chi(\lambda) := \frac{\dim E_\lambda}{N^n},$$

where $(\mathbb{C}^N)^{\otimes n} = \bigoplus_{\lambda \vdash n} E_\lambda$.

Jack deformation

Fix $\alpha \in \mathbb{R}_{>0}$ and expand Jack polynomials $J_\lambda^{(\alpha)}$ in power-sum basis:

$$J_\lambda^{(\alpha)} = \sum_\pi \theta_\pi^{(\alpha)}(\lambda)\, p_\pi.$$

We define irreducible Jack character $\chi_\lambda^{(\alpha)}$:

$$\chi_\lambda^{(\alpha)}(\pi) := \alpha^{-\frac{\|\pi\|}{2}} \frac{z_\pi}{n!}\, \theta_\pi^{(\alpha)}(\lambda),$$

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Jack deformation - examples

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- Jack-Plancherel measure

$$\chi(\pi) := \begin{cases} 1 & \text{if } \pi = 1^n, \\ 0 & \text{otherwise} \end{cases} \leftrightarrow \mathbb{P}_\chi(\lambda) := \frac{n!}{\prod_{(x,y)\in\lambda} h_\alpha(x,y) h'_\alpha(x,y)}$$

- Jack-Schur-Weyl measure

$$\chi(\pi) := N^{\ell(\pi)-|\pi|} = N^{-\|\pi\|} \quad \leftrightarrow$$

$$\mathbb{P}_\chi(\lambda) := n! \prod_{(x,y)\in\lambda} \frac{N + \sqrt{\alpha}(x-1) - \sqrt{\alpha}^{-1}(y-1)}{N \cdot h_\alpha(x,y) h'_\alpha(x,y)}$$

$$= n! \prod_{(x,y)\in\lambda} \frac{N + (\sqrt{\alpha}\, x - \sqrt{\alpha}^{-1}\, y) + (\sqrt{\alpha}^{-1} - \sqrt{\alpha})}{N \cdot h_\alpha(x,y) h'_\alpha(x,y)}.$$

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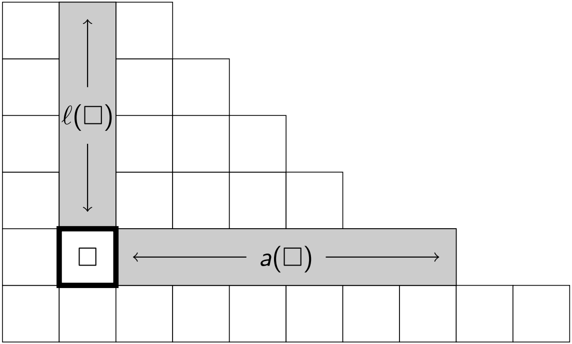
Jack deformation of hook-length formula



$$h_\alpha(\square) := \sqrt{\alpha}\; a(\square) + \sqrt{\alpha}^{-1}\; \ell(\square) + \sqrt{\alpha},$$

$$h'_\alpha(\square) := \sqrt{\alpha}\; a(\square) + \sqrt{\alpha}^{-1}\; \ell(\square) + \sqrt{\alpha}^{-1}.$$

Main result

Theorem (D., Śniady 2017)

For each n let $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ be a reducible Jack character, and let $\alpha = \alpha(n)$ be such that

$$\gamma := \sqrt{\alpha}^{-1} - \sqrt{\alpha} = g\sqrt{n} + g' + o(1)$$

for some $g, g' \in \mathbb{R}$. We impose that the sequence (χ_n) fulfills some technical assumptions about its asymptotic behavior; we will specify their details later.

Let λ_n be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$ associated with $\chi := \chi_n$. Then the sequence (λ_n) of Young diagrams converges to some limit shape in the limit $n \to \infty$ when the number of the boxes tends to infinity.
Furthermore, the fluctuations of λ_n around the limit shape are asymptotically Gaussian.

α-anisotropic Young diagrams

Definition

Anisotropic Young diagram $T_{w,h}(\lambda)$ - polygon obtained from the Young diagram λ by a horizontal stretching of ratio w and a vertical stretching of ratio h (each box 1×1 is replaced by a box of dimension $w \times h$).

$\lambda \mapsto T_{2,\frac{1}{2}}(\lambda)$

In order to study the shape of random Young diagrams $\lambda_n \in \mathbb{Y}_n$ sampled by some Jack-deformed measure, the right scaling is the following:

$$\Lambda_n := T_{\sqrt{\frac{\alpha}{n}},\sqrt{\frac{1}{\alpha n}}}\lambda_n.$$

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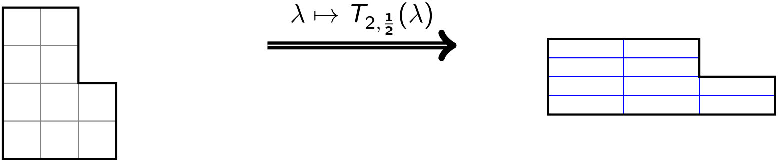

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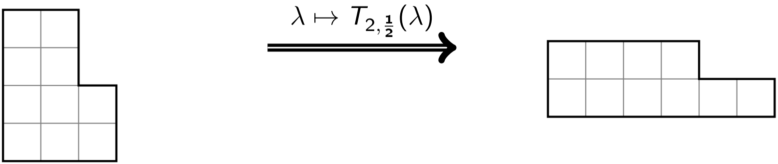

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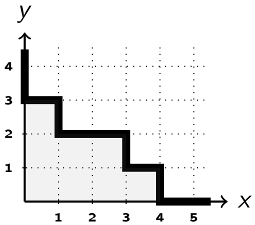
Young diagrams as continuous objects

French convention:



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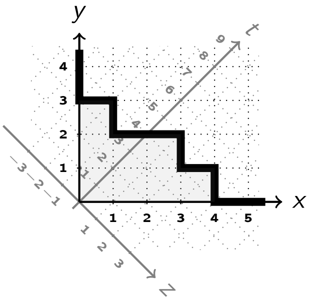

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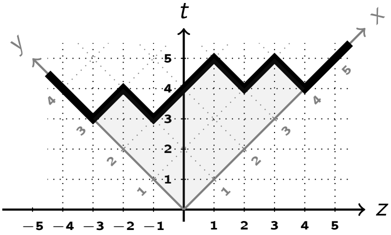

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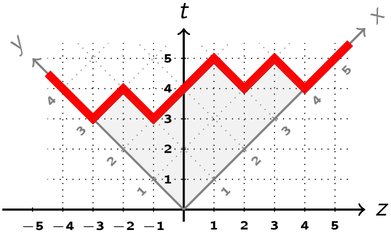

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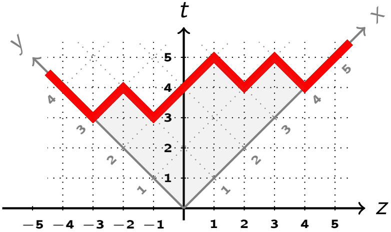

Definition

A profile of a Young diagram λ is a function $\omega_\lambda : \mathbb{R} \to \mathbb{R}_+$ such that its graph is a profile of λ drawn in Russian convention.

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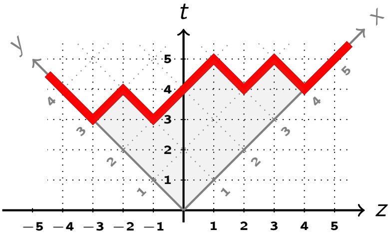

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When we claim that a sequence $(\lambda_n)_n$ of Young diagrams $\lambda_n \in \mathbb{Y}_n$ converges to some limit shape, we actually mean that the sequence of profiles ω_{Λ_n} converges.

Asymptotic shape of large Jack-deformed Young diagrams

Theorem (D., Śniady 2017; $\alpha = 1$ Biane 2002)

For each n let $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ be a reducible Jack character, and let $\alpha = \alpha(n)$ be such that

$$\gamma := \sqrt{\alpha}^{-1} - \sqrt{\alpha} = g\sqrt{n} + g' + o(1)$$

for some $g, g' \in \mathbb{R}$.
Let λ_n be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$ associated with reducible Jack-characters $\chi := \chi_n$ that fulfill some technical assumptions about its asymptotic behavior (presented in details later on).
Then there exists some deterministic function $\omega_{\Lambda_\infty} \colon \mathbb{R} \to \mathbb{R}$ with the property that

$$\lim_{n\to\infty} \omega_{\Lambda_n} = \omega_{\Lambda_\infty},$$

where the convergence holds true with respect to the supremum norm, in probability.

Examples

We recall that $\gamma = g\sqrt{n} + g' + o(1)$.

Example

When $\alpha > 0$ is fixed, that is $g = 0$ then the limit shape ω_{Λ_∞} does not depend on α!.

- Jack-Plancherel measure (D., Féray 2016)

$$\omega_{\Lambda_\infty}(x) = \begin{cases} |x| & \text{if } |x| \geq 2; \\ \frac{2}{\pi}\left(x \cdot \arcsin\frac{x}{2} + \sqrt{4-x^2}\right) & \text{otherwise.} \end{cases}$$

- Jack-Schur-Weyl measure with $\sqrt{n} \sim cN$ (D., Śniady 2017)

$$\omega_{\Lambda_\infty}(x) \text{ – explicit function depending on } c.$$

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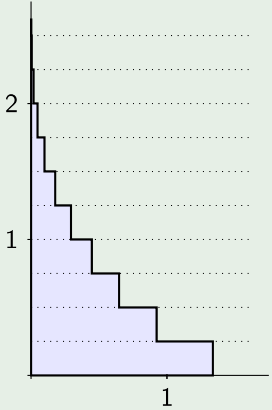
Example



An interesting choice is when $\alpha(n) = \frac{1}{c^2 n}$ for some $c > 0$, that is $g = c, g' = 0$. Then the anisotropic Young diagram Λ_n is a collection of rectangles of the same height g and of the widths $\frac{\lambda_1}{gn}, \frac{\lambda_2}{gn}, \ldots,$ and the limit shape ω_{Λ_∞} clearly depends on g!

The limit shape of random Young diagrams distributed according to the Jack–Plancherel measure in the double scaling limit for $c = \frac{1}{4}$.

Fluctuations

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How to "measure" fluctuations around the limit shape ω_{Λ_∞}?

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We know that $\omega_{\Lambda_n} \to \omega_{\Lambda_\infty}$, so we define

$$\Delta_n := \sqrt{n}\left(\omega_{\Lambda_n} - \omega_{\Lambda_\infty}\right).$$

We would like to show that Δ_n converges to some function Δ_∞, so informally speaking,

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We need to study suitable test functions:

$$Y_k := \frac{k-1}{2}\int u^{k-2}\, \Delta_n(u) du, \quad k \geq 2.$$

Central limit theorem

Theorem (D., Śniady 2017; $\alpha = 1$ Śniady 2006)

For each n let $\chi_n\colon \mathcal{P}_n \to \mathbb{R}$ be a reducible Jack character, and let $\alpha = \alpha(n)$ be such that

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for some $g, g' \in \mathbb{R}$.
Let λ_n be a random Young diagram with the probability distribution $\mathbb{P}_{\chi_n}$ associated with reducible Jack-characters $\chi := \chi_n$ that fulfill some technical assumptions about its asymptotic behavior (presented in details later on).

Then the random vector Δ_n converges in distribution to some (non-centered) Gaussian random vector Δ_∞ as $n \to \infty$.

Equivalently, the family of random variables $(Y_k)_{k\geq 2}$ converges as $n \to \infty$ to a (non-centered) Gaussian distribution.

Question

Problem

What are the proper assumptions about asymptotic behavior of reducible Jack characters which provide the law of large numbers and the central limit theorem?

Approximate factorization property

We extend the domain of $\chi_n \colon \mathcal{P}_n \to \mathbb{R}$ to the set $\bigsqcup_{0 \leq k \leq n} \mathcal{P}_k$ of partitions of sufficiently small numbers by setting

$$\chi_n(\pi) := \chi_n(\pi, 1^{n-|\pi|}) \qquad \text{for } |\pi| \leq n.$$

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The general idea of our assumptions is the following:

- the characters do not grow too fast:

$$\chi_n(\pi) = O(n^{-\frac{\|\pi\|}{2}}),$$

- characters on cycles have subleading terms of a proper order:

$$\chi_n((l))\; n^{\frac{l-1}{2}} = a_{l+1} + \frac{b_{l+1} + o(1)}{\sqrt{n}} \qquad \text{for } n \to \infty,$$

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Note that $\chi_n(\pi) = \mathbb{E}\big(\chi_{(\circ)}(\pi)\big)$ is, by definition, the expectation of the irreducible Jack characters $\chi_\lambda(\pi)$ taken with the probability $\mathbb{P}_{\chi_n}(\lambda)$.

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Cumulants $\kappa_\ell^{\mathbb{E}}(x_1, \dots, x_\ell)$ of random variables $x_1, \dots, x_\ell$ - natural generalization of a variance:

$$\begin{cases} \mathbb{E}(x_1) = \kappa_1^{\mathbb{E}}(x_1), \\ \mathbb{E}(x_1 x_2) = \kappa_2^{\mathbb{E}}(x_1, x_2) + \kappa_1^{\mathbb{E}}(x_1)\kappa_1^{\mathbb{E}}(x_2), \\ \mathbb{E}(x_1 x_2 x_3) = \kappa_3^{\mathbb{E}}(x_1, x_2, x_3) + \kappa_1^{\mathbb{E}}(x_1)\kappa_2^{\mathbb{E}}(x_2, x_3) \\ \qquad + \kappa_1^{\mathbb{E}}(x_2)\kappa_2^{\mathbb{E}}(x_1, x_3) + \kappa_1^{\mathbb{E}}(x_3)\kappa_2^{\mathbb{E}}(x_1, x_2) \\ \qquad + \kappa_1^{\mathbb{E}}(x_1)\kappa_1^{\mathbb{E}}(x_2)\kappa_1^{\mathbb{E}}(x_3), \\ \qquad \vdots \end{cases}$$

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Cumulants $\kappa_\ell^\chi(\pi_1 \dots \pi_\ell)$ of random variables $\chi_{(\circ)}(\pi_1), \dots, \chi_{(\circ)}(\pi_\ell)$ - natural generalization of a variance:

$$\begin{cases} \chi(\pi_1) = \kappa_1^\chi(\pi_1), \\ \chi(\pi_1\pi_2) = \kappa_2^\chi(\pi_1, \pi_2) + \kappa_1^\chi(\pi_1)\ \kappa_1^\chi(\pi_2), \\ \chi(\pi_1\pi_2\pi_3) = \kappa_3^\chi(\pi_1, \pi_2, \pi_3) + \kappa_1^\chi(\pi_1)\ \kappa_2^\chi(\pi_2, \pi_3) \\ \qquad + \kappa_1^\chi(\pi_2)\ \kappa_2^\chi(\pi_1, \pi_3) + \kappa_1^\chi(\pi_3)\ \kappa_2^\chi(\pi_1, \pi_2) \\ \qquad + \kappa_1^\chi(\pi_1)\ \kappa_1^\chi(\pi_2)\ \kappa_1^\chi(\pi_3), \\ \vdots \end{cases}$$

Approximate factorization property revisited

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Examples (Of measures with AFP, thus CLT)

- Jack-Plancherel measure ($\alpha > 0$ fixed, D., Féray 2016)

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More examples

Theorem

Let $\left(\chi_n^1\right), \left(\chi_n^2\right)$ be two families of reducible Jack characters with approximate factorization property. Then all the families consists of reducible Jack characters with approximate factorization property:

- *the* restriction $\left(\chi_{q,n}^i\right) := \left(\left(\chi_{q_n}^i\right)^{\downarrow_n^{q_n}}\right)$, *where* $q_n \geq n$ *and* $\lim_{n\to\infty} \frac{q_n}{n} = q$;
- *the* induction $\left(\chi_{q,n}^{(i}\right) := \left(\left(\chi_{q_n}^i\right)^{\uparrow_n^{q_n}}\right)$, *where* $q_n \leq n$ *and* $\lim_{n\to\infty} \frac{q_n}{n} = q$;
- *the* outer product
$$\left(\chi_n\right) := \left(\chi^1_{q_n^{(1)}} \circ \chi^2_{q_n^{(2)}}\right),$$
where $q_n^{(1)} + q_n^{(2)} = n$ *and the limits* $q^{(i)} := \lim_{n\to\infty} \frac{q_n^{(i)}}{n}$ *exist;*
- *the* tensor product
$$\left(\chi_n\right) := \left(\chi_n^1 \cdot \chi_n^2\right).$$

The main tool

Our main tool for proving above theorems are certain results on the structure of the algebra of polynomial functions $\mathscr{P}$.

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We define the normalized Jack character $\mathrm{Ch}_\pi^{(\alpha)} : \mathbb{Y} \to \mathbb{Q}[\sqrt{\alpha}, \sqrt{\alpha}^{-1}]$:

$$\mathrm{Ch}_\pi^{(\alpha)}(\lambda) := \begin{cases} |\lambda|^{\underline{|\pi|}} \, \chi_\lambda^{(\alpha)}(\pi) & \text{if } |\lambda| \geq |\pi|; \\ 0 & \text{if } |\lambda| < |\pi|. \end{cases}$$

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The algebra of polynomial functions $\mathscr{P}$ is spanned by the elements of the form $\gamma^k \ \mathrm{Ch}_\pi$, where $k \in \mathbb{N}, \pi \in \mathcal{P}$. This algebra is graded:

$$\deg(\gamma^k \, \mathrm{Ch}_\pi) = k + \|\pi\|.$$

Equivalent characterization of characters with AFP

Theorem (D., Śniady 2017; $\alpha = 1$ Śniady 2006)

- *for each integer* $\ell \geq 1$ *and all integers* $l_1, \dots, l_\ell \geq 2$ *the limit*

$$\lim_{n\to\infty} \kappa_\ell^{\chi_n}\big((l_1), \dots, (l_\ell)\big)\ n^{\frac{l_1+\cdots+l_\ell+\ell-2}{2}}$$ *exists and is finite;*

- *for each integer* $\ell \geq 1$ *and all* $x_1, \dots, x_\ell \in \{\mathrm{Ch}_2, \mathrm{Ch}_3, \dots\}$ *the limit*

$$\lim_{n\to\infty} \kappa_\ell^{\chi_n}(x_1, \dots, x_\ell)\ n^{-\frac{\deg x_1+\cdots+\deg x_\ell-2(\ell-1)}{2}}$$ *exists and is finite;*

- *for each integer* $\ell \geq 1$ *and all* $x_1, \dots, x_\ell \in \mathscr{P}$ *the limit*

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- *for each integer* $\ell \geq 1$ *and all* $x_1, \dots, x_\ell \in \mathscr{P}_\bullet$ *the limit*

$$\lim_{n\to\infty} \kappa_{\bullet\ell}^{\chi_n}(x_1, \dots, x_\ell)\ n^{-\frac{\deg x_1+\cdots+\deg x_\ell-2(\ell-1)}{2}}$$ *exists and is finite.*

Two different cumulants:

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$\mathscr{P}$ and $\mathscr{P}_{\bullet}$ are the same rings, but the multiplication is different:

$$(\gamma^p \operatorname{Ch}_\pi) \cdot (\gamma^q \operatorname{Ch}_\sigma) \text{ vs. } (\gamma^p \operatorname{Ch}_\pi) \bullet (\gamma^q \operatorname{Ch}_\sigma) := \gamma^{p+q} \operatorname{Ch}_{\pi\sigma} .$$

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$$\begin{cases} \mathbb{E}_{\chi_n}(x_1) = \kappa_1^{\chi_n}(x_1), \\ \mathbb{E}_{\chi_n}(x_1 \cdot x_2) = \kappa_2^{\chi_n}(x_1,x_2) + \kappa_1^{\chi_n}(x_1)\kappa_1^{\chi_n}(x_2), \\ \mathbb{E}_{\chi_n}(x_1 \cdot x_2 \cdot x_3) = \kappa_3^{\chi_n}(x_1,x_2,x_3) + \kappa_1^{\chi_n}(x_1)\kappa_2^{\chi_n}(x_2,x_3) \\ \qquad + \kappa_1^{\chi_n}(x_2)\kappa_2^{\chi_n}(x_1,x_3) + \kappa_1^{\chi_n}(x_3)\kappa_2^{\chi_n}(x_1,x_2) \\ \qquad + \kappa_1^{\chi_n}(x_1)\kappa_1^{\chi_n}(x_2)\kappa_1^{\chi_n}(x_3), \\ \vdots \end{cases}$$

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$$\mathbb{E}_\chi \operatorname{Ch}_\pi = \begin{cases} |\lambda|^{\underline{|\pi|}} \, \chi(\pi) & \text{if } |\lambda| < |\pi|, \\ 0 & \text{otherwise.} \end{cases}$$

Relation between our main result and the algebra $\mathscr{P}$

We recall that we need to study the random variables:

$$Y_k := \frac{k-1}{2} \int u^{k-2} \, \Delta_n(u) du$$

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Proposition

Functionals $\mathcal{S}_2^{(\alpha)}, \mathcal{S}_3^{(\alpha)}, \dots \in \mathscr{P}$, *and* $\deg(\mathcal{S}_k^{(\alpha)}) = k$.

The outline of the proof

Proof.

- for $\ell \geq 3$ one has

$$\lim_{n\to\infty} \kappa_\ell^{\mathbb{E}_{\chi_n}}(Y_{l_1},\dots,Y_{l_\ell}) = \lim_{n\to\infty} \kappa_\ell^{\chi_n}(\mathcal{S}_{l_1}^{(\alpha)},\dots,\mathcal{S}_{l_\ell}^{(\alpha)})n^{\frac{\ell-l_1-\cdots-l_\ell}{2}} = 0;$$

- $\lim_{n\to\infty} \kappa_2^{\mathbb{E}_{\chi_n}}(Y_{l_1},Y_{l_2}) = \lim_{n\to\infty} \kappa_2^{\chi_n}(\mathcal{S}_{l_1}^{(\alpha)},\mathcal{S}_{l_2}^{(\alpha)})n^{\frac{2-l_1-l_2}{2}}$ exists and is finite;
- if

$$\mathbb{E}_{\chi_n}\left(\mathcal{S}_l^{(\alpha)}\right) n^{-\frac{l}{2}} = a_l + \frac{b_l + o(1)}{\sqrt{n}}, \text{ then}$$

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Perspectives

- What can we say about the limit shape of the Jack-Plancherel measure (or other measures given by reduced characters) in the double scaling limit?
- In order to find a covariance of normal distribution in the double scaling limit, we need to find a formula for the top-degree of normalized Jack characters indexed by a partition with two rows - conjecturally we need to understand the combinatorics of unhandled maps with two faces.
- Let $x_1 \geq x_2 \geq \dots$ be a sequence of eigenvalues of GβE. Then, after proper normalization, their joint distribution is known (β-Tracy-Widom). What about the joint distribution of properly normalized $\left(\lambda_{(n)}\right)_1 \geq \left(\lambda_{(n)}\right)_2 \geq \dots$ with respect to Jack-Plancherel measure (conjecturally should be the same!)?

Thank you

THANK YOU FOR YOUR ATTENTION!