Asymptotic results in enumerative and algebraic combinatorics
Excercises
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Maciej Dołęga, IM PAN

1. The purpose of this exercise is to get familiar with the RSK algorithm and its beautiful properties.

(a) Open SageMath and try to play a little bit with the RSK algorithm. When you represent permutations as the permutation matrices, you can see that there are some natural operations on the set of permutation matrices inherited from the symmetries of a square (the dihedral group naturally acts on permutation matrices). Make experiments in SageMath and try to formulate some conjectures describing the action of the RSK applied to permutations obtained by these symmetries. Can you deduce a statistic on permutations that corresponds to the length of the first column of the associated (via RSK) Young diagram?

(b) Let $\sigma \in \mathfrak{S}_n$, and let λ be the associated Young diagram via RSK. Prove that $\ell(\sigma) = \lambda_1$. Can you formulate a similar interpretation of λ_2 (and $\lambda_3, \lambda_4, \dots$ - experiment in SageMath!)?

2. Let d_λ be the number of standard Young tableaux (SYT) of shape λ. Prove the hook-length formula:

$$d_\lambda = \frac{n!}{\prod_{(i,j)\in\lambda} \mathrm{hook}_\lambda(i,j)} \tag{1}$$

following the probabilistic argument below:

(a) Let e_λ be the RHS of (1). Prove that (1) is equivalent to proving

$$\sum_{\mu\colon \mu\nearrow\lambda} \frac{e_\mu}{e_\lambda} = 1 \tag{2}$$

with the initial condition $e_\emptyset = 1$, where $\mu \nearrow \lambda$ means that μ is obtained from λ by removing a box.

(b) Let $\mathbb{O}_\lambda$ be the set of outer corners of λ, i.e. the set of boxes such that one can remove from λ to obtain a smaller Young diagram μ. Define a probability measure on $\mathbb{O}_\lambda$ by the following process, called the *hook-walk*: choose a box $\square$ in λ uniformly at random. Then choose the next box from the set of boxes belonging to the hook with the corner at $\square$, again uniformly (i.e. with probability $(\mathrm{hook}_\lambda(\square)-1)^{-1}$, because we do not allow to choose the same box $\square$ that we have just chosen), and continue until you choose a box belonging to $\mathbb{O}_\lambda$. Prove that the probability that this process terminates at $(i,j)\in\mathbb{O}_\lambda$ is equal to $\frac{e_{\lambda\setminus(i,j)}}{e_\lambda}$, and deduce (2).

(c) Notice that this process can be used to generate a uniform standard Young tableau of a given shape. Try to code it in SageMath, and experiment with generating various random Young tableaux. Maybe you can observe something interesting?

3. In this excercise we will see that the asymptotic growth of $\mathbb{E}(\ell(\sigma^{(n)}))$ is of order $\sqrt{n}$, as $n\to\infty$, where $\sigma^{(n)}$ is a uniformly random permutation from the symmetric group $\mathfrak{S}_n$.

(a) Let $n>rs$, and let $\sigma\in\mathfrak{S}_n$ be a permutation. Prove that either the length of the longest increasing subsequence $\ell(\sigma)>r$, or the length of the longest decreasing subsequence $d(\sigma)>s$.

(b) Deduce that $\mathbb{E}(\ell(\sigma^{(n)}))\geq\sqrt{n}$.

[Hint:] use the inequality $\frac{a+b}{2}\geq\sqrt{ab}$.

(c) Prove that for any $\alpha>\exp(1)$ one has

$$\mathbb{P}(\ell(\sigma^{(n)})>\alpha)\leq C\exp(-c\sqrt{n})$$

for some $C,c>0$ that do not depend on n (they might depend on α).

[Hint:] Consider the following random variables: $X_k(\sigma^{(n)})$ = number of increasing subsequences of length k in a random permutation $\sigma^{(n)}$, and use the fact that $\mathbb{P}(X_k(\sigma^{(n)})\geq 1)\leq\mathbb{E}(X_k(\sigma^{(n)})$. Apply this to find a bound for $\mathbb{P}(\ell(\sigma^{(n)})>k)$ for $k=[\sqrt{n}(1+\delta)\exp(1)]$.

4. The purpose of this exercise is to partially describe how to prove that

$$\ell\left(\frac{\sigma^{(n)}}{\sqrt{n}}\right)\xrightarrow{p}\Lambda$$

for some constant Λ. It follows from the fact that $\ell(\sigma^{(n)})$ can be understood using the Poisson point process, and from the fact that $\mathrm{Var}(\ell(\sigma^{(n)}))\leq C\sqrt{n}$ for some C. We are not going to discuss the relationship with the Poisson point process, but we will prove the bound for the variance. In order to do this, we will need the following theorem.

Theorem 1 (Elfron–Stein–Steele inequality) *Let, $g \in C(\mathbb{R}^n), g_j \in C(\mathbb{R}^{n-1})$ be continuous functions for $1 \le j \le n$, and let $X_1, \dots, X_n$ be random variables. Then*

$$\mathrm{Var}(Z) \le \sum_{1 \le j \le n} \mathbb{E}(Z - Z_j)^2,$$

where $Z := g(X_1, \dots, X_n)$, and $Z_j := g_j(X_1, \dots, \hat{X}_j, \dots, X_n)$.

(a) For any real numbers $x_1, \dots, x_n$ let $\ell(x_1, \dots, x_n)$ denote the length of the longest increasing subsequence in the word $x_1, \dots, x_n$. Note that $\ell(\sigma^{(n)})$ has the same distribution as $\ell(X_1, \dots, X_n)$, where X_i are independent r.v. uniformly distributed on the interval $[0, 1]$. Let $Z_j = \ell(X_1, \dots, \hat{X}_j, \dots, X_n)$. Show that $Z - Z_j = \mathbf{1}_{E_j}$, where $\mathbf{1}_{E_j}$ is the indicator function (i.e. equal to 1 or 0, depending on whether E_j is true or not) of the following event E_j: X_j participates in all maximal-length increasing subsequences in $X_1, \dots, X_n$.

(b) Prove that $\sum_{1 \le j \le n} \mathbf{1}_{E_j} \le Z$, and deduce that $\mathrm{Var}(\ell(\sigma^{(n)})) \le C\sqrt{n}$ by applying Theorem 1.

5. The purpose of this exercise is to define the *Plancherel growth process* that allows to sample large Young diagrams w.r.t the Plancherel measure.

(a) Prove that for $\mu \nearrow \lambda \vdash n$ one has

$$\mathbb{P}(\lambda^{(n)} = \lambda | \lambda^{(n-1)} = \mu) = \frac{d_\lambda}{n d_\mu}.$$

(b) Deduce that $\mathbb{E}\ell(\sigma^{(n)}) - \mathbb{E}\ell(\sigma^{(n-1)}) \le \sqrt{n}^{-1}$, and conclude that $\mathbb{E}\ell(\sigma^{(n)}) \le 2\sqrt{n}$.

[Hint:] Apply the identity obtained in (a) to analyse the difference $\mathbb{E}\ell(\sigma^{(n)}) - \mathbb{E}\ell(\sigma^{(n-1)})$ via RSK. Use the Cauchy–Schwartz identity.

(c) Let $\mathbb{I}_\lambda$ denote the set of inner corners of λ, i.e. the set of corners that can be added to λ to obtain a larger Young diagram. For a box (i, j) define its content $c(i, j) := j - i$. Prove that for $\mu \nearrow \lambda$ one has

$$\frac{d_\lambda}{n d_\mu} = \frac{\prod_{o \in \mathbb{O}_\mu} (c(x) - c(o))}{\prod_{x \ne i \in \mathbb{I}_\mu} (c(x) - c(i))},$$

where $x = \lambda \setminus \mu$.

(d) Use the formula proved in (c) and (a) to generate random Plancherel-distributed Young diagrams in SageMath, using the Plancherel growth process.

6. Fix a partition $\lambda \vdash n$, and let $N \geq \ell(\lambda)$. Define $\ell_i := \lambda_i + n - i$ for $1 \leq i \leq N$.

(a) Prove the following formula

$$[s_\lambda]p_{(k,1^{n-k})} = (n-k)! \sum_{j=1}^{N} \frac{\mathrm{Vand}(\ell_1, \dots, \ell_j - k, \dots, \ell_N)}{\prod_{i \neq j} \ell_i!(\ell_j - k)!},$$

where $\mathrm{Vand}(x_1, \dots, x_k) := \det(x_i^{j-1})_{1 \leq i,j \leq k}$ is the Vandermonde determinant. [Hint:] Use the bialternant formula.

(b) Apply this formula to find an alternative proof of the hook-length formula (1).

7. Recall that for a probability measure μ on $\mathbb{R}$ with all moments finite, the Cauchy transform $G_\mu(z)$ is defined as

$$G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{z - x} = \sum_{i \geq 0} z^{-i-1} M_i^\mu,$$

where $M_i^\mu := \int_{\mathbb{R}} x^i \, d\mu(x)$ is the i-th moment. Define the i-th cumulant κ_i^μ, and the i-th free cumulant R_i^μ by the following formal power series:

$$K_\mu(z) := \log \int \exp(zx) \, d\mu(x) = \sum_{i \geq 1} \kappa_i^\mu \frac{z^i}{i!},$$

$$R_\mu(z) := G_\mu^{-1}(z) - z^{-1} = \sum_{i \geq 1} R_i^\mu z^{i-1}.$$

(a) Prove the following moment-cumulant formulae:

$$M_n^\mu = \sum_{\pi \in \mathrm{Part}(n)} \prod_{B \in \pi} \kappa_{|B|}^\mu,$$

$$M_n^\mu = \sum_{\pi \in \mathrm{NPart}(n)} \prod_{B \in \pi} R_{|B|}^\mu,$$

where $\mathrm{Part}(n)$ denotes the set of partitions of size n, i.e. $\pi \in \mathrm{Part}(n)$ consists of pairwise disjoint subsets of $[n] := \{1, 2, \dots, n\}$ whose union is the whole set $[n]$. Similarly, $\mathrm{NPart}(n)$ denotes the set of noncrossing partitions of size n, i.e. partitions with the property that there are no $i < j < k < l$ such that

i,k belong to the same block, which is different than the block that contains j,l (make a drawing to see why they are "non-crossing").

(b) Prove that the Gaussian distribution $\mu(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{x^2}{2})$ is the unique probability measure with the sequence of cumulants $\kappa_n = \delta_{n,2}$, and with the sequence of moments $M_{2n} = (2n-1)!!$ and 0 for odd indices. Prove that the semicircle law $\mu(x) = \frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(x)$ is the unique probability measure with the sequence of free cumulants $R_n = \delta_{n,2}$, and with the sequence of moments $M_{2n} = \frac{1}{n+1}\binom{2n}{n}$ and 0 for odd indices. Could you give a combinatorial interpretation of these identities?

8. This excercise serves as an introduction to *the algebra of polynomial functions on Young diagrams.* Define $X_\lambda = \{c(\square) \colon \square \in \mathbb{I}_\lambda\}$, and $Y_\lambda = \{c(\square) \colon \square \in \mathbb{O}_\lambda\}$. Check that X_λ interlaces with Y_λ. Recall that the transition measure μ_λ associated with λ is uniquely determined by its Cauchy transform:

$$G_{\mu_\lambda}(z) = \frac{\prod_{y\in Y_\lambda}(z-y)}{\prod_{x\in X_\lambda}(z-x)}.$$

For a symmetric function f, we define the so-called plethystic substitution $f[X-Y]$. We do this by declaring the value on the power-sum symmetric functions:

$$p_i[X-Y] = \sum_{x\in X} x^i - \sum_{y\in Y} y^i,$$

and we extend it on the whole algebra $\mathbb{C}[p_1, p_2 \dots] \ni f$.

(a) Prove that $M_k(\lambda) := M_k^{\mu_\lambda} = h_k[X_\lambda - Y_\lambda]$, where $h_k = s_{(k)}$ is the complete homogeneous symmetric function. This allows us to consider moments $M_k(\lambda)$ as functions on the set of Young diagrams.

(b) Conclude that the algebra generated by $(M_i)_{i\geq 2}$ is the same as the algebra generated by $(R_i)_{i\geq 2}$ (over $\mathbb{C}$) (we call it the *algebra of polynomial functions on Young diagrams*).