REPRESENTATION THEORY OF FINITE GROUPS

G - finite group.

<u>Idea:</u> G is a group of symmetries of some geometric object.

$$\rho : G \xrightarrow{} GL(V)$$

$\uparrow$ group homomorphism $\uparrow$ vector space $/K$

ρ is called **representation**

Another language (R-modules).

R - ring. M is an (left) R-module if $(M,+)$ is an abelian group equiped with $\cdot : R \times M \to M$ s.t.

1. $r\cdot(x+y) = r\cdot x + r\cdot y$
2. $(r_1+r_2)\cdot x = r_1\cdot x + r_2 \cdot x$
3. $r_1\cdot(r_2\cdot x) = (r_1\cdot r_2)\cdot x$
4. $1\cdot x = x$. $\forall x,y \in M,\ r_1, r_2, r \in R$.

The group algebra $K[G] = \operatorname{span}_K(g : g \in G)$

with the multiplication $\left(\sum a_g \cdot g\right)\cdot\left(\sum b_g \cdot g\right) =$

$$= \sum_{g,h} a_g \cdot b_h (g\cdot h).$$

Representations of G over K $\equiv$ $K[G]$ - modules

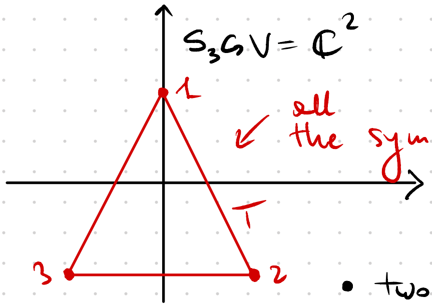
(G-modules in short)

$\rho : G \to GL(V) \longleftrightarrow V$ with the structure

$$\sum_g a_g g(v) := \sum_g a_g \cdot \rho(g) v$$

Examples: ✪ S_3 – permutation group.

$S_3 \circlearrowleft V = \mathbb{C}^2$



all the symmetries of T

$\rho : S_3 \hookrightarrow GL(\mathbb{C}^2)$

irreducible

- two 1-dim: $S_3 \ni \sigma \mapsto 1$; $S_3 \ni \sigma \longmapsto \mathrm{sgn}(\sigma)$

✪ $S_n \circlearrowleft \mathbb{C}^n$ by permuting coordinates:

$$\rho(\sigma)(x_1, \dots, x_n) := (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

This is a representation of dimension n, but it is not irreducible:

$$\mathbb{C}^n = \underbrace{\mathbb{C}(1, \dots, 1)}_{= V_1} \oplus \underbrace{\mathbb{C}(1, \dots, 1)^{\perp}}_{= V_2}$$

Then $\rho(S_n) V_1 \subseteq V_1$, $\rho(S_n) V_2 \subseteq V_2$

✪ Suppose that G acts on the set X.

Then X has a structure of G-module, called the permutation representation

For instance $S_n \circlearrowleft \{1, \dots, n\} =: X$

$$K[X] = \left\{ \sum_{i=1}^{n} c_i \cdot i \;:\; c_i \in K \right\}$$

$\pi \in S_n$ $$\pi\left(\sum_{i=1}^{n} c_i \cdot i\right) = \sum_{i=1}^{n} c_i \, \pi(i) \in K[X]$$

(*) G acts on itself by left multiplication. Therefore $K[G]$ is a $K[G]$-module and defines a repr. called the (left) regular representation

(*) G - abelian group

$\rho : G \to Gl(V)$ - irr. repr.

Then $\dim(V) = 1$.

P-f: Suppose that $\phi : V \to V$

s.t. $\phi \circ \rho = \rho \circ \phi$ (i.e. $\forall g \in G\ \forall v \in V$ $\phi(\rho(g)v) = \rho(g)\phi(v)$)

Then, by Schur Lemma, $\phi = \lambda \cdot Id_V$ for some $\lambda \in \mathbb{C}$.

Fix $g \in G$, and let $\phi := \rho(g)$. Then

$\phi \circ \rho = \rho \circ \phi \Rightarrow \rho(g) = \lambda_g \cdot Id_V$

$\Rightarrow \dim(V) = 1$ because ρ is irreducible.

Def: V - an R-module. $W < V$ is a submodule if it is an R-module (with an induced structure) and $W \subseteq V$. Every module has two trivial submodules: $\{0\}$ and V.

V is called reducible if it contains a non-trivial submodule. Otherwise it is called irreducible.

$\rho_1 : G \to Gl(V)$ is isomorphic to $\rho_2 : G \to Gl(W)$ if $\exists\ \phi : V \to W$ s.t.

$\phi \circ \rho_1 = \rho_2 \circ \phi$

$V \cong W$ if $\exists\ \phi : V \to W$ which is $K[G]$-homomorphism

MASCHKE'S THEOREM

Th: Let V - finite dim. G-module. Then $V = W^{(1)} \oplus \cdots \oplus W^{(d)}$, where $W^{(i)}$ - irreducible G-module.

P-f: Induction on $\dim V =: d$

$d = 1$ ✓

$d > 1$. V - Irr. $\Rightarrow$ done.

Suppose $W < V$ ← submodule.

Let $\langle , \rangle$ be some scalar product on V.

Define $\langle v, w \rangle' := \frac{1}{|G|} \sum_{g \in G} \langle gv, gw \rangle$.

Then $\langle v, w \rangle' = \langle gv, gw \rangle' \quad \forall g \in G, \ \forall w, v \in V$.

Define $W^{\perp} := \{ v \in V : \langle v, w \rangle' = 0 \ \forall w \in W \}$

$W^{\perp}$ is a G module and $V = W \oplus W^{\perp}$.

Therefore $W = W^{(1)} \oplus \cdots \oplus W^{(k_1)}$

$W^{\perp} = W^{(k_1+1)} \oplus \cdots \oplus W^{(k_1+k_2)}$ and $V = W^{(1)} \oplus \cdots \oplus W^{(k_1+k_2)}$ □

SCHUR'S LEMMA (very simple, super strong):

Lemma: V, W - irreducible R-modules. $f \in \mathrm{Hom}_R(V, W)$

Then (1) either $f \equiv 0$, or f - isomorphism.

(2) If $R = \mathbb{C}[G]$, and $V = W$ then $f = \lambda \cdot \mathrm{Id}$, $\lambda \in \mathbb{C}$.

P-f: (1) Consider $\ker(f) < V$. (R-module → ex.)

V - irr $\Rightarrow$ $\ker(f) = 0$ or $\ker(f) = V$.

Suppose $\ker(f) = 0$, consider $\mathrm{Im}(f) < W$ (R-mod → ex.)

$\mathrm{Im}(f) \equiv W$. □

(2). Let $\lambda \in \mathbb{C}$ be an eigenvalue of f (it exists because $\mathbb{C}$ is alg. closed.)

Consider $f' = f - \lambda \cdot \mathrm{Id}$. $f' \in \mathrm{Hom}_{\mathbb{C}[G]}(V,V)$

$\Rightarrow f' = 0$ (because $\ker(f') = \{v \in V : f(v) = \lambda v\} \neq 0$). $\square$

G_{irr} – set of irreducible rep. of G up to $\cong$ over $\mathbb{C}$.

GOAL: Understand G_{irr}

CHARACTERS

Let $\rho : G \longrightarrow Gl(V)$ a repr.

We define a character of ρ as

$$\chi_\rho(g) := \underset{\substack{\uparrow \\ \text{trace}}}{\mathrm{Tr}}(\rho(g))$$

$$\chi_\rho : G \longrightarrow \mathbb{C}.$$

Idea: Classification of irr. rep.

$\Longleftrightarrow$ classification of characters

Lemma: (1) $\chi_\rho(\mathrm{id}) = \dim V$

(2) $\chi_\rho(g) = \chi_\rho(hgh^{-1}) \quad \forall h, g \in G$

(3) ρ_1, ρ_2 – equivalent rep. $\Rightarrow \chi_{\rho_1} = \chi_{\rho_2}$.

Sometimes it is convenient to think that

$$\mathbb{C}[G] \equiv \{ f : G \longrightarrow \mathbb{C} \}$$
$$\sum a_g \cdot g \longmapsto f(g) = a_g.$$

Define $\langle \cdot , \cdot \rangle$ on $\mathbb{C}[G]$ by

$$\langle \psi , \chi \rangle := \frac{1}{|G|} \sum_g \psi(g) \chi(g^{-1})$$
$$= \frac{1}{|G|} \sum_g \psi(g) \overline{\chi(g)}.$$

Lemma: ρ_1, ρ_2 - irred. rep. of G.

$$\langle \chi_{\rho_1}, \chi_{\rho_2} \rangle = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2 \\ 0 & \text{otherwise} \end{cases}$$

CLASSIFICATION THEOREM:

Let $C(G) \equiv$ Conjugacy classes of G.

★ G_{irr} is indexed by $C(G)$
$$\mu \longmapsto V_\mu \in G_{irr}$$

★ $\mathbb{C}[G] = \bigoplus_{\mu \in C(G)} \dim(V_\mu) V_\mu$

★ $|G| = \sum_{\mu \in C(G)} \dim(V_\mu)^2$.

Ex: We saw 3 irr. rep. of S_3: 2 1-dim, one 1 2-dim. Since
$$1^2 + 1^2 + 2^2 = 6 = |S_3|,$$
these are all from $(S_3)_{irr}$.