REPRESENTATION THEORY OF THE SYMMETRIC GROUP

Recall: S_n^{irr} is parametrized by $C(S_n)$.

$\sigma_1 \sim \sigma_2 \iff$ they have the same cycle-type.

Cycle types are parametrized by partitions:

Partition of $n \in \mathbb{Z}_{\geq 1}$: way of writing n as a non-ordered sum of positive integers.

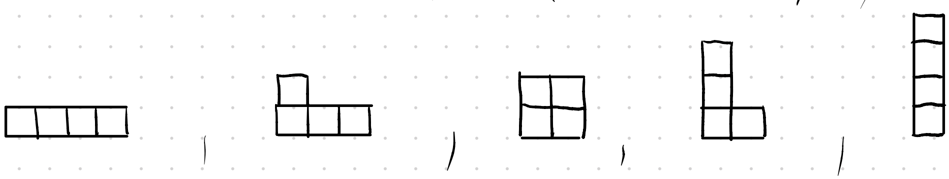
$$\lambda = (\lambda_1, \dots, \lambda_\ell) \quad \lambda_i \geq \lambda_{i+1} \quad \sum_{i=1}^{\ell} \lambda_i = n$$

$\ell(\lambda) := \ell$.

Young diagram — graphical representation of a partition:

Ex: Partitions of 4:

$\lambda = (4)$, $\lambda = (3,1)$, $\lambda = (2,2)$, $\lambda = (2,1,1)$, $\lambda = (1,1,1,1)$



IRREDUCIBLE CHARACTERS OF S_N via SYMMETRIC FUNCTIONS

Recall: V – S_n-module, W – S_m-module $\Rightarrow$ $V \otimes W$ – $(S_n \times S_m)$-modules

Define $V \boxtimes W$ an S_{n+m}-module.

$$V \boxtimes W := V \otimes W \uparrow_{S_n \times S_m}^{S_{n+m}}.$$

outer product

Quick comment on induced representations:

Let $H < G$, and $\rho : H \longrightarrow Gl(V)$.

$G = g_1 H \cup g_2 H \cup \cdots \cup g_k H$

$\rho\uparrow_H^G : G \longrightarrow \mathrm{End}(k \cdot V)$ $\qquad \left(\rho\uparrow_H^G(g)\right)_{i,j} = \rho(g_i^{-1} g g_j)$,

Induced repr.

where $\rho(g) := 0$ for $g \notin H$.

More abstractly: $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} V$.

Product $\boxtimes$ induces an outer product of characters

$$\chi_V \circ \chi_W := \chi_{V \boxtimes W}.$$

$CF := \bigoplus CF_n$ $\leftarrow$ class functions on S_n

$(CF, \circ)$ – a ring

GOAL: understand $(CF, \circ)$.

Def: Symetric polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$

Is a polynomial such that $f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$
$\forall \sigma \in S_n$

Let Λ_n^k – set of homogenous sym. pol. of degree k in n variables

$$\Lambda_n^k \longleftarrow \Lambda_{n+1}^k$$
$$f(x_1, \dots, x_n, 0) \longleftarrow\!\shortmid f(x_1, \dots, x_n, x_{n+1})$$

Then $\Lambda^k := \varprojlim_n \Lambda_n^k$, and

$\Lambda := \bigoplus_{k \geq 0} \Lambda^k$ is a ring of symmetric functions.

Ex: $p_k(x_1, \dots, x_n) := \sum_{i=1}^{n} x_i^k$ – power-sum symmetric polynomial

$$p_k = \sum_{i=1}^{\infty} x_i^k \quad \left(p_k(x_1, \dots, x_n) = p_k(x_1, \dots, x_n, 0, 0, \dots) \right).$$

Fact $\Lambda = \mathbb{C}[p_1, p_2, \dots]$

Schur polynomial: Fix λ, and let $n \geq \ell(\lambda)$.

$$s_\lambda(x_1, \dots, x_n) = \frac{\det\left(x_i^{\lambda_j + n - j}\right)_{1 \leq i,j \leq n}}{\det\left(x_i^{n-j}\right)_{1 \leq i,j \leq n}}$$

Jacobi's bialternant formula
Weyl's character formula

$T : \lambda \longrightarrow \mathbb{N}_{\geq 1}$ - filling of cells of λ by positive integers

T - Young tableaux $\equiv$

When T is filled bijectively by $\{1, \dots, |\lambda|\}$, we denote

$$T \in Y(\lambda)$$

When the entries are weakly increasing in each row (left to right) and strictly increasing in each column (bottom to top) then T is called a standard tableau, and we denote it $T \in SSYT(\lambda)$.

If $T \in SSYT(\lambda) \cap Y(\lambda)$ then T is called standard (denote $T \in SYT(\lambda)$)

Let $s_\lambda = \sum_{T \in SSYT(\lambda)} x^T$.

Then $s_\lambda(x_1, \dots, x_n) = s_\lambda(x_1, \dots, x_n, 0, 0, \dots)$

<u>Ex:</u> $\lambda = (2,1)$

$i < j < k$

$$\begin{array}{|c|c} \hline j & \\ \hline i & i \\ \hline \end{array} + \begin{array}{|c|c} \hline j & \\ \hline i & j \\ \hline \end{array} + \begin{array}{|c|c} \hline k & \\ \hline i & j \\ \hline \end{array} + \begin{array}{|c|c} \hline j & \\ \hline i & k \\ \hline \end{array}$$

$$\Rightarrow s_{(2,1)} = 2 \sum_{i<j<k} x_i x_j x_k + \sum_{i<j} \left(x_i^2 x_j + x_i x_j^2 \right)$$

$$\det \begin{pmatrix} x_1^4 & x_1^2 & 1 \\ x_2^4 & x_2^2 & 1 \\ x_3^4 & x_3^2 & 1 \end{pmatrix} \Big/ \prod_{i>j} (x_i - x_j)$$

$$ch : CF_n \longrightarrow \Lambda_n$$

Frobenius characteristic:

$$ch(f) := \frac{1}{n!} \sum_{\sigma \in S_n} f(\sigma)\, p_{ct(\sigma)} = \sum_{\mu \vdash n} z_\mu^{-1} f(\mu) p_\mu$$

FROBENIUS – SCHUR CORRESPONDENCE

Theorem: $ch : (CF, \circ) \rightarrow (\Lambda, \cdot)$ is an isomorphism.

Moreove $ch(\chi_\lambda) = s_\lambda$

irreducible character of S^λ

Corollary: $s_\lambda = \sum_{\mu \vdash n} \frac{\chi_\lambda(\mu)}{z_\mu} \cdot p_\mu$, where

$\chi_\lambda(\mu) := \chi_\lambda(\sigma)$ for any σ of cycle-type μ.

Corollary 2: Murnaghan–Nakayama Rule

Def: T – rim-hook tableaux of shape λ and type α if

- T is a filling of a diagram λ with α_i numbers i
- the shape of boxes filled by $\leq i$ is a Young diagram
- the shape of boxes filled by i is connected and does not contain subdiagram 2×2

$\forall i$

Then $\chi_\lambda(\mu) = \sum_{T \in RH(\lambda,\mu)} (-1)^{ht(T)}$

Example: $\lambda = (2,1)$

$$\chi_\lambda(\mu) = \begin{cases} \mu = (1,1,1) & RH = \left\{ \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}\,,\; \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} \right\} \quad \color{red}{2} \\ \mu = (2,1); & RH = \left\{ \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array}\,,\; \begin{array}{|c|c} \hline 1 & \\ \hline 1 & 2 \\ \hline \end{array} \right\} \quad \color{red}{1-1=0} \\ \mu = (3) & RH = \left\{ \begin{array}{|c|c} \hline 1 & \\ \hline 1 & 1 \\ \hline \end{array} \right\} \quad \color{red}{-1} \end{cases}$$

$$z_\mu = \begin{cases} 3! = 6 & \text{for } \mu = (1,1,1) \\ 2 & \text{for } \mu = (2,1) \\ 3 & \text{for } \mu = (3) \end{cases}$$

$$\Rightarrow \quad s_\lambda = \frac{2}{6} \cdot p_1^3 + \frac{0}{2} p_2 p_1 - \frac{1}{3} p_3$$

$$= \frac{1}{3}\left(\left(x_1 + x_2 + x_3 + x_4 + \dots\right)^3 - \left(x_1^3 + x_2^3 + \dots\right)\right)$$

$$= 2 \sum_{i<j<k} x_i x_j x_k + \sum_{i<j} \left(x_i^2 x_j + x_i x_j^2\right).$$

Corollary: $\dim(S^\lambda) = \# SYT(\lambda)$

Pf: $\dim(S^\lambda) = \chi_\lambda(1^n) = \# RH(\lambda, 1^n)$

$$= \# SYT(\lambda)$$

Corollary: Magic formula:

$$\sum_{\lambda \vdash n} \# SYT(\lambda)^2 = n! \qquad \text{Pf: Ex :-)}$$

BONUS: SPECHT MODULES

For any subgroup $H < S_n$ define

$$H^+ = \sum_{h \in H} h \in \mathbb{C}[S_n]$$

$$H^- = \sum_{h \in H} \operatorname{sgn}(h) \in \mathbb{C}[S_n].$$

Let C_T, R_T denote the Young subgroups of S_n that do not exchange columns/rows resp. of T where $T \in Y(\lambda)$.

For $T_1, T_2 \in Y(\lambda)$ $T_1 \sim T_2$ if $\exists \pi \in R_{T_1} : T_2 = \pi \cdot T_1$ and let $\{T\}$ denote the equivalence class.

Define $M^\lambda := \operatorname{Span}_{\mathbb{C}} \{ \{T\} : T \in Y(\lambda) \}$

and $S^\lambda := \operatorname{Span}_{\mathbb{C}} \{ e_T : T \in Y(\lambda) \} < M^\lambda,$

where $e_T := C_T^- \cdot \{T\}.$

S^λ is called Specht module

Ex: $\lambda = (n-1, 1)$

Generic $T \in Y(\lambda)$ looks like that:

$$T = \begin{array}{|c|c|c|c|c|c|c|} \cline{1-1} i \\ \hline \sigma(1) & \sigma(2) & \cdots & \sigma(i-1) & \sigma(i+1) & \cdots & \sigma(n) \\ \hline \end{array}, \text{ where}$$

$\sigma \in S_{\{1,\dots,i-1,i+1,\dots,n\}}$.

Therefore $R_T = S_{\{1,\dots,i-1,i+1,\dots,n\}}$.

$C_T = S_{\{i,\sigma(1)\}}$

$$M^\lambda = \mathrm{span}_{\mathbb{C}}\left\{ \left\{ \begin{matrix} i \\ 1\,2 \cdots i-1\;\, i+1 \cdots n \end{matrix} \right\} \; i \in [1,\dots,n] \right\} \approx \mathbb{C}\{\{i\} : i \in [1,\dots,n]\}$$

Let T be generic and compute e_T.

$$e_T = \left\{ \begin{matrix} i \\ \sigma(1) \cdots \sigma(n) \end{matrix} \right\} - \left\{ \begin{matrix} \sigma(1) \\ i \;\; \sigma(2) \cdots \sigma(n) \end{matrix} \right\}.$$

$$= \left\{ \begin{matrix} i \\ 1 \cdots i-1\;\, i+1 \cdots n \end{matrix} \right\} - \left\{ \begin{matrix} j \\ 1 \cdots j-1\;\, j+1 \cdots n \end{matrix} \right\}$$

Therefore $S^\lambda = \mathrm{span}_{\mathbb{C}}\{\{i\} - \{j\} \mid i \neq j\}$

$= \mathrm{span}_{\mathbb{C}}\{\{i+1\} - \{i\} \mid i \in [1,\dots,n-1]\}$.

Theorem: • $S^\lambda = \mathrm{span}_{\mathbb{C}}\{\underset{\substack{\uparrow \\ \text{basis}}}{e_T} \mid T \in SYT(\lambda)\}$

• $S_n^{irr} = \{S_\lambda\}_{\lambda \in Y_n}$.