ASYMPTOTIC REPRESENTATION THEORY PT. I

<u>Problem:</u> What can we say about "typical" irreducible representation of S_n when n is very large?

"Typical" $\equiv$ sampled w.r.t. reasonable model.

Plancherel model:

G – finite $\quad \mathbb{P}$ – prob. measure on G_{irr}

$$\mathbb{P}(V_\lambda) := \frac{(\dim V_\lambda)^2}{|G|}$$

Theorem: (abbrev. version) Vershik – Kerov, Logan–Shepp '77

Let V_λ – Plancherel random irr. rep. of S_n

Then $V_\lambda \longrightarrow V_\Omega$ – deterministic object as $n \to \infty$

What is that?

New ideas and some probability theory.

Convergence of the associated shape.

① Analogy with random matrix theory

Let H_N be a Hermitian matrix:

$$H_N \in M_{N\times N}(\mathbb{C}), \quad H_N^T = \overline{H_N}.$$

Then $\mathrm{Spec}(H_N) \subseteq \mathbb{R}_N$

Define $\mu_{H_N} := \frac{1}{N} \sum_{i=1} \delta_{\lambda_i}$, where

spectral measure

$\mathrm{Spec}(H_N) = \{\lambda_1 \geq \dots \geq \lambda_N\}$

Gaussian Unitary Ensemble — $H_N = (h_{i,j})_{1\leq i,j\leq N}$

s.t. $h_{i,i} \sim N(0,1)$

$h_{i<j} = \underbrace{u_{i,j}}_{\sim N(0,\frac{1}{2})} + i\cdot \underbrace{v_{i,j}}_{\sim N(0,\frac{1}{2})}, \quad h_{i>j} = \overline{h_{i<j}}.$

Let $\mu_{(H_N/\sqrt{N})}$ – spectral measure (random measure) probability

Theorem (Wigner's semicircle law '55)

$$\mu_{(H_N/\sqrt{N})} \xrightarrow[N\to\infty]{(d)} \mu_{s-c} := \underbrace{\frac{1}{2\pi}\sqrt{4-x^2}\,\chi_{[-2,2]}(x)}_{\text{UNIVERSAL OBJECT}}$$

PLAYS THE SAME ROLE IN FREE PROBABILITY AS THE NORMAL DISTR. IN CLASSICAL PROBABILITY

METHOD OF MOMENTS:

Compactly supported prob. measures are characterized by moments.

Stieltjes transform (of $\mu \in \mathcal{P}_c(\mathbb{R})$):

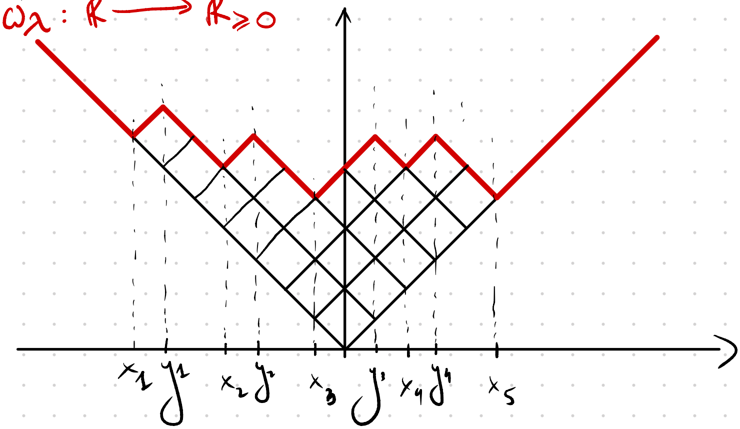
- $$C_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x) = z^{-1} + \sum_{i=1}^{\infty} M_i(\mu)\, z^{-i-1}$$

Idea of Kerov: Young diagrams $\xleftrightarrow{1-1}$ interlacing sequences

$x_1 < y_1 < x_2 < \dots < y_{k-1} < x_k$

$\lambda = (5,5,4,2,2,1,1)$

$\omega_\lambda : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$



Young diagram $\sim$ 1-Lipschitz function s.t. $\omega(x) = |x|$ for $|x|$ large

$$\mu_\lambda \in \mathcal{P}_c(\mathbb{R}) \text{ s.t. } G_{\mu_\lambda}(z) = \frac{\prod_{i=1}^{k-1}(z-y_i)}{\prod_{i=1}^{k}(z-x_i)}$$

Idea 2: Continuous Young diagrams $\equiv$ Probability measures

Def: $$\mathcal{Y}_{[a,b]} = \left\{ \omega : \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0} \;\middle|\; \omega - 1\text{-Lipschitz},\ \operatorname{supp}(\omega(x) - |x|) \subseteq [a,b] \right\}$$

Continuous Young diagrams

Markov - Krein Correspondence:

Theorem: $$\int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x) = z^{-1} \exp\left(-\int_{\mathbb{R}} \frac{\omega(x)-|x|}{2(z-x)}\, dx\right)$$

defines a homeomorphism between

$$\left\{ \begin{array}{c} \mu \in \mathbb{P}_{[a,b]}(\mathbb{R}) \\ \text{weak convergence} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \omega \in \mathbb{Y}_{[a,b]} \\ \text{with the } \|\cdot\|_\infty \text{ norm} \end{array} \right\}$$

Fact: The continuous Young diagram associated with μ_{S-C} is given by

$$\omega_\infty(x) = \begin{cases} \frac{2}{\pi}\left(x \arcsin \frac{x}{2} + \sqrt{4-x^2}\right) & |x| \le 2 \\ |x| & |x| > 2 \end{cases}$$

Theorem: (Vershik-Kerov; Logan-Shepp '77)

Let $\lambda^{(n)} \sim \mathbb{P}^{(n)}_{\text{Planch}}$, and let $\tilde{\omega}_{\lambda^{(n)}}$ — rescaled $\omega_{\lambda^{(n)}}$ s.t. each box 1×1 in $\omega_{\lambda^{(n)}}$ is replaced by a box $\frac{1}{\sqrt{n}} \times \frac{1}{\sqrt{n}}$.

Then
$$\|\tilde{\omega}_{\lambda^{(n)}} - \omega_\infty\|_\infty \xrightarrow[n\to\infty]{\mathbb{P}^{(n)}_{\text{Planch}}} 0.$$

CUMULANTS VS. FREE CUMULANTS

$$\mu \in \mathbb{P}_c(\mathbb{R})$$

$$K_\mu(z) := \log \int_{\mathbb{R}} \exp(zx)\, d\mu(x) = 1 + \sum_{i \geq 1} k_i(\mu) \frac{z^i}{i!}$$

$$R_\mu(z) := C_\mu^{\langle -1 \rangle}(z) - \frac{1}{z} = \sum_{i \geq 1} R_i(\mu) z^{i-1}$$

compositional inverse

$k_i(\mu)$ — i-th cumulant of μ

$R_i(\mu)$ — i-th free cumulant of μ

Combinatorially:

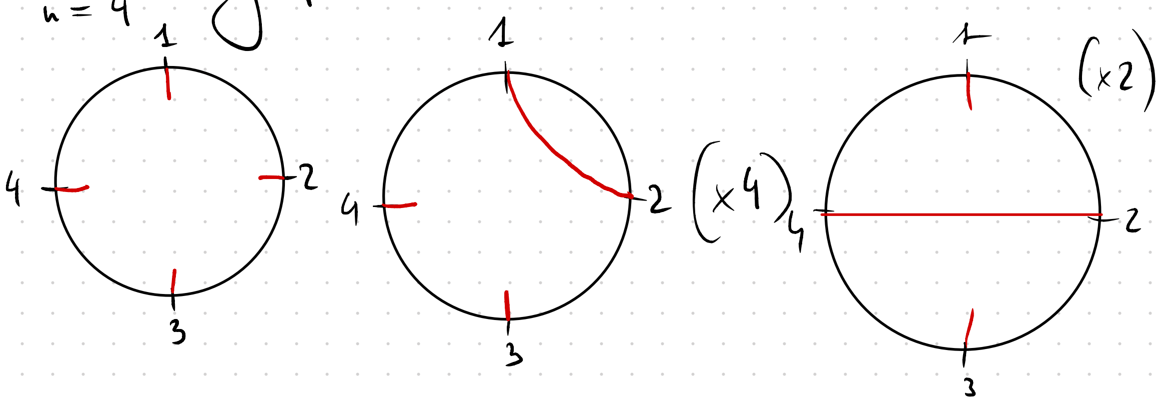
$$M_n(\mu) = \sum_{\pi \in \mathrm{Part}(n)} \prod_{B \in \pi} k_{|B|}(\mu)$$

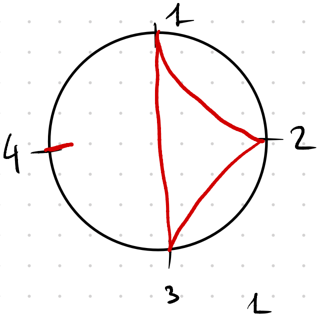
$$M_n(\mu) = \sum_{\pi \in NC(n)} \prod_{B \in \pi} R_{|B|}(\mu)$$

$NC(n) \subseteq \mathrm{Part}(n)$ — partitions of $\{1, \dots, n\}$

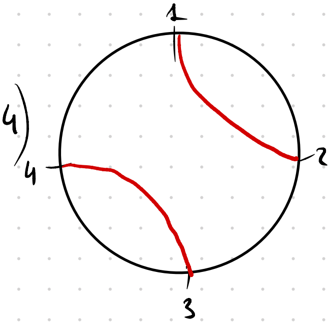
non-crossing partitions

$n = 4$

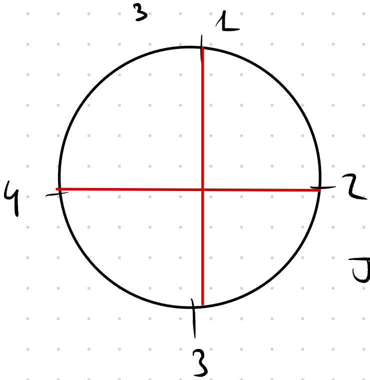




$(\times 4)$

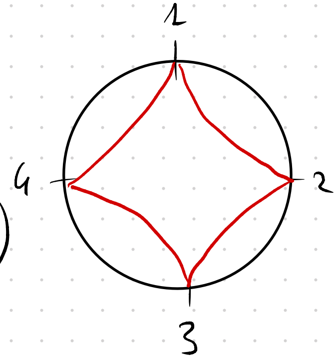


$(\times 2)$



$\Leftarrow$ CROSSING

$\pi = \{\{1,3\},\{2,4\}\} \notin NC(4)$



Therefore $M_4 = k_1^4 + 6\,k_2k_1^2 + 4k_3k_1 + 3\,k_2^2 + k_4$

$$= R_1^4 + 6\,R_2R_1^2 + 4\,R_3R_1 + 2R_2^2 + R_4$$

Gaussian distribution μ_{Gauss} is uniquely determined by $k_n(\mu_{Gauss}) = 0 \quad \forall n > 2$

Semi-circle distribution μ_{S-C} is uniquely determined by $R_n(\mu_{S-C}) = 0 \quad \forall n > 2$

THEOREM OF BIANE:

$\rho : S_n \longrightarrow GL(V)$

ρ defines a prob. measure on S_n^{irr}: $\mathbb{P}_\rho(\lambda) := \langle \chi_\rho, \chi_\lambda \rangle \dfrac{\chi_\lambda(id)}{\chi_\rho(id)}$

Theorem: (Biane '98)

Let $\rho^{(n)}$ be a seq. of rep. of S_n s.t.

- $\lim_{n\to\infty} \dfrac{\chi_{\rho^{(n)}}\left((k,1^{n-k})\right)}{\chi_{\rho^{(n)}}(id)}\sqrt{n}^{\,k-1} = r_{k+1}$ exists
- $\forall\, \sigma_1, \sigma_2$ with disjoint support

$$\frac{\chi_{\rho^{(n)}}(\sigma_1\cdot\sigma_2) - \chi_{\rho^{(n)}}(\sigma_1)\,\chi_{\rho^{(n)}}(\sigma_2)}{\chi_\rho(id)} = o\left(\sqrt{n}^{\left(-|ct(\sigma_1)|+\ell(ct(\sigma_1))-|ct(\sigma_2)|+\ell(ct(\sigma_1))\right)}\right)$$

Then $\exists\, \omega_\infty \in Y_{[a,b]}$ s.t.

$$\|\tilde{\omega}_{\lambda^{(n)}} - \omega_\infty\|_\infty \xrightarrow[n\to\infty]{\mathbb{P}_{\rho^{(n)}}} 0.$$

Moreover $r_k = R_k(\mu_\infty)$, where $\mu_\infty \xleftrightarrow{\text{Markov-Krein}} \omega_\infty$