Theorem: (Biane '98)

Let $\rho^{(n)}$ be a seq. of rep. of S_n s.t.

- $\lim_{n\to\infty} \frac{\chi_{\rho^{(n)}}((k,1^{n-k}))}{\chi_{\rho^{(n)}}(id)} \sqrt{n}^{\,k-1} = r_{k+1}$ exists
- $\forall \sigma_1, \sigma_2$ with disjoint support

$$\frac{\chi_{\rho^{(n)}}(\sigma_1\cdot\sigma_2) - \chi_{\rho^{(n)}}(\sigma_1)\chi_{\rho^{(n)}}(\sigma_2)}{\chi_\rho(id)} = o\left(\sqrt{n}^{\left(-|ct(\sigma_1)| + \ell(ct(\sigma_1)) - |ct(\sigma_2)| + \ell(ct(\sigma_2))\right)}\right)$$

Ex: (1) $\rho^{(n)}$ - regular rep. of $S_n \longrightarrow \mathbb{P}_{\rho^{(n)}} = \mathbb{P}^n_{Planch}$

$$\chi_{\rho^{(n)}}(k,1^{n-k}) = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases} \Rightarrow \checkmark$$

$$r_k = \begin{cases} 1 & k=1 \\ 0 & k>1 \end{cases} \Longrightarrow r_k = R_k(\Omega_{S-C})$$

$$\chi_{\rho^{(n)}}(\sigma_1\cdot\sigma_2) - \chi_{\rho^{(n)}}(\sigma_1)\chi_{\rho^{(n)}}(\sigma_2) = 0 \checkmark$$

$V = \mathbb{C}^N$. Then $S_n \curvearrowright V\otimes\cdots\otimes V$

$$e_1\otimes\cdots\otimes e_n \mapsto e_{\sigma(1)}\otimes\cdots\otimes e_{\sigma(n)}$$

$\rho^{(n)}_N$ — Schur-Weyl rep.

Let $ct(\sigma) = \mu$.

Then $\chi^N_{\rho^{(n)}} = N^{\ell(\mu)} \Rightarrow \lim_{n\to\infty} \frac{\chi^N_{\rho^{(n)}}(k,1^{n-k})}{\chi^N_{\rho^{(n)}}(id)} \sqrt{n}^{\,k-1}$

$$= \lim_{n\to\infty} \left(\frac{\sqrt{n}}{N}\right)^{k-1} = c^{k-1} \text{ if } \sqrt{n} \sim c\cdot N.$$

Again $\chi^N_{\rho(n)}(\sigma_1 \cdot \sigma_2) = \chi^N_{\rho(n)} \cdot \chi^N_{\rho(n)}$ ✓

Therefor μ^c_∞ is characterized by

$$R_k(\mu^c_\infty) = c^{k-1} \quad \left(\text{Free Poisson} \; ; \; \mu^{(c=0)}_\infty = \mu_{S-C} \right)$$

Main idea of the proof:

Let $\chi_\lambda : S_n \longrightarrow \mathbb{C}$.

When $n \gg 0$ then understanding χ_λ becomes extremely hard (Murnaghan-Nakayama rule is very inefficient)

<u>BUT</u> fix $\sigma \in S_k$.

Then $\sigma \in S_n$ for any $n \geq k$.

$\chi_{(\cdot)}(\sigma) : \mathbb{Y}_n \longrightarrow \mathbb{C}$ can be understood when $n \gg 0$.

(Ivanov, Kerov, Olshanski, Vershik)

<u>Def:</u> $Ch_\mu : \mathbb{Y} \longrightarrow \mathbb{C}$ s.t.

$$Ch_\mu(\lambda) = \begin{cases} 0 & \text{if } |\lambda| < |\mu| \\ \underbrace{n(n-1)\cdots(n-|\mu|+1)}_{|\mu|} \dfrac{\chi_\lambda(\mu\, 1^{n-|\mu|})}{\chi_\lambda(1^n)} & \end{cases}$$

where $|\lambda| = n$.

<u>Theorem:</u> (Kerov - Olshanski '94)

$\mathrm{Span}_{\mathbb{Q}} \{ Ch_\mu \mid \mu \in \mathbb{Y} \}$ is an algebra $\mathcal{A}$.

Moreover $\mathcal{A} = \mathbb{Q}[R_2(\cdot), R_3(\cdot), R_4(\cdot), \dots]$,

where $R_i(\lambda) := R_i(\mu_\lambda)$.

__Corollary:__ For each μ there exists a polynomial $P_\mu(R_2, R_3, \dots)$ s.t.

$$Ch_\mu = P_\mu(R_2, R_3, \dots)$$

__Ex:__

$$Ch_{(1)} = R_2$$

$$Ch_{(2)} = R_3$$

$$Ch_{(3)} = R_4 + R_2$$

$$Ch_{(4)} = R_5 + 5R_3$$

$$Ch_{(5)} = R_6 + 15R_4 + 5(R_2)^2 + 8R_2$$

__Theorem (Biane '98)__

$$Ch_{(k)} = R_{k+1} + \text{smaller degree terms (where } \deg(R_i) := i)$$

__Sketch of the proof:__

Step 1: Let $\mu \vdash n$. $p_\mu = \sum_{\lambda \vdash n} \chi_\lambda(\mu) s_\lambda$

Therefore $\chi_\lambda(1^n) = [s_\lambda] p_{1^n}$

$$\chi_\lambda(k, 1^{n-k}) = [s_\lambda] p_{1^{n-k}} \cdot p_k.$$

Fix λ, $N \geq \ell(\lambda)$ and let $l_i := \lambda_i + N - i$

Recall: $s_\lambda(x_1, \dots, x_N) = \dfrac{\det\left(x_i^{l_j}\right)}{\det\left(x_i^{N-j}\right)}$

$$\Rightarrow [s_\lambda] p_1^n = [x_1^{l_1} \cdots x_N^{l_N}] \left(\sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) x_{\sigma(1)}^{N-1} \cdots x_{\sigma(N)} \right) \left(\sum_{i=1}^N x_i \right)^n$$

$$= \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) [x_1^{l_1 - N + \sigma(1)} \cdots x_N^{l_N - N + \sigma(N)}] \left(\sum x_i \right)^n$$

$$= \sum_{\sigma \in S_N} \mathrm{sgn}(\sigma) \frac{n!}{\prod_{i=1}^N (l_i - N + \sigma(i))!}$$

$$= n! \det \left(\frac{1}{(l_i - N + j)!} \right)$$

$$= \frac{n!}{\prod l_i!} \det\left(l_i \cdots (l_i - N + j + 1) \right) = \frac{n!}{\prod l_i!} \prod_{i<j} (l_i - l_j)$$

$$= \frac{n!}{\prod l_i!} \mathrm{Vand}(l_1, \dots, l_N)$$

Similar computations give

$$[s_\lambda] p_1^{n-k} \cdot p_k = (n-k)! \sum_{j=1}^N \frac{\mathrm{Vand}(l_1, \dots, l_j - k, \dots, l_N)}{\prod_{i \neq j} l_i! \, (l_j - k)!}$$

$$\Rightarrow Ch_k(\lambda) = \sum_{j=1}^N \frac{l_j!}{(l_j - k)!} \frac{\prod_{i \neq j} (l_i - l_j + k)}{\prod_{i \neq j} (l_i - l_j)}$$

Step 2: Let $\phi_\lambda(x) = \prod_{i=1}^{N} (x - l_i)$

Then, previous formula is equivalent to

$$Ch_k(\lambda) = -\frac{1}{k}[x^{-1}]\, x(x-1)\cdots(x-k+1)\frac{\phi_\lambda(x-k)}{\phi_\lambda(x)}$$

Step 3:

Recognize that $\frac{z\phi_\lambda(z-1)}{\phi_\lambda(z)} = \frac{1}{C_{\mu_\lambda}(z+N-1)}$

$$\Rightarrow Ch_{(k)}(\lambda) = \frac{-1}{k}[x^{-1}]\frac{1}{C_{\mu_\lambda}(x+N-1)\cdots C_{\mu_\lambda}(x+N-k)}$$

$$\approx -\frac{1}{k}[x^{-1}]\, C_{\mu_\lambda}(x)^{-k} \underset{\substack{\uparrow \\ \text{Lagrange} \\ \text{inversion formula}}}{=} [z^k]\, R_{\mu_\lambda}(z)$$

$$\| $$

$$R_{k+1}(\lambda) \quad \square$$

Conjecture: (Kerov '00)

Ch_k is a polynomial in $R_2, R_3, \ldots$

with NON-NEGATIVE INTEGER coeff.

(Nowadays called Kerov polynomial)

Thm: (D., Féray, Śniady '10)

$$[R_2^{s_2} R_3^{s_3} \cdots]\, Ch_k = \#(\sigma_0, \sigma_0, f) \quad \text{s.t.}$$

(*) $\sigma_0, \sigma_\bullet \in S_k \qquad \sigma_0 \circ \sigma_\bullet = (1, 2, \dots, k)$

$\#\text{cycles}(\sigma_\bullet) = s_2 + s_3 + \cdots$

$\#\text{cycles}(\sigma_0) + \#\text{cycles}(\sigma_\bullet) = 2s_2 + 3s_3 + \cdots$

(*) $q : C(\sigma_\bullet) \longrightarrow \{2, 3, \dots\}$ s.t.

$q^{-1}(i) = s_i$

(*) if $\emptyset \subsetneq C \subsetneq C(\sigma_\bullet)$ then

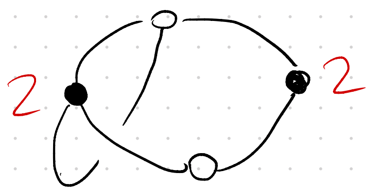
$\#\text{cycles in } \sigma_0 \text{ that intersect } C > \sum_{c \in C} (q(c) - 1)$

Ex:

$$Ch_{(5)} = R_6 + 15 R_4 + \underline{\underline{5 (R_2)^2}} + 8 R_2$$

$\# C(\sigma_0) = 2$

$\# C(\sigma_\bullet) = 2$



$\sigma_0 = (1\,4\,2)(3\,5) \qquad \sigma_\bullet = (1\,4\,3)(2\,5)$

$\sigma_0 = (3\,1\,4)(2\,5) \qquad \sigma_\bullet = (3\,1\,5)(2\,4)$

$\sigma_0 = (2\,5\,4)(1\,3) \qquad \sigma_\bullet = (4\,2\,1)(3\,5)$

$\sigma_0 = (2\,5\,3)(4\,1) \qquad \sigma_\bullet = (2\,5\,4)(1\,3)$

$\sigma_0 = (1\,5\,3)(2\,4) \qquad \sigma_\bullet = (2\,5\,3)(1\,4)$