# Recent Progress in the Geometry of $\mathbb{Q}$-Acyclic Surfaces 

Karol Palka


#### Abstract

We give a survey of results on the geometry of complex algebraic $\mathbb{Q}$-acyclic surfaces including some recent results.


## 1. Introduction

In this article we want to take the reader to the rich and yet not fully explored world of plane-like complex algebraic surfaces. We hope our survey will give a general reader a taste of methods used and will serve as an update on recent results for experts. All varieties considered are complex algebraic.

The story begins with a surprising discovery of Ramanujam [36] of a contractible affine surface nonisomorphic to $\mathbb{C}^{2}$, one of the many nontrivial smooth homotopy planes, i.e., smooth contractible surfaces. Ramanujam discovered an important characterization of $\mathbb{C}^{2}$ by showing that it is the unique smooth homotopy plane which is simply connected at infinity. Since then affine algebraic geometers began the study of smooth (and more generally normal) varieties with the same Betti numbers as $\mathbb{C}^{2}$, so-called $\mathbb{Q}$-homology planes. One of the motivations was the search for more practical characterizations of the complex plane (the computation of the fundamental group at infinity of an affine variety is usually very difficult). In the face of the topological simplicity the intriguing algebraic side of $\mathbb{Q}$-acyclic surfaces is more clearly visible. Because of their homological similarity to the plane, smooth $\mathbb{Q}$-homology planes, and especially smooth homotopy planes, play today an important role as a source of examples or counterexamples when studying working hypotheses as well as more challenging conjectures. They accompany us when studying exotic structures on $\mathbb{C}^{n}$ 's (see [46]), the Cancellation Conjecture ${ }^{1}$ (which was the motivation for Ramanujam and has been proved in dimension two by Fujita and Miyanishi [8]), the Jacobian Conjecture ${ }^{2}$ (see [24, Section 5.2]) and others.

[^0]Today, after almost forty years, mainly due to the tools of the theory of open surfaces, the scheme of the classification, and in most cases the classification itself, are settled. Recently the author had the pleasure of adding his part to the story.

Although we try to avoid notions which are not well-known to any well-versed algebraic geometer, we need to refer the reader for the basics of the theory of open surfaces to [23]. Since the Section 3.4 loc. cit. is a review of smooth $\mathbb{Q}$-homology planes, we concentrate mainly on the singular case, stating the updated results so to include the smooth case if possible (see also [24]).

Acknowledgements. The author would like to thank Professors P. Russell and S. Lu for the invitation to Montréal. He also thanks Prof. P. Russell for stimulating discussions.

## 2. Preliminaries

A normal pair $(\bar{X}, D)$ consists of a complete normal surface $\bar{X}$ and a divisor $D$ contained in the smooth locus of $\bar{X}$ which has smooth components and simple normal crossings (an snc-divisor). A normal pair ( $\bar{X}, D$ ) is said to be smooth if $\bar{X}$ is smooth. If $X$ is a normal surface then an embedding $\iota: X \rightarrow \bar{X}$, where $(\bar{X}, \bar{X} \backslash X)$ is a normal pair, is called a normal completion of $X$. A normal completion is a smooth completion if $\bar{X}$ is smooth. Two normal completions $\iota_{j}: X \rightarrow \bar{X}_{j}, j=1,2$ are isomorphic if there exists a morphism $f: \bar{X}_{1} \rightarrow \bar{X}_{2}$ for which $f \circ \iota_{1}=\iota_{2}$. A surface with isolated singularities is called logarithmic if each of its singular points is of analytical type $\mathbb{C}^{2} / G$ for some finite subgroup $G<\mathrm{GL}(2, \mathbb{C})$. By an $n$-curve we mean a smooth rational curve with self-intersection $n$.

An affine ruling (a $\mathbb{P}^{1}$-ruling, a $\mathbb{C}^{(n *)}$-ruling) is a morphism from a surface onto a smooth curve with a general fiber isomorphic to $\mathbb{C}^{1}$ (respectively to $\mathbb{P}^{1}, \mathbb{C}^{1}$ with $n$ points deleted). All these morphisms are called rational rulings.

Definition 2.1. Let $p: X \rightarrow B$ be a rational ruling of a normal surface $X$. A triple $(\bar{X}, D, \bar{p})$ is called $a$ completion of $p$ if and only if $(\bar{X}, D)$ is a normal completion of $X$ and $\bar{p}: \bar{X} \rightarrow \bar{B}$ is a $\mathbb{P}^{1}$-ruling onto a smooth curve $\bar{B} \supseteq B$ extending $p$. Such a completion is minimal (we say also that $(\bar{X}, D)$ is $\bar{p}$-minimal) if it does not dominate any other completion of $p$.

If an snc-divisor $T$ (or rather its dual graph) is a chain and $T=T_{1}+T_{2}+\cdots+T_{n}$ is its decomposition into irreducible components so that $T_{i} \cdot T_{i+1}=1$ for $i=$ $1, \ldots, n-1$, we then write $T=\left[-T_{1}^{2}, \ldots,-T_{n}^{2}\right]$. As long as $T$ is not considered as a twig attached to some other divisor containing $T$ there is no preferred choice of the tip ( $T_{1}$ or $T_{n}$ ) of $T$, so in this case $T=\left[-T_{n}^{2}, \ldots,-T_{1}^{2}\right]$ as well. If $T$ is a twig of some fixed bigger divisor then by convention we always choose the tip of $T$ which is a tip of this bigger divisor as $T_{1}$.

The Iitaka dimension of a divisor $F$ on a smooth complete surface $\bar{X}$ (hence projective by the theorem of Zariski) is defined as

$$
\kappa(F)=\sup _{n>0} \operatorname{dim} \operatorname{Im}\left(\Phi_{|n F|}\right) \in\{-\infty, 0,1,2\},
$$

where $\Phi_{|n F|}: \bar{X} \rightarrow \mathbb{P}^{N}$ is the mapping given by the linear system $|n F|$. Then the (logarithmic) Kodaira dimension of a smooth open surface $X$ can be defined by taking some smooth completion $(\bar{X}, D)$ of $X$ and by putting

$$
\bar{\kappa}(X)=\kappa\left(K_{\bar{X}}+D\right)
$$

where $K_{\bar{X}}$ is a canonical divisor on $\bar{X}$. This is well-known to be independent of the smooth completion (see [16] for the properties of $\kappa(F)$ ). The Kodaira dimension of a singular surface is defined to be the Kodaira dimension of any resolution.

If $R$ is an snc-divisor on a complete surface and $Q(R)$ is the intersection matrix of $R$ then we define the discriminant of $R$ by $d(R)=d(-Q(R))$.

## 3. Balanced and standard completions

One of the basic steps to take when dealing with an open surface is to construct a completion and a boundary. Of course, these are not unique, but we want to bring to the attention of the reader some normalizing conditions, which make them more unique and more useful in practice. We came to this problem when trying to distinguish (or to find an isomorphism) between some $\mathbb{Q}$-acyclic surfaces. This type of analysis was done at least partially by many authors, with the most complete treatment in terms of weighted dual graphs in [2] and [5]. Here we present the necessary results using partially our own terminology. For simplicity we restrict ourselves to divisors whose dual graphs contain no loops (forests).

Definition 3.1. A rational chain $D=\left[a_{1}, \ldots, a_{n}\right]$ is balanced if $a_{1}, \ldots, a_{n} \in$ $\{0,2,3, \ldots\}$ or if $D=[1]$. A reduced snc-forest $D$ is balanced if all rational chains contained in $D$ which do not contain branching components of $D$ are balanced. A normal pair $(X, D)$ is balanced if $D$ is balanced.

The word balanced stands here for the property that on one hand we do not allow nonbranching $(-1)$-curves, but on the other hand we do not allow nonbranching $b$-curves with positive $b$. The following operation is responsible for nonuniqueness of balanced completions of a given surface.

Definition 3.2. Let $(X, D)$ be a normal pair. Let $L$ be a 0 -curve which is a nonbranching component of $D$. Make a blowup of a point $c \in L$, such that $c \in L \cap(D-L)$ if $L \cdot(D-L)=2$ and contract the proper transform of $L$. The resulting pair $\left(X^{\prime}, D^{\prime}\right)$, where $D^{\prime}$ is the reduced direct image of the total transform of $D$ is called an elementary transform of $(X, D)$. The point $c \in L$ is the center of the transformation. A composition of elementary transformations of $D$ and its subsequent elementary transforms is called a flow inside $D$.

For example, taking $T=\left[0,0, a_{1}, \ldots, a_{n}\right]$ one can obtain by a flow exactly the chains $\left[0, b, a_{1}, \ldots, a_{n}\right],\left[a_{1}, \ldots, a_{k-1}, a_{k}-b, 0, b, a_{k+1}, \ldots, a_{n}\right]$ and $\left[a_{1}, \ldots, a_{n}, b, 0\right]$, where $1 \leq k \leq n$ and $b \in \mathbb{Z}$. In particular, it is easy to see when two rational chains differ by a flow. The following result is the main property of balanced completions, it follows from [5, Corollary 3.36].

Proposition 3.3. Any normal surface which admits a normal completion with a forest as a boundary has a balanced completion. Two such completions differ by a flow inside the boundary. In particular, all balanced boundaries of a given surface are isomorphic as curves.

Of course, two balanced boundaries of a given surface are in general nonisomorphic as weighted curves (weights are here the self-intersections of their components). One introduces normalizing conditions to deal with this. After [5] we state the following definition.

Definition 3.4. Let $D$ be an snc-divisor. The connected components of the divisor which remains after subtracting all nonrational and all branching components of $D$ are called the segments of $D$. An snc-divisor is standard if and only if any of its connected components is either [1] or has all segments of type $[0],[0,0]$, $[0,0,0],\left[0,0, a_{1}, \ldots, a_{n}\right]$ or $\left[a_{1}, \ldots, a_{n}\right]$ with $a_{1}, \ldots, a_{n} \geq 2$.

It follows from the example above and from the Hodge index theorem that each balanced chain can be carried to a standard form by a flow.

Example 3.5. The affine plane $\mathbb{C}^{2}$ has a balanced completion $\left(\mathbb{F}_{2}, T\right)$, where $\mathbb{F}_{2}$ is the second Hirzebruch surface and $T=[0,2]$. It follows that each balanced boundary of $\mathbb{C}^{2}$ is of type $[0, a]$ for some $a \geq 2$, it is standard if and only if $a=0$. Note that the isomorphism type of the boundary can change when we admit nonbalanced completions, for example the boundary of $\mathbb{C}^{2}$ embedded in $\mathbb{P}^{2}$ is $[-1]$.

More generally, it follows from the above considerations about a flow inside a chain that if a surface has a completion with some rational chain as a boundary then there are at most two weighted graphs which can be dual graphs of a standard boundary of this surface. If there are two then one of them is of type $T=\left[0,0, a_{1}, \ldots, a_{n}\right]$ with some $a_{1}, \ldots, a_{n} \geq 2$ and the other is the reversion of the first one, $T^{\text {rev }}=\left[0,0, a_{n}, \ldots, a_{1}\right]$. Given one weighted dual graph of some standard boundary of some surface each other can be easily described, which gives a method of distinguishing between many open surfaces.

In general two standard boundaries of a given surface can differ by more than the reversion of segments. It is useful to introduce the following, more restrictive normalizing conditions.

Definition 3.6. A balanced snc-forest $D$ is strongly balanced if and only if it is standard and either $D$ contains no segments of type [0], $[0,0,0]$ or for at least one of such segments there is a component $B \subseteq D$ intersecting it, such that $B^{2}=0$. A normal pair $(X, D)$ for which $D$ is a forest is strongly balanced if $D$ is strongly balanced.

## 4. Basic properties

Let $R$ be a ring. An $R$-homology plane is a normal surface $X$ with $H^{*}(X, R) \cong$ $R$. (This is a bit nonstandard, as usually $R$-homology planes are defined as smooth by definition). We say that $X$ is a homotopy plane if $\pi_{i}(X) \cong 0$ for $i>0$. A $\mathbb{Z}$-homology plane with trivial $\pi_{1}$ is a homotopy plane by the theorem of Hurewicz. By a theorem of Whitehead homotopy planes are contractible.

Example 4.1. Let $G<\mathrm{GL}(2, \mathbb{C})$ be a finite subgroup without pseudoreflections. Then $\mathbb{C}^{2} / G$ is a singular logarithmic homotopy plane for which the smooth locus has negative Kodaira dimension. Indeed, the linear contraction of $\mathbb{C}^{2}$ to $0 \in \mathbb{C}^{2}$ descends to a contraction of the quotient and the quotient morphism $\mathbb{C}^{2}-\{0\} \rightarrow \mathbb{C}^{2}-\{0\} / G$ is étale, so $\bar{\kappa}\left(\mathbb{C}^{2}-\{0\} / G\right)=\bar{\kappa}\left(\mathbb{C}^{2}-\{0\}\right)=-\infty$.

We now fix the notation for the rest of the paper. Let $\epsilon: S \rightarrow S^{\prime}$ be a minimal snc-resolution of a singular $\mathbb{Q}$-homology plane $S^{\prime}$. Denote the exceptional divisor of $\epsilon$ by $\widehat{E}$. Let $(\bar{S}, D)$ be a smooth completion of $S$ and let $S_{0}$ be the smooth locus of $S^{\prime}$. Since $S^{\prime}$ is normal, its singular locus consists of a finite number of points.

For topological spaces $A \subseteq X$ we write $H_{i}(X, A)$ for $H_{i}(X, A, \mathbb{Q})$ and $b_{i}(X, A)$ for $\operatorname{dim} H_{i}(X, A)$.

Logarithmic $\mathbb{Q}$-homology planes are known to be affine by an argument of Fujita (cf. $[9,2.4]$ ). They are also known to be rational due to Gurjar - Pradeep - Shastri $[14,15,35]$. On the other hand, the following example shows that nonlogarithmic $\mathbb{Q}$-homology planes can be nonrational.

Example 4.2. Let $C \subseteq \mathbb{P}^{n}$ be a projectively normal embedding of a smooth projective curve. Then the affine cone over $C$ is normal and contractible. It has a standard cylinder resolution, for which the exceptional divisor is isomorphic to $C$. In case $C$ is not rational, the cone is nonrational and nonlogarithmic.

In general we have the following result (cf. [30, Corollary 3.2, Proposition 3.4]).
Theorem 4.3. Every $\mathbb{Q}$-homology plane is affine and birationally ruled.
As for the affiness we note that the above mentioned argument of Fujita works as long as one assumes the inclusion $\iota: D \cup \widehat{E} \rightarrow \bar{S}$ induces an isomorphism $H_{2}(\iota): H_{2}(D \cup \widehat{E}) \rightarrow H_{2}(\bar{S})$. One can prove that this condition is always satisfied. If $\widehat{E}$ is a rational forest this can be seen as follows. First note that $H_{i}(\bar{S}, D \cup \widehat{E}) \cong$ $H^{4-i}\left(S_{0}\right)$ by the Lefschetz duality. One can prove that $b_{2}\left(S_{0}\right)=b_{1}\left(S_{0}\right)=b_{1}(\widehat{E})$, so if $\widehat{E}$ is a rational forest then $b_{3}(\bar{S}, D \cup \widehat{E})=b_{2}(\bar{S}, D \cup \widehat{E})=0$ and $H_{2}(\iota)$ is an isomorphism. For $b_{1}(\widehat{E}) \neq 0$ the argument is more complicated. For the proof of ruledness we refer to loc. cit.

Corollary 4.4. If $S^{\prime}$ is a singular $\mathbb{Q}$-homology plane then its boundary is connected and the homology groups $H_{i}\left(S^{\prime}, \mathbb{Z}\right)$ vanish for $i \geq 2$.

Proof. Since $S^{\prime}$ is affine, by [17] its boundary is connected and $S^{\prime}$ is homotopy equivalent to a CW-complex of real dimension at most two, hence $H_{2}\left(S^{\prime}, \mathbb{Z}\right)$ is torsionfree and $H_{3}\left(S^{\prime}, \mathbb{Z}\right)=H_{4}\left(S^{\prime}, \mathbb{Z}\right)=0$. Since $b_{2}\left(S^{\prime}\right)=0$, we get $H_{2}\left(S^{\prime}, \mathbb{Z}\right)=0$.

The rationality of smooth $\mathbb{Q}$-homology planes has strong consequences for vector bundles.

Theorem 4.5. Let $S^{\prime}$ be a smooth $\mathbb{Q}$-homology plane. Then any vector bundle over $S^{\prime}$ is a sum of a line bundle and a trivial vector bundle. Moreover, Pic $S^{\prime} \cong$ $H_{1}(X, \mathbb{Z})$, so if $S^{\prime}$ is a $\mathbb{Z}$-homology plane then all vector bundles over $S^{\prime}$ are trivial.

Proof. The first part of the theorem is a result of Murthy [29] valid for smooth affine surfaces which are birationally ruled. Since $S^{\prime}$ is rational, we have $\operatorname{Pic} S^{\prime} \cong$ $H^{2}\left(S^{\prime}, \mathbb{Z}\right)$. The groups $H_{1}\left(S^{\prime}, \mathbb{Z}\right)$ and $H_{2}\left(S^{\prime}, \mathbb{Z}\right)$ are finite, so by the universal coefficient theorem $H^{2}\left(S^{\prime}, \mathbb{Z}\right) \cong H_{1}\left(S^{\prime}, \mathbb{Z}\right)$.

We note that the question whether a vector bundle over any smooth contractible threefold is necessarily trivial is open.

## 5. Case $\bar{\kappa}\left(S_{0}\right)=-\infty$ and moduli

Let us start with recalling some known results in case $S_{0}$ has negative Kodaira dimension. This is a strong assumption, which in particular forces $S^{\prime}$ to be logarithmic (and hence rational). This can be seen as follows.

Suppose $S_{0}$ is affine-ruled. The affine ruling extends to a $\mathbb{P}^{1}$-ruling of some smooth completion $(\bar{S}, D+\widehat{E})$ of $S_{0}$, where $\widehat{E}$ is the exceptional divisor of some (not necessarily minimal) snc-resolution of singularities of $S^{\prime}$. The unique section of this extension contained in the boundary is in fact contained in $D$. Indeed, if it is contained in $\widehat{E}$, then $D$ is vertical for this ruling (i.e., all its components intersect trivially with fibers), so since the homology classes of components of $D+\widehat{E}$ generate $H_{2}(\bar{S})$, the intersection form on $\bar{S}$ is semi-negative definite, contradicting the Hodge index theorem. Thus the affine ruling of $S_{0}$ extends to an affine ruling of $S^{\prime}$, so $S^{\prime}$ has at most cyclic singularities by [22, Theorem 1]. There is no bound on the number of singularities, $H_{1}\left(S^{\prime}, \mathbb{Z}\right)$ can be any finite abelian group. The extension has a unique fiber contained in the boundary (the fiber can be assumed to be smooth) and each singular fiber contains a unique component not contained in $D \cup \widehat{E}$ (see [25, Sections 1, 2] for more details). A strongly balanced boundary of $S^{\prime}$ is unique if it is branched and it is unique up to reversion if it is a chain. There can be infinitely many strongly balanced completions (see [30, Example 4.6]). Note that from Proposition 1.2 loc. cit. if follows that $\mathbb{C}^{2}$ is the only smooth $\mathbb{Z}$-homology plane of negative Kodaira dimension (see [8] for the first proof of this) and this characterization implies the positive solution of the two-dimensional Cancellation Conjecture.

Now if $S_{0}$ is not affine-ruled (this can happen only if $S^{\prime}$ is singular) then it follows from an important structure theorem by Miyanishi-Tsunoda [27] that it contains an open subset $U$ with a very special $\mathbb{C}^{*}$-ruling called a Platonic fibration. In fact one shows that $S_{0}=U$ (cf. [19, Theorem 3.1]), which implies that $S^{\prime} \cong$ $\mathbb{C}^{2} / G$ for a finite small noncyclic subgroup $G<\mathrm{GL}(2, \mathbb{C})$. The strongly balanced completion of $S^{\prime}$ is unique.

It is well known that there are only finitely many fake projective planes, i.e., smooth projective surfaces with Betti numbers of $\mathbb{P}^{2}$ (cf. [1, Section V.1]). This is clearly not the case for $\mathbb{Q}$-homology planes (the name fake affine planes is not used). Moreover, there are arbitrarily high-dimensional nontrivial families of $\mathbb{Q}$ homology planes with the same weighted boundary. The following example of a family of singular $\mathbb{Q}$-homology planes is a modification of a similar example in [6, Example 4.16]. We sketch the arguments.

Example 5.1. Let $p: \mathbb{F}_{1} \rightarrow B \cong \mathbb{P}^{1}$ be the $\mathbb{P}^{1}$-ruling of the first Hirzebruch surface and let $N$ be a positive integer. Choose $N+3$ distinct points


Figure 1. Singular fibers in Example 5.1
$x_{\infty}, x_{1}, x_{2}, \ldots, x_{N+2}$ on the negative section. Blow up successively over each $x_{i}$ so to produce singular fibers with reductions $F_{\infty}=[0], F_{1}=[3,1,2,2], F_{2}=[2,1,2]$, $F_{i}^{\prime}=[2,1,2]$ for $i=2, \ldots, N+2$ lying respectively over $x_{\infty}, x_{1}, \ldots, x_{N+2}$. Denote the resulting surface by $V$ and the proper transform of the negative section by $D_{h}$. Note there are chains $\widehat{E}_{1}=[2,2]$ and $\widehat{E}_{2}=[2]$ contained in $F_{1}$ and $F_{2}$ respectively. We can assume that $\widehat{E}_{1}, \widehat{E}_{2}$ do not intersect $D_{h}$. For each $i \in\{3, \ldots, N\}$ choose a point $y_{i}$ on the $(-1)$-curve $D_{i}$ of $F_{i}^{\prime}$ and blow up once. For $i \in\{1, \ldots, N\}$ denote the unique (-1)-curve of $F_{i}$ by $C_{i}$. Let $\bar{S}_{y}$, where $y=\left(y_{3}, \ldots, y_{N+2}\right)$, be the resulting surface and put $D=F_{\infty}+D_{h}+\left(F_{1}-C_{1}-\widehat{E}_{1}\right)+\left(F_{2}-C_{2}-\widehat{E}_{2}\right)+\sum_{i=3}^{N}\left(F_{i}-C_{i}\right)$. One checks that the surface $S_{y}^{\prime}$ obtained by the contraction of $\widehat{E}_{1}$ and $\widehat{E}_{2}$ on $\bar{S}-D$ is a singular $\mathbb{Q}$-homology plane. Clearly, the above family $S_{y}^{\prime}$ is $N$-dimensional. Now if $S_{y}^{\prime} \cong S_{z}^{\prime}$ then the isomorphism lifts to $\bar{S}-D$ and then, since $F_{\infty}$ is the only 0-curve in $D$, extends to $\bar{S}_{y}-F_{\infty} \cong \bar{S}_{z}-F_{\infty}$ by Proposition 3.3, which in turn descends to an automorphism of $U=\mathbb{F}_{1}-F_{\infty}-D_{h} \cong \mathbb{C}^{2}$ fixing fibers. However, if $x, y$ are respectively the horizontal and vertical coordinate on $U$, each automorphism of $U$ fixing fibers can be written as $(x, y) \rightarrow(x, \lambda y+P(x))$ for some $P[x] \in \mathbb{C}[x]$ and its lifting to $V$ acts by $\lambda^{2}$ on $D_{i}$ in some coordinates on $D_{i}$ (the multiplicity of $D_{i}$ in the fiber is 2 ). Thus if we consider an $(N-1)$-dimensional subfamily with fixed $y_{3} \in D_{3}$ then $\lambda^{2}=1$, so the mentioned action on each $D_{i}$ is trivial, hence $y=z$ and different members of this subfamily are nonisomorphic.

## 6. Logarithmic Bogomolov - Miyaoka - Yau

When studying $\mathbb{Q}$-homology planes we often use the logarithmic version of the Bogomolov - Miyaoka - Yau inequality proved by Kobayashi (cf. [18]). Usually this inequality is stated for surfaces of general type in terms of the so-called strongly minimal model. This is not necessary and in fact the inequality works for surfaces of nonnegative Kodaira dimension. For example, the following lemma has been proved in [31, Corollary 2.5] as an easy corollary from an inequality of Bogomolov Miyaoka - Yau type proved by Langer (cf. [20]). The latter generalizes in particular both the inequality of Kobayashi and an inequality of Miyaoka [28]. For the notion of the Zariski decomposition and of the bark Bk $D$ of an effective snc-divisor $D$ see [23, Section 2.3]. Put $D^{\#}=D-\operatorname{Bk} D$.

Lemma 6.1. Let $(X, D)$ be a smooth pair with $\kappa\left(K_{X}+D\right) \geq 0$. Then:

$$
\begin{equation*}
3 \chi(X-D)+\frac{1}{4}\left(\left(K_{X}+D\right)^{-}\right)^{2} \geq\left(K_{X}+D\right)^{2} \tag{i}
\end{equation*}
$$

(ii) For each connected component of $D$, which is a connected component of Bk $D$ (hence contractible to a quotient singularity) denote by $G_{P}$ the local fundamental group of the respective singular point $P$. Then

$$
\chi(X-D)+\sum_{P} \frac{1}{\left|G_{P}\right|} \geq \frac{1}{3}\left(K_{X}+D^{\#}\right)^{2}
$$

We now illustrate the usefulness of the second inequality in our context.
Corollary 6.2. Let $S_{0}$ be the smooth locus of a singular $\mathbb{Q}$-homology plane $S^{\prime}$ and let $\widehat{E}_{1}, \ldots, \widehat{E}_{q}$ be the connected components of the exceptional divisor of the snc-minimal resolution.
(i) If $\bar{\kappa}\left(S_{0}\right)=2$ then $S^{\prime}$ is logarithmic and $q=1$.
(ii) If $\bar{\kappa}\left(S_{0}\right)=0$ or 1 then either $q=1$, or $q=2$ and $\widehat{E}_{1}=\widehat{E}_{2}=[2]$.

Proof. Let $\left(S_{m}, D_{m}\right)$ be the almost minimal model of $(\bar{S}, D+\widehat{E})$. For the notion of almost minimality see [23, 2.3.11]. Since $S^{\prime}$ is affine, $S_{m}-D_{m}$ is isomorphic to an open subset of $S_{0}$ satisfying $\chi\left(S_{m}-D_{m}\right) \leq \chi\left(S_{0}\right)=1-q$. Let $Q$ be the set of singular points which have been created by contracting connected components of $D$ as in Lemma $6.1(\mathrm{ii})$. Since $\# Q \leq q$, the above inequality gives

$$
\frac{1}{3}\left(\left(K_{S_{m}}+D_{m}\right)^{+}\right)^{2} \leq \chi\left(S_{m}-D_{m}\right)+\sum_{P \in Q} \frac{1}{\left|G_{P}\right|} \leq 1-q+\frac{\# Q}{2} \leq 1-\frac{q}{2}
$$

Now if $\bar{\kappa}\left(S_{0}\right)=2$ then $\left(\left(K_{S_{m}}+D_{m}\right)^{+}\right)^{2}>0$ and we get $q=1$ and $0<$ $\sum_{P \in Q} 1 /\left|G_{P}\right|$, so there is a unique singular point on $S^{\prime}$ and it is of quotient type. If $q>1$ and $\bar{\kappa}\left(S_{0}\right)=0,1$ then $\left(\left(K_{S_{m}}+D_{m}\right)^{+}\right)^{2}=0$ and we get $q=2$ and $1=1 /\left|G_{P_{1}}\right|+1 /\left|G_{P_{2}}\right|$, so $\left|G_{P_{1}}\right|=\left|G_{P_{2}}\right|=2$.

## 7. Exceptional $\mathbb{Q}$-homology planes

The structure theorem for $\mathbb{Q}$-homology planes with smooth locus of nongeneral type is based on general structure theorems for open surfaces.

Definition 7.1. A $\mathbb{Q}$-homology plane for which the smooth locus is neither of general type, nor $\mathbb{C}^{1}$ - or $\mathbb{C}^{*}$-ruled is exceptional.

Now the mentioned structure theorems lead to the fact that for exceptional $S^{\prime}$ one has $\bar{\kappa}\left(S_{0}\right)=0$. It was proved by Fujita (cf. [9, 8.64]) that each exceptional smooth $\mathbb{Q}$-homology plane is up to isomorphism one of three surfaces called $Y\{3,3,3\}, Y\{2,4,4\}$ and $Y\{2,3,6\}$ (the Fujita's surfaces of type $H[k,-k]$ with $k \geq 1$ are $\mathbb{C}^{*}$-ruled). The snc-minimal boundary of $Y\{a, b, c\}$ is a rational fork (a tree with one branching component), such that its three maximal twigs consist of $(-2)$-curves and have discriminants equal to $a, b, c$ respectively. A singular exceptional $\mathbb{Q}$-homology plane having a boundary with this property will be denoted by $S Y\{a, b, c\}$. Together with the description of exceptional singular $\mathbb{Q}$-homology planes in [31] we have the following theorem (to be precise one still needs to prove that smooth $Y\{a, b, c\}$ 's are not $\mathbb{C}^{*}$-ruled, but this can be done as in loc. cit.).

Theorem 7.2. If $S^{\prime}$ is a $\mathbb{Q}$-homology plane with smooth locus $S_{0}$ of nongeneral type then $S_{0}$ is affine-ruled or $\mathbb{C}^{*}$-ruled or $S^{\prime}$ is up to isomorphism one of five exceptional $\mathbb{Q}$-homology planes having smooth locus of Kodaira dimension zero: smooth $Y\{3,3,3\}, Y\{2,4,4\}, Y\{2,3,6\}$ and singular $S Y\{3,3,3\}, S Y\{2,4,4\}$. For the last two surfaces $\bar{\kappa}\left(S^{\prime}\right)=0$ and the singular locus consists of a unique point of Dynkin type $\mathrm{A}_{2}$ and $\mathrm{A}_{1}$, respectively.

Since the exceptional $\mathbb{Q}$-homology planes admit boundaries with no nonbranching 0-curves, their snc-minimal (which are also balanced) completions are unique. None of the exceptional surfaces is a $\mathbb{Z}$-homology plane. Writing this review the author noticed that a posteriori there is another (different than the one discovered in loc. cit.) nice description of exceptional singular $\mathbb{Q}$-homology planes. Namely, $Y\{2,4,4\}$ and $Y\{3,3,3\}$ have automorphism groups $\mathbb{Z}_{2}$ and $\mathbb{Z}_{3}$ respectively (the automorphism group of $Y\{2,3,6\}$ is trivial) and the actions have unique fixed points. The quotients are two nonisomorphic $\mathbb{Q}$-homology planes with smooth loci of Kodaira dimension zero. Suppose, say, the quotient $S^{\prime}=Y\{3,3,3\} /$ Aut $Y\{3,3,3\}$ is


Figure 2. Singular Y $\{3,3,3\}$, dual Hesse configuration.
not exceptional. Then its smooth locus $S_{0}$ is $\mathbb{C}^{*}$-ruled, so the Stein factorization of the pull-back of this $\mathbb{C}^{*}$-ruling gives a $\mathbb{C}^{*}$-ruling of the complement of the fixed point of $Y\{3,3,3\}$. Since $Y\{3,3,3\}$ is exceptional, the closures of the fibers meet in the fixed point, hence the closures of the fibers of the $\mathbb{C}^{*}$-ruling of $S_{0}$ meet in the singular point of $S^{\prime}$. Thus the last $\mathbb{C}^{*}$-ruling does not extend to a ruling of $S^{\prime}$. Since the singularity is cyclic, it follows from [30, Theorem 5.4] that $\bar{\kappa}\left(S_{0}\right)=-\infty$, a contradiction. (One can also get a contradiction with the fact that $Y\{3,3,3\}$ does not contain infinitely many contractible curves, cf. Theorem 10.1).

We recall here the construction of $S Y\{3,3,3\}$, mainly because of its beautiful connection with classical geometry. A projective configuration of type $\left(a_{c}, b_{d}\right)$ is an arrangement of $b$ lines in a projective space and $a$ points on these lines, such that each point belongs to $c$ lines and each line contains $d$ points. Clearly, $a c=b d$ for such a configuration.

Example 7.3. Up to a projective automorphism there exists a unique projective configuration $\mathfrak{H}$ of type $\left(12_{3}, 9_{4}\right)$ (the uniqueness is easy to show using information on the automorphism group of the configuration which we give below). The dual configuration $\mathfrak{H}^{*}$ is the famous Hesse configuration $\left(9_{4}, 12_{3}\right)$ of flexes of an elliptic curve. It is known that $\operatorname{Aut}(\mathfrak{H}) \cong \operatorname{Aut}\left(\mathfrak{H}^{*}\right)$ has order 216 and is isomorphic to the group of special affine transformations of $\mathbb{F}_{3}^{2}$, i.e., $\mathbb{F}_{3}^{2} \rtimes / \mathrm{SL}\left(2, \mathbb{F}_{3}\right)$ (cf. [3, Section 4]). Choose three points $Q_{1}, Q_{2}, Q_{3}$ of $\mathfrak{H}$ none two of which lie on a common line of $\mathfrak{H}$ and choose some line $U$ incident to $Q_{3}$. By taking the dual choice in $\mathfrak{H}$ we check using linear algebra that $\operatorname{Aut}(\mathfrak{H})$ acts transitively on the set of such choices and the stabilizer $\Gamma<\operatorname{Aut}(\mathfrak{H})$ has order three. For $i=1,2$ let $T_{1, i}, T_{2, i}, T_{3, i}$ be the lines incident to $Q_{i}$ and let $E_{1}, E_{2}, U$ be the lines incident to $Q_{3}$. Blow up once in each of eight points not incident to $U$. Let $\bar{S}$ be the resulting complete surface. We denote divisors and their proper transforms by the same letters. Let $B$ be the exceptional curve over $Q_{2}$, put $D=B+T_{1,1}+T_{1,2}+T_{2,1}+T_{2,2}+T_{3,1}+T_{3,2}$ and $\widehat{E}=E_{1}+E_{2}$. Clearly, all components of $D-B+\widehat{E}$ are $(-2)$-curves, $D$ and $\widehat{E}$ are disjoint and $D$ is a fork. Put $S^{\prime}=(\bar{S}-D) / \widehat{E}$. One can show that $\operatorname{Aut}\left(S^{\prime}\right) \cong \Gamma$.

To see that $S^{\prime}$ is $\mathbb{Q}$-acyclic note that $b_{1}(\bar{S})=0$ and $b_{2}(\bar{S})=b_{2}(D \cup \widehat{E})=9$. Now since $d(D+\widehat{E}) \neq 0$, the natural morphism $H_{2}(D \cup \widehat{E}) \rightarrow H_{2}(\bar{S})$ is an isomorphism. Put $S=\bar{S} \backslash D$. The homology exact sequence of the pair $(\bar{S}, D)$ and the Lefschetz
duality give $b_{1}(S)=b_{3}(S)=b_{4}(S)=0$ and $b_{2}(S)=\# \widehat{E}$. We know from the above that $H_{2}(\widehat{E}) \rightarrow H_{2}(S)$ is a monomorphism, so the homology exact sequence of the pair $(S, \widehat{E})$ gives that $S^{\prime}$ is $\mathbb{Q}$-acyclic.

We check easily that in both cases $K_{\bar{S}}+D^{\#}=K_{\bar{S}}+D^{\#}+\widehat{E}^{\#}$ intersects trivially with all components of $D+\widehat{E}$, hence $\bar{\kappa}\left(S_{0}\right)=\bar{\kappa}\left(S^{\prime}\right)=0$. See [31, Section 5] for an explicit realization of $\mathfrak{H}$ and for a proof that the constructed $\mathbb{Q}$-homology plane does not admit a $\mathbb{C}^{*}$-ruling.

## 8. Nonlogarithmic $\mathbb{Q}$-homology planes

Let us start with a generalization of Example 4.2.
Example 8.1. Let $U$ be an affine cone over a projectively normal curve and let $G$ be a finite group acting on it so that the vertex of the cone is a fixed point, the action is free on its complement and respects the set of lines of the cone. Then the quotient $S^{\prime}=U / G$ is a normal contractible surface.

What is surprising is that all nonlogarithmic $\mathbb{Q}$-homology planes arise in the above way. Namely, we have the following theorem (see [30, Corollary 5.8]).

Theorem 8.2. Every singular $\mathbb{Q}$-homology plane containing a nonquotient singularity is a quotient of an affine cone over a smooth projective curve by an action of a finite group which is free off the vertex of the cone and respects the set of lines through the vertex. In particular, it is contractible, has negative Kodaira dimension, has a unique singular point and its smooth locus is $\mathbb{C}^{*}$-ruled. The snc-minimal completion of such a surface is unique.

Proof (sketch). By Lemma 6.2 and Theorem 7.2 the smooth locus of a nonlogarithmic $\mathbb{Q}$-homology plane $S^{\prime}$ is $\mathbb{C}^{1}$ - or $\mathbb{C}^{*}$-ruled. By the results of Section 5 , $\bar{\kappa}\left(S_{0}\right) \geq 0$, so in fact $S_{0}$ is $\mathbb{C}^{*}$-ruled. One can show (cf. Theorem 3.6 loc. cit.) that if this ruling extends to a $\mathbb{C}^{*}$-ruling of $S^{\prime}$ then $S^{\prime}$ is necessarily logarithmic, so we can further assume that this is not the case. This means that there is a $\mathbb{P}^{1}$-ruling $p: \bar{S} \rightarrow \mathbb{P}^{1}$ of some completion $(\bar{S}, D+\widehat{E})$ of $S_{0}$ as before, such that for a general fiber $f$ we have $f \cdot D=f \cdot \widehat{E}=1$. We can assume that the completion is $p$-minimal. Then each singular fiber of $p$ is a so-called columnar fiber, which means that it is a rational snc-chain of discriminant zero, its components have negative self-intersections, it contains a unique $(-1)$-curve $C_{i}$ which is also a unique $S_{0}$-component of the fiber (i.e., a component not contained in $D \cup \widehat{E}$ ) and it is intersected by the horizontal components of $D$ and $\widehat{E}$ in tips. It follows that $D$ and $\widehat{E}$ contain unique branching components $D_{h}$ and $\widehat{E}_{h}$. In particular, the boundary contains no nonbranching 0 -curves, so $S^{\prime}$ has a unique snc-minimal (hence also balanced) normal completion. Write the singular fibers as $D_{i}+C_{i}+E_{i}$, where $D_{i} \subseteq D$ and $E_{i} \subseteq \widehat{E}$ (see Figure 3). Contracting singular fibers we get a $\mathbb{P}^{1}$-bundle over a complete curve, so all nonlogarithmic singular $\mathbb{Q}$-homology planes can be reconstructed starting from such a bundle by producing columnar singular fibers, taking out $D$ and contracting $\widehat{E}$. The mentioned bundle admits a usual $\mathbb{C}^{*}$ action fixing pointwise the images of $D_{h}$ and $\widehat{E}_{h}$. This action induces a $\mathbb{C}^{*}$-action on $S^{\prime}$ with the singular point as the unique fixed point. Thus by [34, Theorem 1.1] $S^{\prime}$ is a quotient of an affine cone over a smooth projective curve by an action of a finite group. In fact without knowing this global description the contractibility


Figure 3. Construction of nonlogarithmic $\mathbb{Q}$-homology planes.
follows also from $[9,5.9,4.19]$ and the Whitehead theorem, as one can show that $\pi_{1}\left(\widehat{E}_{h}\right) \rightarrow \pi_{1}(S)$ is an isomorphism and $\pi_{1}(S) \rightarrow \pi_{1}\left(S^{\prime}\right)$ is an epimorphism. Since $S=\bar{S}-D$ is affine-ruled, we get $\bar{\kappa}\left(S^{\prime}\right)=-\infty$.

We see that one can obtain any nonlogarithmic $\mathbb{Q}$-homology plane by creating its snc-minimal completion $\bar{S}$ of the resolution as in the proof above by starting with a nontrivial $\mathbb{P}^{1}$-bundle over some smooth complete curve $B$ and by blowing up some number $n$ of columnar fibers. Then exactly one of two forks, say $\widehat{E}$, separated by vertical ( -1 )-curves is negative definite, hence by [10] it contracts analytically to a singularity. What is quite surprising here is that this contraction is always algebraic (cf. the contraction criterion [30, Corollary 2.6]). Suppose $\widehat{E}_{h} \cong B \cong \mathbb{P}^{1}$. Although $\widehat{E}$ is a rational tree, the singularity does not have to be a rational singularity. Indeed, using Artin's criterion one can show that if $\widehat{E}_{h}^{2}+n \leq 0$ then $S^{\prime}$ has a rational and if $\widehat{E}_{h}^{2}+n \geq 2$ a nonrational singularity (cf. [30, Corollary 5.8]). The case $\widehat{E}_{h}^{2}+n=1$ is more subtle (cf. [34, Corollary 5.8]).

## 9. $\mathbb{C}^{*}$-ruled $\mathbb{Q}$-homology planes

By the results described in previous sections the classification of $\mathbb{Q}$-homology planes of nongeneral type reduces now to the classification of logarithmic (and hence rational) $\mathbb{Q}$-homology planes which are $\mathbb{C}^{*}$-ruled, or in other words, for which the smooth locus is $\mathbb{C}^{*}$-ruled and the ruling extends to a $\mathbb{C}^{*}$-ruling of $S^{\prime}$. Note that the existence of a $\mathbb{C}^{*}$-ruling implies that $\bar{\kappa}\left(S_{0}\right) \neq 2$ by the 'easy addition theorem' (cf. [23, 2.1.5]). This case was analyzed in [25], where one can find a description of singular fibers and a computation of $\bar{\kappa}\left(S_{0}\right)$ and $H_{1}\left(S^{\prime}, \mathbb{Z}\right)$ in terms of these fibers. We note here that in [30, Theorem 6.8] we have redone some incorrect computations of $\bar{\kappa}\left(S_{0}\right)$ from loc. cit. (identified then with the Kodaira dimension of $S^{\prime}$ ) and we have computed $\bar{\kappa}\left(S^{\prime}\right)$ too. We discuss two issues here.

First, it is practically useful to know when a $\mathbb{C}^{(n *)}$-ruled surface is a $\mathbb{Q}$-homology plane. To formulate a criterion we need to recall the definition of some numbers characterizing rational rulings. Having a fixed $\mathbb{P}^{1}$-ruling of a smooth complete surface $X$ and a reduced divisor $T$ we define

$$
\Sigma_{X-T}=\sum_{F \nsubseteq T}(\sigma(F)-1),
$$

where $\sigma(F)$ is the number of $(X-T)$-components (i.e., irreducible components not contained in $T$ ) of a fiber $F$ (cf. [9, 4.16]). $T_{h}$ is the horizontal part of $T$, which consists of components of $T$ intersecting nontrivially with a general fiber. If $T_{h}=0$ then $T$ is vertical. The numbers $h$ and $\nu$ are defined respectively as $\# T_{h}$ and as the number of fibers contained in $T$.

Lemma 9.1. Let $(\bar{S}, T)$ be a smooth pair and let $p: \bar{S} \rightarrow \mathbb{P}^{1}$ be a $\mathbb{P}^{1}$-ruling. Assume the following conditions are satisfied:
(i) there exists a unique connected component $D$ of $T$ which is not vertical,
(ii) $D$ is a rational tree,
(iii) $\Sigma_{\bar{S}-T}=h+\nu-2$,
(iv) $d(D) \neq 0$.

Then the surface $S^{\prime}$ defined as the image of $\bar{S}-D$ after contraction of connected components of $T-D$ to points is a rational $\mathbb{Q}$-homology plane and $p$ induces a rational ruling of $S^{\prime}$. Conversely, if $p^{\prime}: S^{\prime} \rightarrow B$ is a rational ruling of a rational $\mathbb{Q}$-homology plane $S^{\prime}$ (not necessarily singular) then any completion $(\bar{S}, T, p)$ of the restriction of $p^{\prime}$ to the smooth locus of $S^{\prime}$ has the above properties.

The conditions (iii) - (iv) are equivalent to the fact that $H_{2}(D \cup \widehat{E}) \rightarrow H_{2}(\bar{S})$ is an isomorphism, similar criteria were used by many authors. What is important, in case of a $\mathbb{C}^{*}$-ruling the most problematic condition (iv) can be replaced by an easier and more geometric condition (cf. [30, Lemma 6.1]).

Second, there is a question of uniqueness of a $\mathbb{C}^{*}$-ruling of $S^{\prime}$. Let us assume $S^{\prime}$ is singular. In case $\bar{\kappa}\left(S_{0}\right)=1$ it is easy to prove that there is a unique $\mathbb{C}^{*}$-ruling of $S^{\prime}$ and it is induced by the $\mathbb{C}^{*}$-ruling of $S_{0}$ given by the linear system of some multiple of the logarithmic canonical divisor of $S_{0}$. In case $\bar{\kappa}\left(S_{0}\right)=0$ generically there are two $\mathbb{C}^{*}$-rulings of $S^{\prime}$, but there may be zero (exceptional $\mathbb{Q}$-homology planes), one or three as well (cf. Theorem 6.12 loc. cit.). We will see that this information is important for example when computing the number of contractible curves on $S^{\prime}$.

Assume that $S^{\prime}$ admits a $\mathbb{C}^{*}$ - ruling but no affine-ruling. Then it has a unique balanced completion, unless it admits an untwisted $\mathbb{C}^{*}$-ruling with base $\mathbb{C}^{1}$ (untwisted means that for some completion of the surface the ruling extends to a $\mathbb{P}^{1}$-ruling for which the horizontal part of the boundary consists of two irreducible components). In the latter case $S^{\prime}$ has infinitely many balanced completions, but there are exactly two which are strongly balanced (cf. Theorem 6.11 loc. cit.).

## 10. Contractible curves

It is known that the logarithmic Bogomolov - Miyaoka - Yau inequality imposes restrictions on the number of topologically contractible curves on $S^{\prime}$. According to the author's knowledge this number is known except the cases when $S^{\prime}$ is nonlogarithmic or when $S^{\prime}$ is singular and $\bar{\kappa}\left(S_{0}\right)=0$ (see $[12,13,44,45]$ ). In the nonlogarithmic case it is easily seen to be infinity by Theorem 8.2 and, as we sketch below, in the last case it can be deduced from the knowledge on the number (and types) of $\mathbb{C}^{*}$-rulings of $S_{0}$ [30, Theorem 6.12]. The final result is as follows.

Theorem 10.1. Let $l$ be the number of topologically contractible curves on a $\mathbb{Q}$-homology plane $S^{\prime}$. Let $S_{0}$ be the smooth locus of $S^{\prime}$. Then:
(i) if $\bar{\kappa}\left(S_{0}\right)=2$ then $l=0$,
(ii) if $S^{\prime}$ is exceptional (hence $\left.\bar{\kappa}\left(S^{\prime}\right)=\bar{\kappa}\left(S_{0}\right)=0\right)$ then $l=0$,
(iii) if $S^{\prime}$ is nonlogarithmic (hence $\bar{\kappa}\left(S^{\prime}\right)=-\infty, \bar{\kappa}\left(S_{0}\right)=0,1$ ) then $l=\infty$,
(iv) if $\bar{\kappa}\left(S_{0}\right)=-\infty$ then $l=\infty$,
(v) in other cases $\left(S^{\prime}\right.$ is $\mathbb{C}^{*}$-ruled and $\left.\bar{\kappa}\left(S_{0}\right)=0,1\right) l=1$ or 2 .

Proof (sketch). If $\bar{\kappa}\left(S_{0}\right)=-\infty$ then $S^{\prime}$ is affine-ruled or isomorphic to $\mathbb{C}^{2} / G$ for some $G<\operatorname{GL}(2, \mathbb{C})$, so $l=\infty$. If $S^{\prime}$ is nonlogarithmic then by Theorem 8.2 $l=\infty$. We can therefore assume that $\bar{\kappa}\left(S_{0}\right) \geq 0$ and $S^{\prime}$ is logarithmic. Suppose $S^{\prime}$ contains a contractible curve $L$. It follows from the logarithmic Bogomolov-Miyaoka-Yau inequality that $\bar{\kappa}\left(S_{0}-L\right) \leq 1$ (cf. [12]). We show that $S_{0}-L$ is $\mathbb{C}^{*}$ ruled. This is the case if $\bar{\kappa}\left(S_{0}-L\right)=1$, as the ruling is given by some multiple of the logarithmic canonical divisor or $S_{0}-L$. We may therefore assume $\bar{\kappa}\left(S_{0}-L\right)=0$. Since $S^{\prime}$ is rational, $\operatorname{Pic}\left(S_{0}\right)=\operatorname{Coker}(\operatorname{Pic}(D+\widehat{E}) \rightarrow \operatorname{Pic} \bar{S})$ is torsion, so the class of $L$ in $\operatorname{Pic}\left(S_{0}\right)$ is torsion. Then there is a morphism $f: S_{0}-L \rightarrow \mathbb{C}^{*}$. Taking its Stein factorization one gets a $\mathbb{C}^{*}$-ruling of $S_{0}-L$, so we are done. Since $\bar{\kappa}\left(S_{0}\right) \geq 0$, any $\mathbb{C}^{*}$-ruling of $S_{0}-L$ extends to a $\mathbb{C}^{*}$-ruling of $S_{0}$. Since $S^{\prime}$ is logarithmic, each $\mathbb{C}^{*}$ ruling of $S_{0}$ extends in turn to a $\mathbb{C}^{*}$-ruling of $S^{\prime}$. Therefore, any contractible curve on $S^{\prime}$ is vertical for some $\mathbb{C}^{*}$-ruling of $S^{\prime}$. In particular, if $l>0$ then $S^{\prime}$, and hence $S_{0}$, is necessarily $\mathbb{C}^{*}$-ruled, hence $S^{\prime}$ cannot be exceptional. The analysis of fibers (Suzuki's formula) leads to the corollary that there exist one or two contractible vertical curves for a given $\mathbb{C}^{*}$-ruling. Now if $\bar{\kappa}\left(S_{0}\right)=1$ then this ruling is unique (given by a multiple of the logarithmic canonical divisor of $S_{0}$ ), hence $l=1,2$. Consider now the case $\bar{\kappa}\left(S_{0}\right)=0$. Here the problem is more difficult, as there may be more $\mathbb{C}^{*}$-rulings of $S^{\prime}$. In case $S^{\prime}$ is smooth it was shown in [13] that $l=1$. In case $S^{\prime}$ is singular we have computed the number and types of possible $\mathbb{C}^{*}$-rulings of $S_{0}$ in [30, Theorem 6.12 (one can do the same if $S^{\prime}$ is smooth in a similar way). Since this number is finite we see that $l$ is finite and nonzero. Looking more closely at the computations one deduces that $l \leq 2$ (cf. Corollary 6.13. loc. cit).

Example 10.2. Let $\pi: S \rightarrow \mathbb{C}^{2}$ be the restriction of the projection $(x, y, z) \rightarrow$ $(x, y)$ to the surface $S=\left\{(x, y, z) \in \mathbb{C}^{3}: z^{n}=f(x, y)\right\}$, where $n \geq 2$. Then $\pi$ is a branched cover with the curve $C=\left\{(x, y) \in \mathbb{C}^{2}: f(x, y)=0\right\}$ as the branch locus. Suppose $S$ is a smooth $\mathbb{Q}$-homology plane. Then $C$ is smooth and we have $1-\chi(C)=1-\chi\left(\pi^{-1}(C)\right)=1-\chi(S)+\chi\left(S-\pi^{-1}(C)\right)=n \chi\left(\mathbb{C}^{2}-C\right)=n(1-\chi(C))$, hence $\chi(C)=1$. If $U$ is any smooth affine curve then it is noncomplete, so its Euler characteristic is smaller than the Euler characteristic of the smooth completion $\bar{U}$, hence $\chi(U) \leq 1-2 g(\bar{U})$, where $g$ is the genus. In case $U$ is irreducible it follows that $\chi(U) \geq 0$ only if $U \cong \mathbb{C}^{1}$ or $U \cong \mathbb{C}^{*}$. In particular, some component of $C \cong \pi^{-1}(C)$ is an affine line, so by Theorem 10.1 $S$ is not of general type. In fact either $\bar{\kappa}(S)=-\infty$ or $S$ is $\mathbb{C}^{*}$-ruled. Smooth $\mathbb{Q}$-homology planes of this kind have been classified in [21].

## 11. Smooth $\mathbb{Q}$-homology planes of general type

Let $S^{\prime}$ be a smooth $\mathbb{Q}$-homology plane of general type, i.e., $\bar{\kappa}\left(S^{\prime}\right)=2$. We know already that $S^{\prime}$ is rational. By Theorem $10.1 S^{\prime}$ contains no contractible curves, which implies that its snc-minimal completion $(\bar{S}, D)$ is almost minimal. We can assume that this completion is balanced. Since $S^{\prime}$ is neither $\mathbb{C}^{1}$ - nor $\mathbb{C}^{*}$-ruled, $D$ contains no nonbranching 0 -curves, so any flow inside $D$ is trivial. It follows that
the balanced completion of $S^{\prime}$ is unique. As for now there is no classification, but there are some partial results.

The first example of such a surface which was shown to be nonisomorphic to $\mathbb{C}^{2}$ is the famous example of Ramanujam ([36]). We know now infinitely many examples of this kind. One method of construction of $\mathbb{Q}$-homology planes of general type is to use $\mathbb{C}^{(n *)}$-rulings with $n \geq 2$. For $n=2$ this was done in [26]. Another method was used by tom Dieck and Petrie.

Definition 11.1. Let $(\bar{S}, D)$ be a completion of $S^{\prime}$ for which there exists a birational morphism $f:(\bar{S}, D) \rightarrow\left(\mathbb{P}^{2}, f_{*} D\right)$. Then $f_{*} D$ is called a plane divisor of $S^{\prime}$. If $f_{*} D$ is a sum of lines then it is a linear plane divisor and we say that $S^{\prime}$ comes from the line arrangement $f_{*} D$.

In [39] it was noticed that an inequality of Hirzebruch bounds the number of types of possible linear plane divisors for smooth $\mathbb{Q}$-homology planes, a list of these divisors has been given. In [42] a general algorithm for recovering smooth $\mathbb{Q}$-homology planes (in fact countable series of them) starting from a given rational divisor on a minimal rational complete surface is described. The conjecture that all smooth $\mathbb{Q}$-homology planes of general type have linear plane divisors is not true by an example of tom Dieck (cf. [40]). Tom Dieck's smooth $\mathbb{Z}$-homology plane has a nontrivial automorphism group (it is necessarily finite, as $S^{\prime}$ is of general type), so is also a counterexample to the earlier conjecture of Petrie [33].

Example 11.2. Let $C \subseteq \mathbb{P}^{2}$ be an irreducible curve. Recall that a singular point $p \in C$ is a cusp if $C$ is locally irreducible at $p$. Since $C \hookrightarrow \mathbb{P}^{2}$ induces a monomorphism on $H_{2}(\cdot, \mathbb{Q})$, by the Lefschetz duality and by the long exact sequence of the pair $\left(\mathbb{P}^{2}, C\right)$ the Betti numbers of $S_{C}=\mathbb{P}^{2} \backslash C$ are $b_{2}\left(S_{C}\right)=b_{1}(C)$ and $b_{i}\left(S_{C}\right)=0$ for $i=1$ and $i>2$. Assume that $C$ is rational and cuspidal, i.e., it has only cusps as singularities. Then $S_{C}$ is a smooth $\mathbb{Q}$-homology plane (these conditions are in fact equivalent). By [43] if $C$ has two cusps then $\bar{\kappa}\left(S_{C}\right) \geq 0$ and if it has more than two cusps then $S_{C}$ is of general type. The literature on plane cuspidal curves is rich, see for example $[4,7]$ and references there.

We now list some conjectural properties of smooth $\mathbb{Q}$-homology planes.
Conjecture. Let $S^{\prime}$ be a smooth $\mathbb{Q}$-homology plane of general type and let $(\bar{S}, D)$ be its minimal smooth completion.
(A) $S^{\prime}$ has a plane divisor consisting of lines and conics.
(B) $S^{\prime}$ admits a $\mathbb{C}^{(3 *)}$-ruling.
(C) $\left(K_{\bar{S}}+D\right)^{2}=-2$, or equivalently $K_{\bar{S}} \cdot\left(K_{\bar{S}}+D\right)=0$.
(D) $S^{\prime}$ is rigid and has unobstructed deformations.
(E) The set of all possible Eisenbud-Neumann diagrams for smooth $\mathbb{Q}$-homology planes is finite.

In [41] conjectures $(\mathrm{A})-(\mathrm{C})$ have been stated and verified for all known smooth $\mathbb{Z}$-homology planes. Sugie [38] analyzed $\mathbb{C}^{(n *)}$-rulings on smooth $\mathbb{Q}$-homology planes and classified possible singular fibers for $n=2$. Flenner and Zaidenberg [6, Proposition 6.12] have shown that part (D), which implies (C), holds for smooth $\mathbb{Q}$ homology planes of general type having linear plane divisors. See [47] for (D), (E) and related conjectures.

Recently, the following result has been proved [11].

Theorem 11.3. Let $S^{\prime}$ be a smooth homotopy plane of general type. Then Aut $S^{\prime}$ is cyclic, its action on $S^{\prime}$ has a unique fixed point and is free off this point.

It is not known whether Aut $S^{\prime}$ has to be cyclic for a singular $\mathbb{Q}$-homology plane with smooth locus of general type.

## 12. Smooth locus of general type

Let $S^{\prime}$ be a singular $\mathbb{Q}$-homology plane with smooth locus $S_{0}$ of general type, i.e., $\bar{\kappa}\left(S_{0}\right)=2$. By Lemma $6.2 S^{\prime}$ has a unique singular point and this point is of quotient type. This means that $\widehat{E}$ is either a chain if this point is a cyclic singularity or a negative definite rational fork if not. Since $S^{\prime}$ is logarithmic, it is rational. By Theorem 10.1 it contains no topologically contractible curves, which implies that the snc-minimal completion $(\bar{S}, D+\widehat{E})$ of $S_{0}$ is almost minimal. Arguing as above we see that in fact this snc-minimal completion is unique and does not contain nonbranching 0-curves. Again, there is no classification, but there are some partial results.

Note that due to the existence of the transfer homomorphism for branched coverings (cf. [37]) the quotient of a smooth $\mathbb{Q}$-homology plane of general type by its automorphism group is a $\mathbb{Q}$-homology plane (with smooth locus of general type).

A priori there is no restriction on the Kodaira dimension of $S^{\prime}$. However, refining the methods of Koras - Russell [19] M. Koras and the author have obtained the following theorem (this is also a part of the thesis of the author written under the supervision of M. Koras, cf. [32]).

Theorem 12.1. Singular $\mathbb{Q}$-homology planes with smooth locus of general type have nonnegative Kodaira dimension.

The cases $\bar{\kappa}\left(S^{\prime}\right)=0,1$ and especially the case $\bar{\kappa}\left(S^{\prime}\right)=2$ have not been analyzed systematically as for now. Also the number of known examples is much smaller than in the case of smooth $\mathbb{Q}$-homology planes. For example, similarly as in the smooth case, one could ask if $S_{0}$ has a completion $(\bar{S}, D+\widehat{E})$ admitting a birational morphism $f: \bar{S} \rightarrow \mathbb{P}^{2}$ with $f_{*}(D+\widehat{E})$ a sum of lines and conics. Even the case when $f_{*}(D+\widehat{E})$ is a sum of lines has not been studied.

In the known examples of singular $\mathbb{Z}$-homology planes of general type the local fundamental group of the singular point (which has order equal to $d(\widehat{E})$ ) is cyclic of order not bigger than six. As for now the following result in this direction has been proved ([11]):

Theorem 12.2. A singular $\mathbb{Z}$-homology plane with smooth locus of general type has a cyclic quotient singularity.

## References

1. W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven, Compact complex surfaces, 2nd ed., Ergeb. Math. Grenzgeb. (3), vol. 4, Springer, Berlin, 2004.
2. D. Daigle, Classification of weighted graphs up to blowing-up and blowing-down, available at arXiv:math/0305029.
3. I. V. Dolgachev, Abstract configurations in algebraic geometry, The Fano Conference (Turin, 2002) (A. Collino, A. Conte, and M. Marchisio, eds.), Univ. Torino, Turin, 2004, pp. 423-462.
4. J. Fernández de Bobadilla, I. Luengo, A. Melle-Hernández, and A. Némethi, On rational cuspidal plane curves, open surfaces and local singularities, Singularity Theory (Marseille, 2005) (D. Chéniot, N. Dutertre, C. Murolo, D. Trotman, and A. Pichon, eds.), World Sci. Publ., Hackensack, NJ, 2007, pp. 411-442.
5. H. Flenner, S. Kaliman, and M. Zaidenberg, Birational transformations of weighted graphs, Affine Algebraic Geometry (T. Hibi, ed.), Osaka Univ. Press, Osaka, 2007, pp. 107-147.
6. H. Flenner and M. Zaidenberg, $\mathbb{Q}$-acyclic surfaces and their deformations, Classification of Algebraic Varieties (L’Aquila, 1992) (C. Ciliberto, E. L. Livorni, and A. J. Sommese, eds.), Contemp. Math., vol. 162, Amer. Math. Soc., Providence, RI, 1994, pp. 143-208.
7. _ On a class of rational cuspidal plane curves, Manuscripta Math. 89 (1996), no. 4, 439-459.
8. T. Fujita, On Zariski problem, Proc. Japan Acad. Ser. A Math. Sci. 55 (1979), no. 3, 106-110.
9. -_, On the topology of noncomplete algebraic surfaces, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 29 (1982), no. 3, 503-566.
10. H. Grauert, Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann. 146 (1962), 331-368.
11. R. V. Gurjar, M. Koras, M. Miyanishi, and P. Russell, A homology plane of general type can have at most a cyclic quotient singularity, available at arXiv:1012.4120.
12. R. V. Gurjar and M. Miyanishi, Affine lines on logarithmic Q-homology planes, Math. Ann. 294 (1992), no. 3, 463-482.
13. R. V. Gurjar and A. J. Parameswaran, Affine lines on Q-homology planes, J. Math. Kyoto Univ. 35 (1995), no. 1, 63-77.
14. R. V. Gurjar and C. R. Pradeep, Q-homology planes are rational. III, Osaka J. Math. 36 (1999), no. 2, 259-335.
15. R. V. Gurjar, C. R. Pradeep, and A. R. Shastri, On rationality of logarithmic Q-homology planes. II, Osaka J. Math. 34 (1997), no. 3, 725-743.
16. S. Iitaka, Algebraic geometry, Grad. Texts in Math., vol. 76, Springer, New York, 1982.
17. K. K. Karčjauskas, A generalized Lefschetz theorem, Funkcional. Anal. i Priložen. 11 (1977), no. 4, 80-81 (Russian); English transl., Functional Anal. Appl. 11 (1977), no. 4, 312-313.
18. R. Kobayashi, Uniformization of complex surfaces, Kähler Metric and Moduli Spaces (T. Ochiai, ed.), Adv. Stud. Pure Math., vol. 18, Academic Press, Boston, MA, 1990, pp. 313394.
19. M. Koras and P. Russell, Contractible affine surfaces with quotient singularities, Transform. Groups 12 (2007), no. 2, 293-340.
20. A. Langer, Logarithmic orbifold Euler numbers of surfaces with applications, Proc. London Math. Soc. (3) 86 (2003), no. 2, 358-396.
21. A. Maharana, $Q$-homology planes as cyclic covers of $\mathbb{A}^{2}$, J. Math. Soc. Japan 61 (2009), no. 2, 393-425.
22. M. Miyanishi, Singularities of normal affine surfaces containing cylinderlike open sets, J. Algebra 68 (1981), no. 2, 268-275.
23. , Open algebraic surfaces, CRM Monogr. Ser., vol. 12, Amer. Math. Soc., Providence, RI, 2001.
24._, Recent developments in affine algebraic geometry: from the personal viewpoints of the author, Affine Algebraic Geometry (T. Hibi, ed.), Osaka Univ. Press, Osaka, 2007, pp. 307378.
24. M. Miyanishi and T. Sugie, Homology planes with quotient singularities, J. Math. Kyoto Univ. 31 (1991), no. 3, 755-788.
25. , Q-homology planes with $\mathbf{C}^{* *}$-fibrations, Osaka J. Math. 28 (1991), no. 1, 1-26.
26. M. Miyanishi and S. Tsunoda, Noncomplete algebraic surfaces with logarithmic Kodaira dimension $-\infty$ and with nonconnected boundaries at infinity, Japan. J. Math. (N.S.) 10 (1984), no. 2, 195-242.
27. Y. Miyaoka, The maximal number of quotient singularities on surfaces with given numerical invariants, Math. Ann. 268 (1984), no. 2, 159-171.
28. M. P. Murthy, Vector bundles over affine surfaces birationally equivalent to a ruled surface, Ann. of Math. (2) 89 (1969), 242-253.
29. K. Palka, On the classification of singular $\mathbf{Q}$-acyclic surfaces, available at arXiv:0806.3110v3.
30. _ Exceptional singular Q-homology planes, Ann. Inst. Fourier (Grenoble), available at arXiv:0909.0772.
31. K. Palka and M. Koras, Singular Q-homology planes of negative Kodaira dimension have smooth locus of non-general type, available at arXiv:1001.2256.
32. T. Petrie, Algebraic automorphisms of smooth affine surfaces, Invent. Math. 95 (1989), no. 2, 355-378.
33. H. Pinkham, Normal surface singularities with $C^{*}$ action, Math. Ann. 227 (1977), no. 2, 183-193.
34. C. R. Pradeep and A. R. Shastri, On rationality of logarithmic Q-homology planes. I, Osaka J. Math. 34 (1997), no. 2, 429-456.
35. C. P. Ramanujam, A topological characterisation of the affine plane as an algebraic variety, Ann. of Math. (2) 94 (1971), 69-88.
36. L. Smith, Transfer and ramified coverings, Math. Proc. Cambridge Philos. Soc. 93 (1983), no. 3, 485-493.
37. T. Sugie, Singular fibers of homology planes with pencils of rational curves, Mem. Fac. Educ. Shiga Univ. III Nat. Sci. 49 (1999), 29-40.
38. T. tom Dieck, Linear plane divisors of homology planes, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 37 (1990), no. 1, 33-69.
39. $\qquad$ , Symmetric homology planes, Math. Ann. 286 (1990), no. 1-3, 143-152.
40. _ Optimal rational curves and homotopy planes, Bol. Soc. Mat. Mexicana (2) 37 (1992), no. 1-2, 115-138.
41. T. tom Dieck and T. Petrie, Homology planes and algebraic curves, Osaka J. Math. 30 (1993), no. 4, 855-886.
42. I. Wakabayashi, On the logarithmic Kodaira dimension of the complement of a curve in $P^{2}$, Proc. Japan Acad. Ser. A Math. Sci. 54 (1978), no. 6, 157-162.
43. M. G. Zaidenberg, Isotrivial families of curves on affine surfaces, and the characterization of the affine plane, Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987), no. 3, 534-567 (Russian); English transl., Math. USSR-Izv. 30 (1988), no. 3, 503-532.
44. $\qquad$ , Additions and corrections to the paper "Isotrivial families of curves on affine surfaces, and the characterization of the affine plane", Izv. Akad. Nauk SSSR Ser. Mat. 55 (1991), no. 2, 444-446 (Russian); English transl., Math. USSR-Izv. 38 (1992), no. 2, 435-437.
45. $\qquad$ , Exotic algebraic structures on affine spaces, Algebra i Analiz 11 (1999), no. 5, 3-73 (Russian); English transl., St. Petersburg Math. J. 11 (2000), no. 5, 703-760.
46. _, Selected problems, available at arXiv:math/0501457.

Department of Mathematics and Statistics, McGill University, 805 Sherbrooke St. West, Montréal, QC H3A 2K6, Canada and Institute of Mathematics, Polish Academy of Sciences, ul. Śniadeckich 8, 00-956 Warsaw, Poland

E-mail address: palka@impan.pl


[^0]:    2010 Mathematics Subject Classification. Primary 14R05; Secondary 14J17, 14J26.
    Key words and phrases. Acyclic surface, homology plane, $\mathbb{Q}$-homology plane.
    The author was supported by Polish Grant NCN 201608640.
    This is the final form of the paper.
    ${ }^{1}$ The $m$-dimensional Cancellation Conjecture. If $X \times \mathbb{C}^{n} \cong \mathbb{C}^{n+m}$ then $X \cong \mathbb{C}^{m}$.
    ${ }^{2}$ The Jacobian Conjecture. A polynomial map $f: \mathbb{C}^{n} \rightarrow \mathbb{C}^{n}$ with nowhere vanishing Jacobian determinant $|J(f)|$ has a polynomial inverse.

