ALMOST MINIMAL MODELS OF LOG SURFACES

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ABSTRACT. Given a generalized log canonical or a $\mathbb{Q}$-factorial log surface (X, D) defined over an algebraically closed field of arbitrary characteristic we define and construct its almost minimal model, whose underlying surface has singularities not worse than X and which differs from a minimal model by a contraction of some curves supported in the boundary only. For boundaries of type rD, where D is reduced and $r \in [0,1] \cap \mathbb{Q}$, we show that if X is smooth or $r \in [0, \frac{1}{2}] \cup \{1\}$ then the construction respects $(1-r)$-divisorial log terminality and $(1-r)$-log canonicity. When X is singular and $r \in (\frac{1}{2}, 1)$ this is false in general.

1. MAIN RESULTS

An important tool in the study of quasi-projective surfaces and of log surfaces is the logarithmic version of the Minimal Model Program (MMP) [KM98], which finds a birational model whose log canonical divisor has uniform numerical properties. It is well known that in dimensions bigger than two a minimal model of a smooth projective variety can be singular [Mat02, Example 3.1.3]. The same problem appears naturally for quasi-projective surfaces and log surfaces with nonzero boundary [KM98, Example 3.49]. Whenever the MMP is used to understand the geometry of a smooth variety, passing to a singular model makes the analysis more difficult. To avoid this, for log smooth surfaces with reduced boundary Miyanishi developed the notion of an *almost minimal model* [Miy01, 2.3.11]. It is related to a minimal model by a well described morphism, which we call a *peeling*, see Definition 3.11, contracting only some curves supported in the boundary. For a log smooth surface with reduced boundary an almost minimal model, unlike the minimal model, is smooth and one shows that it is in fact log smooth. Understanding the process of almost minimalization gives an effective tool to analyze log surfaces. In particular, it helps to obtain various structure theorems, see [Miy01, §2-3].

We show that the idea of almost minimalization can be used more widely. Given an MMP run $f\colon (X, D) \longrightarrow (\overline{X}, \overline{D})$ on a log surface defined over an algebraically closed field of arbitrary characteristic we define its *almost minimalization* $f^{\#}\colon X \longrightarrow X'$ as the unique K_X-MMP over $\overline{X}$, see Definition 3.5, and we call $(X', f^{\#}_{*}D)$ an *almost minimal model* of (X, D). By construction, when measured in terms of log discrepancies, X' is not more singular than X (see Lemma 2.9). However, describing log singularities of an almost minimal model requires a detailed analysis of the almost minimalization morphism, which amounts to the analysis of reordering of contractions of log exceptional curves. For this we introduce general notions of peeling, squeezing, redundant and almost log exceptional curves, see Section 3C. The non-almost-minimality of a log surface is witnessed by the existence of an almost log exceptional curve, necessarily not supported in the boundary, see Corollary 3.8. Almost log exceptional curves keep some extremal properties of log exceptional curves, which makes them especially important. In particular, their intersection with the boundary is well controlled.

Given a quasi-projective surface one can make it into a log surface in numerous ways. The freedom comes from a choice of a completion and from a choice of coefficients of boundary components. In practice we concentrate on uniform boundaries, that is, the ones of type rD, where D is reduced and $r \in [0,1] \cap \mathbb{Q}$. If (X, rD) is $(1-r)$-divisorially log terminal ($(1-r)$-dlt) or $(1-r)$-log canonical ($(1-r)$-lc), see Definition 2.5, then so is its image under the contraction of every log exceptional curve, hence its minimal model, too. In general these properties of log singularities are not respected by the process of almost minimalization, see Example 5.9. Nevertheless, we prove that they are inherited by almost minimal models in case X is smooth or $r \in [0, \frac{1}{2}] \cup \{1\}$. For X singular and $r \in (\frac{1}{2}, 1)$ we construct counterexamples.

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Theorem 1.1. *Let (X,D) be a log surface with reduced boundary and let $r\in[0,1]\cap\mathbb{Q}$. Assume that (X,rD) is $(1-r)$-dlt ($(1-r)$-lc) and one of the following holds:*
(1) X is smooth,
(2) $r\in[0,\frac{1}{2}]\cup\{1\}$.
Then every almost minimal model of (X,rD) is $(1-r)$-dlt (respectively, $(1-r)$-lc). Moreover, if $\frac{1}{r}\in\mathbb{N}$ then each intermediate model in the process of almost minimalization is $(1-r)$-lc.

The theorem implies for instance the following result by Miyanishi, see [Miy01, p. 105].

Corollary 1.2. An almost minimal model of a log smooth surface with a reduced boundary is log smooth.

Proof. Let (X,D) be a log smooth surface with a reduced boundary. By Lemma 2.9 log discrepancies of the underlying surfaces improve in the process of almost minimalization. Since X is terminal, each of the intermediate models is terminal, hence smooth. Since (X,D) is dlt, the theorem implies that an almost minimal model is dlt, hence log smooth. □

The new parameter r gives additional flexibility to the theory. Previously we treated the case $r=\frac{1}{2}$ [Pal19]. This instance of the construction turned out to be especially useful for the difficult class of rational surfaces of log general type, which share many properties with projective surfaces of general type and at the same time have rich birational geometry. The analysis of the process of almost minimalization for $r=\frac{1}{2}$ and complex affine $X\setminus D$ was a key tool in the recent proof of the Coolidge-Nagata conjecture [Pal14], [KP17] and in obtaining classification results for rational cuspidal curves [PP17], [PP19], [KP22]. Recently, it allowed to obtain strong classification results for $\mathbb{Q}$-acyclic surfaces [Pełż21], an interesting class of plane-like surfaces studied for a long time, see [Miy01, §3.4].

The content of the article is as follows. We discuss necessary properties of the MMP in the class of *generalized MR log canonical surfaces* introduced by Fujino, which contains $\mathbb{Q}$-factorial and log canonical surfaces. We discuss the geometry of log exceptional curves of the first and second kind; see Definitions 2.3, 2.13. In Section 3A we discuss the process of reordering of contractions of an MMP run of the first and second kind, including a characterization of MMP runs for surfaces as birational morphisms increasing discrepancies of the contracted curves, see Corollary 3.2. We discuss the uniqueness of almost minimalization and its properties for compositions in Section 3B. To effectively describe the process we generalize Miyanishi's 'theory of peeling', defining the peeling of a boundary as a composition of a maximal sequence of contractions of log exceptional curves supported in the boundary and its images, see Definition 3.11. We then describe how to construct an almost minimal model in steps, successively contracting redundant components of D and almost log exceptional curves, which are proper transforms of log exceptional curves, see Definition 3.12 and Remark 3.18. In Section 3E we discuss analogous properties for contractions of log exceptional curves of the second kind, that is, the ones intersecting the log canonical divisor trivially.

In Section 4 we review some properties of log canonical surface singularities and in Section 4C we work out characterizations of peeling, redundant and almost log exceptional curves of the first and second kind for a reduced boundary in a generality which later allows to use it in the analysis of uniform boundaries. Since almost log exceptional and redundant curves are not necessarily log exceptional themselves, log singularities can get worse in the process of almost minimalization. For uniform boundaries we are able to give a complete description of such situations, and hence we are able to control the behavior of log singularities under almost minimalization. Theorem 1.1 is proved at the end of Section 5.

In Section 5D we introduce the notion of a weighted Kodaira dimension for quasi-projective surfaces and in Section 6 we make an explicit discussion of the case $r\leqslant\frac{1}{2}$.

Contents

2. Preliminaries

We work over an algebraically closed field $\Bbbk$ of arbitrary characteristic. A *curve* is an irreducible and reduced variety of dimension 1.

2A. *Log Minimal Model Program for surfaces*

Given a normal surface, a *boundary* is a Weil $\mathbb{Q}$-divisor whose coefficients of irreducible components are between 0 and 1. A *log surface* (X, D) consists of a normal projective surface X and a boundary D such that $K_X + D$ is $\mathbb{Q}$-Cartier. The Minimal Model Program works in the class of $\mathbb{Q}$-factorial surfaces and in the class of log canonical surfaces, see [Fuj12], [Tan14] and [FT12], cf. [KK94]. Recently it has been generalized to the class of GMRLC log surfaces [Fuj21, Theorem 1.5], which contains both of them.

Given a log surface (X, D) and a proper birational morphism from a normal surface $f\colon Y \longrightarrow X$, the *log pullback* of D is defined as the unique Weil $\mathbb{Q}$-divisor D_Y on Y such that

$$K_Y + D_Y \sim_{\mathbb{Q}} f^*(K_X + D) \quad \text{and} \quad f_*D_Y = D. \tag{2.1}$$

Definition 2.1. Let (X, D) be a log surface. If there exists a normal $\mathbb{Q}$-factorial algebraic surface Y and a proper birational morphism $Y \longrightarrow X$ such that D_Y is a boundary then we say that (X, D) is a *generalized MR log canonical (GMRLC) surface.*

A log surface is *log smooth* if X is smooth and D has simple normal crossings. Given a projective morphism $f\colon X \longrightarrow S$ from a normal surface X onto an algebraic variety S and a $\mathbb{Q}$-Cartier divisor D on X we say that D is *f-nef* (*f-ample*) if $D \cdot E \geqslant 0$ (respectively, $D \cdot E > 0$) for every curve E contracted by f (cf. [KM98, Theorem 1.44]). We say that D is *f-semi-ample* if there exists a morphism $g\colon X \longrightarrow Y$ over S such that $D \sim g^*A$ for some $f \circ g^{-1}$-ample $\mathbb{Q}$-Cartier divisor A on Y (cf. [Fuj11a, Lemmas 4.13, 4.14]). We will use the following facts.

Lemma 2.2 ([Fuj21, 4.3, 4.4]). *Let (X, D) be a GMRLC surface.*
(1) If $f\colon X \longrightarrow Z$ is a proper birational morphism onto a normal surface such that $-(K_X + D)$ is f-nef then (Z, f_*Z) is GMRLC.
(2) If D' is a boundary such that $D' \leqslant D$ then (X, D') is GMRLC. In particular, K_X and all components of D are $\mathbb{Q}$-Cartier.

By a *contraction* we mean a morphism between normal varieties which has connected fibers. Writing a birational contraction between log surfaces as $f\colon (X, D) \longrightarrow (X', D')$ we always assume that $D' = f_*D$. The exceptional divisor $\operatorname{Exc} f$ is the sum of curves contracted by f.

Definition 2.3 (A log exceptional curve). Let (X, D) be a log surface. A curve $\ell \subseteq X$ is *log exceptional* if

$$\ell \cdot (K_X + D) < 0 \quad \text{and} \quad \ell^2 < 0. \tag{2.2}$$

By the logarithmic contraction theorem [Fuj21, Theorem 5.5] there exists a birational contraction of ℓ, which we denote by ctr_ℓ. It is known that ℓ is a rational curve and $\ell\cdot(K_X+D)\geqslant -2$ [Fuj21, Theorem 5.6]. We emphasize that the cone theorem and the contraction theorem hold for arbitrary projective log surfaces, also in the relative form, but in this generality the log exceptional curve does not have to be $\mathbb{Q}$-Cartier, and then the direct image of a log canonical divisor can be non-$\mathbb{Q}$-Cartier.

We have the following Corollary from Lemma 2.2.

Remark 2.4 (Direct images of $\mathbb{Q}$-Cartier divisors)**.** Let ℓ be a log exceptional curve on a log surface (X,D). The following hold.

(1) ℓ is $\mathbb{Q}$-Cartier if and only if the direct image of K_X+D under ctr_ℓ is $\mathbb{Q}$-Cartier.
(2) If ℓ is $\mathbb{Q}$-Cartier then its contraction maps $\mathbb{Q}$-Cartier divisors to $\mathbb{Q}$-Cartier divisors.
(3) If (X,D) is GMRLC then ℓ is $\mathbb{Q}$-Cartier.

Proof. Let $f\colon (X,D)\longrightarrow (X',D')$ be the contraction of ℓ.

(1) Since K_X+D is $\mathbb{Q}$-Cartier, the claim follows from the linear equivalence $K_X+D\sim f^*(K_{X'}+D')+a\ell$, where $a>0$.

(2) Let G be a $\mathbb{Q}$-Cartier divisor on X. Then $G+a\ell$ for some $a\in\mathbb{Q}$ is a $\mathbb{Q}$-Cartier divisor trivially intersecting ℓ, hence by the contraction theorem [Fuj21, Theorem 5.5(iii)] $G+a\ell=f^*C$ for some $\mathbb{Q}$-Cartier divisor C on X'. Then $f_*G=C$ is $\mathbb{Q}$-Cartier.

(3) By Lemma 2.2 the divisor $K_{X'}+D'$ is $\mathbb{Q}$-Cartier, hence (1) implies that ℓ is $\mathbb{Q}$-Cartier. ☐

If ℓ is a log exceptional curve on a GMRLC log surface then $K_X+\ell$ is $\mathbb{Q}$-Cartier by Lemma 2.2(2) and Remark 2.4. Since the coefficient of ℓ in D is at most 1 and $D\geqslant 0$, we get $\ell\cdot(K_X+\ell)<0$. If the log surface is log canonical of $\mathbb{Q}$-factorial then it follows that $\ell\cong\mathbb{P}^1$, see [Tan14, Theorem 3.19], cf. [KK94, Lemma 2.3.5].

By a *partial MMP run on log surface* (X,D) (a *partial* (K_X+D) -MMP) we mean a composition of a sequence of birational contractions

$$(X,D)=(X_1,D_1)\xrightarrow{\varphi_1}\dots\xrightarrow{\varphi_n}(X_{n+1},D_{n+1}) \tag{2.3}$$

between log surfaces such that each φ_i is a contraction of a log exceptional curve on (X_i,D_i). An *MMP run* is a maximal partial MMP run.

Assume that (X,D) is GMRLC. Then each (X_i,D_i) is a GMRLC log surface by Lemma 2.2(1) and each $\operatorname{Exc}\varphi_i$ is automatically $\mathbb{Q}$-Cartier by Remark 2.4. Moreover, if (X,D) is $\mathbb{Q}$-factorial (respectively, log canonical) then each (X_i,D_i) is $\mathbb{Q}$-factorial (respectively log canonical). The output of an MMP run is *minimal*, i.e. has no log exceptional curve. By [Fuj21, Theorem 1.5] on a minimal GMRLC log surface either the log canonical divisor is semi-ample or its negative is f-ample for some contraction of positive relative dimension and relative Picard rank 1.

We note that a $\mathbb{Q}$-factorial algebraic surface is automatically quasi-projective [Fuj12, Lemma 2.2] and that every log terminal surface is $\mathbb{Q}$-factorial, see Remark 2.7.

2B. *Discrepancies*

Singularities of log surfaces and their changes under a run of an MMP are conveniently measured in terms of log discrepancies, see [KM98, §2.3]. Given a log surface (X,D) and an irreducible component E of D we denote by $\mathrm{coeff}_E(D)$ the coefficient of E in the irreducible decomposition of D. Given a proper birational morphism from a normal surface $f\colon Y\longrightarrow X$ for a prime divisor E on Y we define the *coefficient* of E over (X,D) and the *log discrepancy* of E over (X,D), as, respectively (see (2.1))

$$c(E;X,D)=\mathrm{coeff}_E(D_Y)\quad\text{and}\quad \mathrm{ld}(E;X,D)=1-\mathrm{coeff}_E(D_Y). \tag{2.4}$$

They depend only on the valuation of the field of rational function on X associated with E, not on f. Let $\mathcal{E}(f)$ denote the set of prime divisors contracted by f. Then $\operatorname{Exc} f=\sum_{E\in\mathcal{E}(f)}E$. We have a linear equivalence over $\mathbb{Q}$:

$$K_Y+f_*^{-1}D+\operatorname{Exc} f\sim f^*(K_X+D)+\sum_{E\in\mathcal{E}(f)}\mathrm{ld}(E;X,D)E. \tag{2.5}$$

We write $\mathrm{ld}(E)$ instead of $\mathrm{ld}(E;X,D)$ if X and D are clear from the context. We call $\mathrm{ld}_Y(X,D)=\sum_{E\in\mathcal{E}(f)}\mathrm{ld}(E;X,D)E$ the *log discrepancy divisor*. If $\mathcal{S}$ is a set of divisors over X then we put $\mathrm{ld}(\mathcal{S})=$

$\inf\{\mathrm{ld}(E) \colon E \in \mathcal{S}\}$. A *divisor over* X is a divisor on some Y as above. It is *exceptional* if its image on X has codimension bigger than 1. We define the log discrepancy of (X, D) as

$$\mathrm{ld}(X, D) = \inf\{\mathrm{ld}(E; X, D) : E \text{ is an exceptional divisor over } X\}. \tag{2.6}$$

and the *total log discrepancy of* (X, D) as

$$\mathrm{tld}(X, D) = \inf\{\mathrm{ld}(E; X, D) : E \text{ is a divisor over } X\}. \tag{2.7}$$

Thus $\mathrm{tld}(X, D) = \min(\mathrm{ld}(X, D), \{\mathrm{ld}(E) \colon E \text{ is a component of } D\})$. We put $\mathrm{ld}(X) = \mathrm{ld}(X, 0)$.

Definition 2.5 (ε-lc surfaces)**.** Let $\varepsilon \in [0, 1] \cap \mathbb{Q}$. A log surface (X, D) is ε*-log canonical* (ε-lc) if $\mathrm{tld}(X, D) \geqslant \varepsilon$. It is ε*-divisorially log terminal* (ε-dlt) if it is ε-log canonical and $\mathrm{ld}(\mathcal{E}(f)) > \varepsilon$ for some log resolution f.

In particular, if (X, D) is ε-lc then the coefficients of components of D do not exceed $1-\varepsilon$. However, even if (X, D) is ε-dlt, the boundary D can have components with coefficient equal to $1-\varepsilon$. Note also that for $\varepsilon \neq 0$ an ε-lc log surface is log terminal, hence $\mathbb{Q}$-factorial, see Remark 2.7.

Remark 2.6. When we blow up the point of intersection of two components of a boundary of a log smooth surface with log discrepancies u_1 and u_2 then the new exceptional curve has log discrepancy $u_1 + u_2$. It follows that the infimum in (2.7) can be computed on the set consisting of components of D and exceptional curves of any fixed log resolution. Hence ε-divisorial log terminality for $\varepsilon \neq 0$ and ε-log canonicity can be checked using any log resolution.

Remark 2.7 (Surface singularities and MMP)**.** Let X be a normal surface. Since the intersection matrix of a resolution of singularities of X is negative definite [Mum61], every Weil $\mathbb{Q}$-divisor has a uniquely determined pullback intersecting trivially all exceptional curves. This determines uniquely a $\mathbb{Q}$-valued intersection product of Weil $\mathbb{Q}$-divisors on X consistent with the projection formula. Then the formula (2.5) defining log discrepancies extends to pairs (X, D) for which $K_X + D$ is not necessarily $\mathbb{Q}$-Cartier. In particular, it allows to define *numerically dlt (numerically lc)* surfaces as the ones for which log discrepancies of some log resolution are positive (non-negative), see [KM98, §4.1]. However, it turns out that in both cases $K_X + D$ is in fact $\mathbb{Q}$-Cartier. Numerically dlt surfaces have rational singularities and are $\mathbb{Q}$-factorial; see [KM98, 4.11, 4.12], [FT12, 6.3, 6.4]. Since the dual graphs of minimal resolutions of numerically lc surfaces are classified, see [Kol92, §3] and [KM98, §4.1], the rationality of numerically dlt surfaces can be verified directly, too [Art66, Theorem 3], and then the $\mathbb{Q}$-factoriality follows by [Lip69, 17.1].

We need the following results. Note that for surfaces intersections of Weil divisors with curves are well defined by Remark 2.7, so the numerical definitions of f-nef and f-ample divisors extend to Weil $\mathbb{Q}$-divisors. We note that the two lemmas below work in any dimension under the assumption that $K_X + D$, $K_{X'} + f_*D$ and A are $\mathbb{Q}$-Cartier. For surfaces this assumption can be dropped.

Lemma 2.8 ([KM98, 3.38])**.** *Let* $f \colon X \longrightarrow X'$ *be a proper birational morphism between normal surfaces and let* D *be a* $\mathbb{Q}$*-divisor on* X*. If* $-(K_X + D)$ *is* f*-nef then for every exceptional divisor* E *over* X' *we have*

$$\mathrm{ld}(E; X', f_*D) \geqslant \mathrm{ld}(E; X, D).$$

If additionally $-(K_X+D)$ *is* f*-ample and* f *is not an isomorphism over the generic point of the center of* E *on* X' *then the inequality is strict.*

Lemma 2.9 (Negativity lemma [KM98, 3.39])**.** *Let* $f \colon X \longrightarrow X'$ *be a proper birational morphism between normal surfaces and let* A *be a* $\mathbb{Q}$*-divisor on* X*. If* f_*A *is effective and* $-A$ *is* f*-nef then* A *is effective and* $\operatorname{Supp} A = f^{-1}(\operatorname{Supp} f(A))$*.*

Corollary 2.10 (f-nef canonical divisor)**.** Let $f \colon X \longrightarrow Y$ be a birational morphism between normal surfaces such that K_X is f-nef. Then $f^*K_Y \sim K_X + E$ for some effective divisor E such that $\operatorname{Supp} E \subseteq \operatorname{Supp} \operatorname{Exc} f$. Moreover, if $\operatorname{Exc} f$ is connected then $\operatorname{Supp} E = \operatorname{Supp} \operatorname{Exc} f$, unless $E = 0$.

Proof. Apply Lemma 2.9 to $A = f^*K_Y - K_X$. □

Lemmas 2.8 and 2.2(1) give the following corollary.

Corollary 2.11. Let (X, D) be an ε-lc log surface for some $\varepsilon \geqslant 0$ and $f \colon X \longrightarrow X'$ a birational morphism onto a normal surface such that $-(K_X + D)$ is f-nef. Then (X, f_*D) is an ε-lc log surface. If (X, D) is ε-dlt and $-(K_X + D)$ is f-ample then (X', f_*D) is ε-dlt.

It follows from Corollary 2.11 that the outcome of an MMP run on an ε-dlt log surface is ε-dlt. Example 2.12 reminds that in the above corollary it is important to have a bound on boundary coefficients, as a contraction of a boundary component may decrease the global log discrepancy (but not the total log discrepancy).

Example 2.12 (Global log discrepancy may decrease)**.** Let $X = \mathbb{F}_n$ for $n \geqslant 3$. Let $X \longrightarrow X'$ be the contraction of the negative section D. We have $(K_X + D) \cdot D < 0$, $f_*D = 0$ and $\operatorname{ld}(D; X', f_*D) = \frac{2}{n} > 1 - \operatorname{coeff}_D(D) = \operatorname{ld}(D; X, D)$, in agreement with Lemma 2.9. At the same time we have $\operatorname{ld}(X', f_*D) = \frac{2}{n} < 1 = \operatorname{ld}(X, D)$.

2C. ***Curves of the second kind***

Sometimes it is useful to contract curves intersecting the log canonical divisor trivially, too.

Definition 2.13. A curve $\ell \subseteq X$ is *log exceptional of the second kind* on (X, D) if

$$\ell \cdot (K_X + D) = 0 \quad \text{and} \quad \ell^2 < 0. \tag{2.8}$$

Such curves appear in the construction of a log canonical model. By [Tan14, Theorem 3.19], on log canonical and on $\mathbb{Q}$-factorial surfaces they are either isomorphic to $\mathbb{P}^1$ or $\mathcal{O}(m(K_X + \ell))|_\ell \cong \mathcal{O}_\ell$ for every positive m for which $m(K_X + \ell)$ is Cartier. As an example the reader may consult [Miy01, 2.4.4-12], where a description of snc-minimal reduced boundary divisors consisting of log exceptional curves of the second kind is given.

Remark 2.14. A log exceptional curve ℓ of the second kind on a GMRLC log surface (X, D) is $\mathbb{Q}$-Cartier, unless $\ell \cap D = \emptyset$ and $\ell \cdot K_X = 0$.

Proof. If ℓ is a component of D then it is $\mathbb{Q}$-Cartier by Lemma 2.2. Assume it is not a component of D. For $f = \operatorname{ctr}_\ell \colon X \longrightarrow X'$ we have $K_X = f^*K_{X'} + a\ell$ for some $a \geqslant 0$, so $a\ell$ is a $\mathbb{Q}$-Cartier divisor. If $\ell \cdot K_X \neq 0$ then $a > 0$ and hence ℓ is $\mathbb{Q}$-Cartier. We may thus assume that $\ell \cdot K_X = 0$. Then $\ell \cdot D = 0$. It follows that ℓ is disjoint from D. $\square$

Let ℓ be a curve on a GMRLC log surface (X, D). We say that ℓ is an *isolated reduced component of D of elliptic type* if ℓ is a connected component of D such that $\operatorname{coeff}_\ell(D) = 1$ and $\ell \cdot (K_X + \ell) = 0$. By Remark 2.14 ℓ is $\mathbb{Q}$-Cartier.

Lemma 2.15. *Let ℓ be a log exceptional curve of the first or second kind on a GMRLC log surface (X, D). Then there exists a contraction $\operatorname{ctr}_\ell \colon (X, D) \longrightarrow (Y, B)$ of ℓ, unless one of the following holds:*

(1) ℓ is an isolated reduced component of D of elliptic type,
(2) ℓ is a non-$\mathbb{Q}$-Cartier K_X-trivial curve disjoint from D.

If the contraction exists then (Y, B) is a GMRLC log surface and $\kappa(K_X + D) = \kappa(K_Y + B)$.

Proof. Assume that the contraction exists. Then (Y, B) is a GMRLC log surface by Lemma 2.2(1). Let n be an integer such that $n(K_Y + B)$ is Cartier. Since $\pi_*(K_X + D) = K_Y + B$, we get $h^0(n(K_X + D)) \leqslant h^0(n(K_Y + B))$. Write $\pi^*(K_Y + B) = K_X + D - a\ell$ for some $a \in \mathbb{Q}$. Since ℓ is log exceptional of the first or second kind, we have $a \geqslant 0$, so $h^0(n(K_Y + B)) \leqslant h^0(n(K_X + D) - na\ell) \leqslant h^0(n(K_X + D))$.

By Remark 2.14 and by (2) we may assume that ℓ, and hence $K_X + \ell$, is $\mathbb{Q}$-Cartier. Let $c = \operatorname{coeff}_\ell(D)$. Assume that $\ell \cdot (K_X + \ell) \geqslant 0$. Then $0 = \ell \cdot (K_X + D) = \ell \cdot (K_X + \ell) + (1-c)(-\ell^2) + \ell \cdot (D - c\ell)$, so since each term is non-negative, ℓ is an isolated reduced elliptic component of D, which gives (1). We may now assume that $\ell \cdot (K_X + \ell) < 0$. Then the existence of the contraction follows from the logarithmic contraction theorem, see [Fuj21, Theorem 5.5] (note that (X, ℓ) is not necessarily GMRLC). $\square$

We define a *partial MMP run of the second kind* as a composition of a sequence (2.3) of birational contractions between log surfaces such that each φ_i is a contraction of a log exceptional curve of the first or second kind on (X_i, D_i). An MMP run of the second kind is a maximal partial MMP of the second kind. If (X, D) is GMRLC then by Lemma 2.2(1) each (X_i, D_i) is GMRLC.

We note that the existence of a contraction of a log exceptional curve of the second kind as in Lemma 2.15 is subtle in general, see Example 2.16, so the MMP of the second kind may stop even though a log exceptional curve of the second kind exists. Since log exceptional curves of the second kind have trivial log discrepancy, it follows that if the initial log surface is log canonical then so is the final one. On the other hand, the process does not respect $\mathbb{Q}$-factoriality, which is why it is better to work with GMRLC surfaces.

The following example shows that the exception in Lemma 2.15(1) may occur. Part (1) is due to Hironaka, see [Har77, Example 5.7.3].

Example 2.16 (Problems with contractions of curves of the second kind)**.**

(1) A log exceptional curve of the second kind may be non-contractible. Let $\sigma\colon X \longrightarrow \mathbb{P}^2$ be a blowup in distinct points $p_1, \ldots, p_n \in \mathbb{P}^2$, $n \geqslant 10$ lying on an elliptic curve and let $\ell \subseteq X$ be the proper transform of that curve. We have $K_X + \ell \sim 0$ and $\ell^2 = 9 - n < 0$, so ℓ is log exceptional of the second kind on (X, ℓ). If there exists a contraction of ℓ then the points are $\mathbb{Z}$-linearly dependent in the group law of $\sigma(\ell)$. Indeed, denoting the contraction by $\pi\colon X \longrightarrow Y$ and taking a very ample divisor H on Y not passing through $\pi(\ell)$ we see that the divisor $\sigma_*\pi^*H$ meets the elliptic curve only in the chosen points, hence induces a nontrivial relation between $p_1, \ldots, p_n$. Choosing the points to be independent we get ℓ which is not contractible (in the category of schemes). Such choice is possible as long as the rank of the elliptic curve, $\dim_{\mathbb{Q}}(\mathrm{Pic}^0(\sigma(\ell)) \otimes \mathbb{Q})$, is not smaller than n, for instance infinite. For algebraically closed fields the latter holds if and only if the field is not an algebraic closure of a finite field, see [FJ74, Theorem 10.1] and [Tan14, Fact 2.3].
(2) A contraction of a log exceptional curve of the second kind may destroy $\mathbb{Q}$-factoriality. Let $\pi\colon X \longrightarrow Y$ be the minimal resolution of a projective cone over an elliptic curve $E \subseteq \mathbb{P}^2$ and let ℓ be the exceptional divisor. Being smooth, X is $\mathbb{Q}$ factorial. Clearly, $\ell \cong E$ and $\ell \cdot (K_X + \ell) = 0$ by adjunction, so ℓ is log exceptional of the second kind. We have $\mathrm{Cl}(Y) \cong \mathrm{Cl}(E)$ [Har77, Exercise II.6.3(a)] and the line over any non-torsion point of E is a non-$\mathbb{Q}$-Cartier Weil divisor, hence Y is not $\mathbb{Q}$-factorial. Note that K_Y is $\mathbb{Q}$-Cartier by adjunction, [KM98, Theorem 5.50].

Remark 2.17. Let us note that in Example 2.16(1) it is also possible to choose the centers of the blowup so that the contraction of ℓ exists. To see this let the centers be the points of intersection of the cubic $\sigma(\ell)$ with some curve $C \subseteq \mathbb{P}^2$ of degree at least 4 meeting it normally. Then the linear system of $C' := \sigma_*^{-1}C$ contains $\ell + (\deg C - 3)\sigma_*^{-1}L$, where L is a line not passing through the points of $C \cap \sigma(\ell)$, hence contains no curve in the base locus. It follows that some $|nC'|$, $n > 0$ has no base points. Since for any curve $U \neq \ell$ we have $U \cdot C' = U \cdot \ell + (\deg C - 3)\sigma(U) \cdot L > 0$, we see that $|nC'|$ is the contraction of ℓ.

3. Almost minimalization

3A. *Relative MMP and reordering contractions*

Although a run of an MMP on a log surface (X, D) improves log singularities (Lemma 2.8), singularities of the underlying surface X may easily get worse for contractions of exceptional curves contained in the boundary. For instance, even if X is smooth, its image may be singular, see Example 3.14. Passing to almost minimal models allows to delay such contractions. Before going into details, we need some preparations.

Recall that given a projective morphism of normal varieties $f\colon X \longrightarrow Y$ and a divisor D on X one defines the *relative MMP over* Y. For surfaces we simply require that in the sequence (2.3) proper transforms of the contracted curves are contained in the fibers of f. Then each (X_i, D_i) has an induced projective morphism $f_i\colon X_i \longrightarrow Y$. The following result will be used frequently, cf. [Kol13, 1.35] and [Fuj11b].

Lemma 3.1 (The minimal model of a birational morphism)**.** *Let (X, D) be a GMRLC log surface and $f\colon X \longrightarrow Y$ a birational morphism onto a normal surface. Then there is a unique $(K_X + D)$-MMP run over Y, which we denote by $f_{D,\min}$. Moreover, $f_{D,\min}$ factors through the contraction of f-exceptional curves E with $\mathrm{ld}(E; Y, f_*D) > \mathrm{ld}(E; X, D)$.*

Proof. (1) Let ℓ, m be distinct log exceptional curves on (X, D) contracted by f and let $g\colon (X, D) \longrightarrow (X', D')$ be the contraction of m. By Lemma 2.2 (X', D') is a (GMRLC) log surface. Since $\mathrm{Exc}\, f$ is negative definite, $g(\ell)^2 < 0$. We have $g^*(K_{X'} + D') = K_X + D - um$, where $u = (m \cdot (K_X + D))/m^2 > 0$, hence $g(\ell) \cdot (K_{X'} + D') = \ell \cdot g^*(K_{X'} + D') \leqslant \ell \cdot (K_X + D) < 0$. Thus the image of ℓ is log exceptional on (X', D'), too. By induction we infer that the exceptional divisor of $f_{D,\min}$ is determined uniquely, hence $f_{D,\min}$ is unique by the normality of a minimal model.

Put $B = f_*D$. We have $K_X + f_*^{-1}B + \operatorname{Exc} f - f^*(K_Y + B) = \sum_{E\in\mathcal{E}(f)} \operatorname{ld}(E; Y, B)E$. Since $\operatorname{ld}(E; X, D) = 1 - \operatorname{coeff}_E(D)$, we can write this as

$$K_X + D - f^*(K_Y + B) = \sum_{E\in\mathcal{E}(f)} (\operatorname{ld}(E; Y, B) - \operatorname{ld}(E; X, D))E. \tag{3.1}$$

Put $a_E := \operatorname{ld}(E; Y, B) - \operatorname{ld}(E; X, D)$. By the uniqueness of $f_{D,\min}$ it is sufficient to show that there exists a partial $(K_X + D)$-MMP over Y which contracts exactly the components with $a_E > 0$. We may assume that $\mathcal{E}^+(f) := \{E \in \mathcal{E}(f) : a_E > 0\}$ is nonempty. Since the intersection matrix of the divisor $A := \sum_{E\in\mathcal{E}^+(f)} a_E E$ is negative definite, we get $0 > A^2 = \sum_{E\in\mathcal{E}^+(f)} a_E E \cdot A$, hence $a_E E \cdot A < 0$ for some $E \in \mathcal{E}^+(f)$. Then $E \cdot (K_X + D) \leqslant E \cdot A < 0$, so E is log exceptional on (X, D). We obtain the desired partial $(K_X + D)$-MMP over Y by induction. □

A birational morphism of log surfaces $f : (X, D) \longrightarrow (Y, B)$ is called *log crepant* (respectively, crepant) if $f^*(K_Y + B) = K_X + D$ (respectively, $f^*K_Y = K_X$). For the definition of an MMP run of the second kind see Section 2C.

Corollary 3.2 (A characterization of MMP runs)**.** Let (X, D) be a GMRLC log surface. Let $f : X \longrightarrow Y$ be a birational morphism onto a normal surface and let $\mathcal{E}(f)$ denote the set of prime divisors contracted by f. Then the following hold.

(1) f is an MMP run of the first kind on (X, D) if and only if $\operatorname{ld}(E; Y, f_*D) > \operatorname{ld}(E; X, D)$ for all $E \in \mathcal{E}(f)$.
(2) If $\operatorname{ld}(E; Y, f_*D) \geqslant \operatorname{ld}(E; X, D)$ for all $E \in \mathcal{E}(f)$ then $f = \sigma \circ f'$, where f' is an MMP run of the second kind on (X, D) and $\sigma : (X', f'_*D) \longrightarrow (Y, B)$ is a crepant morphism whose exceptional curves are non-$\mathbb{Q}$-Cartier and disjoint from f'_*D.

Proof. By Lemma 2.8 we may assume that $\operatorname{ld}(E; Y, f_*D) \geqslant \operatorname{ld}(E; X, D)$ and by Lemma 3.1 we may assume that $\operatorname{ld}(E; Y, f_*D) = \operatorname{ld}(E; X, D)$ for each $E \in \mathcal{E}(f)$. Lemma 2.2 implies that (Y, f_*D) is a GMRLC log surface. By Lemma 2.15 we may assume that each log exceptional curve of the second kind on (X, D) is as in (1) or (2). In particular, it follows that $\operatorname{Exc} f = E_1 + E_2$, where E_1 and E_2 are disjoint, all components of E_1 are connected components of D of elliptic type and all components of E_2 are non-$\mathbb{Q}$-Cartier, K_X-trivial and disjoint from D. Let E be a component of E_1. Gluing $X - E$ with $Y - f(\operatorname{Exc} f - E)$ we see that there exists a contraction of E; denote it by τ. It is log crepant, its image is again a log surface. Thus we may contract components of E_1 one by one, hence we may in fact assume that $E_1 = 0$, in which case the claim is trivial. □

Corollary 3.3 (Reordering MMP contractions)**.** Assume that $f : (X, D) \longrightarrow (X', D')$ is a partial MMP run of the first (second) kind on a GMRLC log surface and that $f' : (X, D) \longrightarrow (Y, B)$ is a birational morphism onto a GMRLC log surface with $\operatorname{Exc} f' \leqslant \operatorname{Exc} f$. Then $f \circ (f')^{-1} : (Y, B) \dashrightarrow (X', D')$ is a partial MMP run of the first kind (respectively, of the second kind composed with a crepant morphism whose exceptional curves are non-$\mathbb{Q}$-Cartier and disjoint from the direct image of B). Moreover,

$$\kappa(K_X + D) = \kappa(K_Y + B) = \kappa(K_{X'} + D'). \tag{3.2}$$

Proof. The rational map $\alpha := f \circ (f')^{-1}$ is defined off the image of the exceptional divisor of f'. We argue that it is regular. We may assume that $\operatorname{Exc} f$ is connected, so its image is a point $x' \in X'$ and hence there exists an affine neighborhood $U \subseteq X$ of x'. Let $y \in f'(\operatorname{Exc} f')$. For some open neighborhood V of y it maps $V \setminus \{y\}$ to U. Since $y \in Y$ is a normal point, α is regular at y, see [Eis95, Corollary 11.4].

By assumption (Y, B) is GMRLC. Since f is an MMP run of the first (second) kind, we have $\operatorname{ld}(E; X, D) > \operatorname{ld}(E; X', D')$ (respectively, $\operatorname{ld}(E; X, D) \geqslant \operatorname{ld}(E; X', D')$) for every $E \in \mathcal{E}(\alpha)$ by Lemma 2.8. Since $B = f_*D$, we have $\operatorname{ld}(E; Y, B) = 1 - \operatorname{coeff}_E(B) = 1 - \operatorname{coeff}_E(D) = \operatorname{ld}(E; X, D)$, so the claim about α is a consequence of Corollary 3.2.

Given a birational morphism $\sigma : X_1 \longrightarrow X_2$ and a divisor D on X_1 we have an induced injection $H^0(X_1, D) \hookrightarrow H^0(X_2, \sigma_*D)$. It follows that $\kappa(K_X + D) \leqslant \kappa(K_Y + B) \leqslant \kappa(K_{X'} + D')$. Since $(X, D) \longrightarrow (X', D')$ is a composition of contractions of log exceptional curves of the first and second kind, we have $\kappa(K_X + D) = \kappa(K_{X'} + D')$. □

The corollary says in particular that if f is a partial MMP run then so is $f \circ (f')^{-1}$. The following example reminds that in general this is not true for f'.

Example 3.4 (Reordering contractions)**.** Let ℓ_1 be a (-1)-curve on a smooth projective surface X and let ℓ_2 be a (-2)-curve meeting ℓ_1 once. The contraction of $\ell_1+\ell_2$ is a composition of the contraction of ℓ_1 followed by the contraction of the image of ℓ_2, which is a (-1)-curve, hence both resulting surfaces are smooth. But it is also a composition of the contraction of ℓ_2 onto a singular (canonical) surface Y followed by the contraction of ℓ_1', the image of ℓ_1. Here ℓ_2 is not log exceptional on X, but ℓ_1' is log exceptional on Y.

3B. ***The definition of an almost minimalization***

An inductive construction of an almost minimal model was given by Miyanishi in the special case of log smooth surfaces with reduced boundary, see [Miy01, p. 107]. In [Pal19] we generalized the construction in an analogous way to smooth completions of affine surfaces with half-integral boundaries. Here we give a general simple definition.

Given a GMRLC log surface X (with no boundary) and a birational morphism onto a normal surface $f\colon X \longrightarrow Y$ we denote the unique relative MMP given by Lemma 3.1 by $f^{\#}\colon X \longrightarrow X_f$ and the resulting minimal model of f by $f_{\min} := f_{0,\min}\colon X_f \longrightarrow Y$. This gives a commutative diagram with K_{X_f} being $f_{\min}$-nef:

(3.3)
$$\begin{tikzcd} X \arrow[rr, "f^{\#}"] \arrow[dr, "f"'] & & X_f \arrow[dl, "f_{\min}"] \\ & Y & \end{tikzcd}$$

Given a divisor D on X we put $D_f := f^{\#}_* D$. Note that by Lemma 2.9 we have $f^*_{\min}K_Y \geqslant K_{X_f}$.

Definition 3.5 (Almost minimalization)**.** Let $f\colon (X,D) \longrightarrow (\overline{X},\overline{D})$ be a partial MMP run of a first or second kind on a GMRLC log surface. Then $f^{\#}\colon X \longrightarrow X_f$, that is, the unique K_X-MMP over $\overline{X}$, is called a *(partial) almost minimalization of* f and the log surface (X_f, D_f) is called an *almost minimal model* of (X,D).

Remark 3.6. In Definition 3.5 the divisor K_X and all components of D are $\mathbb{Q}$-Cartier by Lemma 2.2. Since $(\overline{X},\overline{D})$ is GMRLC, the same holds for $K_{\overline{X}}$ and the components of $\overline{D}$. On the other hand, Remark 2.4 implies that K_{X_f} and the components of D_f are $\mathbb{Q}$-Cartier. Since (X,D) is GMRLC, the surface X, and hence the surface X_f, is GMRLC. Moreover, X_f is $\mathbb{Q}$-factorial (log canonical) if X is $\mathbb{Q}$-factorial (respectively, log canonical).

The geometry of (X_f, D_f) is to be studied. It follows from the definition that an almost minimalization is a composition of contractions of lifts of some log exceptional curves on (X,D) and its images under the contractions constituting f. The analysis of how these contractions affect the geometry of the boundary comes down to the analysis of reordering MMP contractions for (X,D). It is well-known that in general changing the order of contractions may lead to worse singularities of the intermediate surfaces, see Example 3.4.

Lemma 3.7 (Relative minimalization and composition)**.** *Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be birational morphisms between normal projective surfaces. Assume that X and Y are GMRLC. Then*

(3.4)
$$(g\circ f)^{\#} = (g\circ f_{\min})^{\#}\circ f^{\#}.$$

Moreover, $(g\circ f_{\min})^{\#}$ factors through $(g^{\#}\circ f_{\min})^{\#}$ and the latter morphism contracts the proper transform of $\operatorname{Exc} g^{\#}$*.*

Proof. Put $h = g\circ f$. As noted above, X_f, Y_g and $X_{g^{\#}\circ f}$ and $X_{g\circ f}$ are GMRLC. By definition $(g\circ f_{\min})^{\#}\circ f^{\#}$ is a K_X-MMP run over Z, hence by uniqueness, see Lemma 3.1, we have $(g\circ f)^{\#} = (g\circ f_{\min})^{\#}\circ f^{\#}$. Moreover $(g^{\#}\circ f_{\min})^{\#}\circ f^{\#}$ is a K_X-MMP over Y_g, hence over Z, so $(g\circ f)^{\#}$ factors through it by uniqueness. This gives a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f^{\#}"] \arrow[dr, "f"'] & X_f \arrow[r, "(g^{\#}\circ f_{\min})^{\#}"] \arrow[d, "f_{\min}"] \arrow[rr, bend left, "(g\circ f_{\min})^{\#}"] & X_{g^{\#}\circ f} \arrow[r] \arrow[d] & X_{g\circ f} \arrow[dl] \\
& Y \arrow[r, "g^{\#}"] \arrow[dr, "g"'] & Y_g \arrow[d] & \\
& & Z &
\end{tikzcd} \tag{3.5}$$

To show that $(g^{\#}\circ f_{\min})^{\#}$ contracts the proper transform of $G := \operatorname{Exc} g^{\#}$ it is sufficient to construct a K_{X_f}-MMP over Y_g which contracts the proper transform of G. Put $\alpha = f_{\min}$. Take a component m of G intersecting K_Y negatively and let $\sigma\colon Y \longrightarrow Y'$ be its contraction. By Corollay 2.10 we have $K_{X_f} = \alpha^*K_Y - E$ for some effective α-exceptional divisor E. Put $m' = \alpha_*^{-1}m$. Then $m' \cdot K_{X_f} \leqslant m' \cdot \alpha^*K_Y = m \cdot K_Y < 0$, so there exists a K_{X_f}-MMP over Y' contracting m' whose outcome is a birational morphism $\alpha'\colon X' \longrightarrow Y'$ for which $K_{X'}$ is α'-nef. Thus we may replace X_f, Y, $\alpha\colon X_f \longrightarrow Y$, $g^{\#}\colon Y \longrightarrow Y_g$ with X', Y', $\alpha'\colon X' \longrightarrow Y'$ and $g^{\#}\circ\sigma^{-1}\colon Y' \longrightarrow Y_g$ and we continue the construction until the whole proper transform of G is contracted. $\square$

Corollary 3.8. If $f\colon (X, D) \longrightarrow (\overline{X}, \overline{D})$ is an MMP run of the first (second) kind on a GMRLC log surface then $f_{\min}$ is an MMP run on (X_f, D_f) of the first (second) kind such that $\operatorname{Exc}\alpha$ consists of some components of D_f and, in case f is of the second kind, of some K_{X_f}-trivial curves disjoint from D_f.

Proof. In the formula (3.1) for $f_{\min}$ we have $a_E \geqslant 0$ for all exceptional curves contracted by $f_{\min}$. Assume some a_E is positive. Then as in the proof of Lemma 3.1 we get a log exceptional curve E on (X_f, D_f). We have $E \cdot D_f < -E \cdot K_{X_f} \leqslant 0$, so E is a component of D. By Remark 3.6 E is $\mathbb{Q}$-Cartier, so its contraction, call it σ, is a part of an MMP run over $\overline{X}$ and maps $\mathbb{Q}$-Cartier divisors to $\mathbb{Q}$-Cartier divisors. We have $\sigma^*(\sigma_*K_{X_f}) = K_{X_f} + uE$ for some $u \geqslant 0$, hence $\sigma_*K_{X_f}$ is nef. By induction we may thus assume that $f_{\min}$ is log crepant. Let E be an exceptional component of $f_{\min}$. We have $E \cdot D_f = -E \cdot K_{X_f} \leqslant 0$. If the inequality is strict then

By Corollary 3.3 we may assume that $f = f_{\min}$. Decompose f as $f = f' \circ \sigma$, where $\sigma\colon X \longrightarrow X'$ is a contraction of a log exceptional curve ℓ of the first (second) kind on (X, D). Since K_X is f-nef, $\sigma^*K_{X'} = K_X + u\ell$ for some $u \geqslant 0$, which implies that $K_{X'}$ is f'-nef. By induction with respect to the number of exceptional components of $\operatorname{Exc} f$ we reduce the proof to the case $f = \sigma$. If ℓ is not a component of D then $0 \leqslant \ell \cdot D \leqslant \ell \cdot (K_X + D) \leqslant 0$, so the assertion is clear. $\square$

Remark 3.9. Let the notation be as in Lemma 3.7. We note that $(g \circ f)^{\#}$ may contract more than $(g^{\#} \circ \alpha)^{\#} \circ f^{\#}$. For instance, if X is minimal over Y (that is, K_X is f-nef) and Y is minimal over Z (that is, K_Y is g-nef) then the latter morphism is the identity. But it can happen that at the same time X is not minimal over Z, so $(g \circ f)^{\#}$ is not an isomorphism; see Example 3.10.

The following example shows that for two birational morphisms $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ such that K_X is f-nef and K_Y is g-nef the divisor K_X does not have to be $(g \circ f)$-nef. By $[a_1, \dots, a_n]$ we denote an snc divisor which is a chain of rational curves with subsequent selfintersection numbers equal to $-a_1, -a_2, \dots, -a_n$, respectively; cf. Section 4A.

Example 3.10 (Relatively nef canonical divisors under composition)**.** Consider a smooth projective surface X with a divisor $[n_1, 1, n_2]$ on it, where n_1, n_2 are positive integers such that $n_1 \geqslant 3$ and $n_2 \geqslant 2 + \frac{4}{n_1 - 2}$. Let A be the (-1)-curve and let D_i, $i = 1, 2$ be the component with self-intersection $-n_i$. Let $h\colon X \longrightarrow Z$ be the contraction of $A + D_1 + D_2$ and let $f\colon X \longrightarrow Y$ be the contraction of $D_1 + D_2$. Then $g = h \circ f^{-1}\colon Y \longrightarrow Z$ is the contraction of f_*A. We have $f^*K_Y = K_X + (1 - \frac{2}{n_1})D_1 + (1 - \frac{2}{n_2})D_2$, hence $f_*A \cdot K_Y = 1 - \frac{2}{n_1} - \frac{2}{n_2} \geqslant 0$. It follows that K_X is f-nef and K_Y is g-nef, but K_X is not $h = g \circ f$-nef. Still, $h^*K_Z - K_X = f^*(g^*K_Z - K_Y) + (f^*K_Y - K_X)$ is effective by Corollary 2.10.

3C. ***Peeling and squeezing***

Let (X,D) be a log surface. Given an MMP run $\psi\colon (X,D)\longrightarrow(\overline{X},\overline{D})$, by Corollary 3.3 its almost minimalization $\psi_{\mathrm{am}}:=\psi^{\#}\colon (X,D)\longrightarrow (X_{\psi},D_{\psi})$ induces an MMP run $\psi_{\min}=\psi\circ\psi_{\mathrm{am}}^{-1}\colon (X_{\psi},D_{\psi})\longrightarrow(\overline{X},\overline{D})$ such that $K_{X_{\psi}}$ is $\psi_{\min}$-nef:

$$\begin{tikzcd} (X,D) \arrow[r, "\psi_{\mathrm{am}}"] \arrow[dr, "\psi"'] & (X_{\psi},D_{\psi}) \arrow[d, "\psi_{\min}"] \\ & (\overline{X},\overline{D}) \end{tikzcd} \tag{3.6}$$

Conversely, using (3.4) we can construct an almost minimal model of (X,D) in steps in parallel to a construction of an MMP run ψ. We now discuss how to conveniently group these steps. Note that $\operatorname{Exc}\psi_{\min}\subseteq D_{\psi}$ by Corollary 3.8. The distinction between log exceptional curves contained and not contained in the boundary leads to the definition of a peeling, cf. [Miy01, 2.3.3.6].

Definition 3.11 (Peeling, almost minimality)**.** Let (X,D) be a log surface.

(1) A *partial peeling* of (X,D) (and of D) is a partial MMP run α on (X,D) for which $\operatorname{Exc}\alpha\subseteq\operatorname{Supp}D$. A *peeling* is a maximal partial peeling, i.e. a partial peeling which cannot be extended to a partial peeling with a strictly bigger number of contracted curves.

Let α be a partial peeling of (X,D). We have a unique decomposition $\alpha=\alpha_{\min}\circ\alpha_{\mathrm{am}}$.

(2) α_{am} is called an *α-squeezing* of (X,D).
(3) α is called *pure* if K_X is α-nef. In this case we say that (X,D) is *α-squeezed*.
(4) If (X,D) is α-squeezed and $(\alpha(X),\alpha_*D)$ is minimal then (X,D) is called *α-almost minimal*.

We say that (X,D) is *almost minimal* (respectively, *squeezed*) if it is α-almost minimal (respectively, α-squeezed) for some pure peeling α.

Definition 3.12 (Redundant and almost log exceptional curves)**.** Let α be a pure partial peeling of a log surface (X,D) and let $\ell\subseteq X$ be a curve. We say that:

(1) ℓ is *α-almost log exceptional* if $\ell\nsubseteq D$ and $\alpha(\ell)$ is log exceptional on $(\alpha(X),\alpha_*D)$,
(2) ℓ is *α-redundant* if $\ell\subseteq D$, $\ell\cdot K_X<0$ and $\alpha(\ell)$ is log exceptional on $(\alpha(X),\alpha_*D)$.

A curve is *almost log exceptional* (redundant) if it is α-almost log exceptional (respectively, α-redundant) for some pure peeling α.

We use terms like 'α-redundant' and 'redundant with respect to α' interchangeably. A peeling restricts to an isomorphism of $X\setminus D$ onto its image. Since a redundant curve intersects K_X negatively, for smooth X it is in particular a (-1)-curve. A redundant component of D or an almost log exceptional curve on (X,D) is in general not log exceptional itself, so while it can be contracted, the contraction is not a part of an MMP run for (X,D). In particular, the effect of the contraction on (log discrepancies of) log singularities requires a more careful analysis. Squeezing should be thought of as a useful preparation for running an almost MMP by pre-contracting some components of D without making the singularities of the underlying surface X worse.

Corollary 3.13.

(1) Let α be a pure partial peeling of (X,D). Then (X,D) is α-squeezed if D contains no α-redundant component. It is α-almost minimal if additionally there are no α-almost log exceptional curves on (X,D).
(2) (X,D) is almost minimal if and only if there exists an MMP run ψ on (X,D) for which ψ_{am} is an isomorphism.

Proof. (1) follows from definitions. (2) If (X,D) is α-almost minimal then we take $\psi=\alpha$. Now ψ_{am} is an isomorphism, because K_X is ψ-nef. Conversely, if ψ_{am} is an isomorphism then K_X is ψ-nef, hence $\operatorname{Exc}\psi\subseteq D$ by Corollary 3.8, so ψ is a pure peeling. ☐

Example 3.14 (Peeling and almost minimality)**.** Let $\mathbb{F}_n$, $n\geqslant 2$ be a Hirzebruch surface. Let C be the negative section and F_i, $i=1,\dots,N$, $N\geqslant 1$ be distinct fibers of the $\mathbb{P}^1$-fibration. Put $D=cC+\sum_{i=1}^{N}w_iF_i$, where $c,w_1,\dots,w_N\in\mathbb{Q}\cap[0,1]$. The peeling morphism is either the identity or it contracts C. In each case the log surface $(\mathbb{F}_n,D)$ is almost minimal. The peeling morphism does contract C if and only if $C\cdot(K+D)<0$, that is, if $n(1-c)+\sum_{i=1}^{N}w_i<2$. For instance, if all coefficients of D are equal to r then the condition reads as $n(1-r)+Nr<2$, so peeling contracts C if and only if $N=1$ and $r\in[1-\frac{1}{n-1},1]$.

Example 3.15 (An almost minimal surface with an α-almost log exceptional curve)**.** Blowing up three times on $\mathbb{P}^1\times\mathbb{P}^1$ we construct a smooth $\mathbb{P}^1$-fibered surface X with a unique degenerate fiber $T_1+T_2+T_3+L$ where T_i, $i=1,2,3$ are (-2)-curves and L is a (-1)-curve meeting T_2. Let F_1,F_2 be two smooth fibers and T_0 be a section meeting T_3 and having self-intersection number equal to 0. Put $D=T_1+T_2+T_3+T_0+F_1+F_2$. The curve L is a unique (-1)-curve on X intersecting D once. Let α be the contraction of T_1+T_2. It is a pure partial peeling. There are two log exceptional curves on $(\alpha(X),\alpha_*D)$: $\alpha(T_3)$ and $\alpha(L)$. Let ψ be the contraction of $T_1+T_2+T_3$. It is an MMP run with singular image $\psi(X)$. (Another MMP run ψ' has $\operatorname{Exc}\psi'=T_1+T_2+L$, in which case $\psi'(X)$ is smooth). We infer that (X,D) is a log smooth almost minimal log surface. Since L is α-almost log exceptional, (X,D) is not α-almost minimal. Thus, on an almost minimal log surface there may exist curves which are almost log exceptional with respect to some pure partial peeling.

Almost log exceptional (respectively, α-redundant) curves will be also called *almost log exceptional* (respectively, *α-redundant*) *curves of the first kind*. Their numerical characterization will be discussed in the next sections. For their analogues of the second kind see Definition 3.25. For convenience the following result is formulated jointly for both kinds.

Lemma 3.16 (Almost log exceptional vs redundant)**.** *Let α be a pure partial peeling of (X,D). If $A\subseteq X$ is an α-almost log exceptional curve of the first or second kind then*

(1) $A\cdot K_X<0$,

(2) for every $s\in(0,1]\cap\mathbb{Q}$ the morphism $\operatorname{ctr}_{\alpha(A)}\circ\alpha$ is a peeling of $(X,D+sA)$,

(3) A is α'-redundant of the first kind on $(X,D+sA)$ for some pure partial peeling α' of $(X,D+sA)$ with $\operatorname{Exc}\alpha'\subseteq\operatorname{Exc}\alpha$.

Proof. Let $(\overline{X},\overline{D})$ be the image of α and let $\varphi\colon\overline{X}\longrightarrow Y$ be the contraction of $\alpha(A)$. By Corollary 2.10 we have $A\cdot K_X\leqslant A\cdot\alpha^*(K_{\overline{X}})=\alpha(A)\cdot K_{\overline{X}}$. Since $A\not\subseteq D$, we have $\alpha(A)\cdot K_{\overline{X}}\leqslant\alpha(A)\cdot(K_{\overline{X}}+\overline{D})\leqslant 0$. For a curve of the first kind the latter inequality is strict and for a curve of the second kind $A\cdot K_X\neq 0$ by definition, which gives (i). By Lemma 2.8 for every prime divisor E contracted by α we have

$$\operatorname{ld}(E;Y,\varphi_*\overline{D})\geqslant\operatorname{ld}(E;\overline{X},\overline{D})>\operatorname{ld}(E;X,D)=\operatorname{ld}(E;X,D+sA).$$

On the other hand,

$$\operatorname{ld}(A;Y,\varphi_*\overline{D})\geqslant\operatorname{ld}(A;\overline{X},\overline{D})=1>1-s=\operatorname{ld}(A,X,D+sA).$$

By Corollary 3.2 the morphism $\varphi\circ\alpha\colon(X,D+sA)\longrightarrow(Y,\varphi_*D)$ is a peeling, hence (ii). Since α is pure, A is the only curve in $\operatorname{Exc}\alpha+A$ intersecting K_X negatively, hence there exists α' as in (iii). $\square$

3D. ***Effective almost minimalization***

The following lemma shows how to conveniently construct an almost minimal model of a log surface (X,D) in parallel to a minimal model. It shows also that the construction generalizes Miyanishi's construction for log smooth surfaces (with the squeezing a being an analogue of an snc-minimalization), see [Miy01, 2.3.11], cf. Remark 3.20

Lemma 3.17 (Effective almost minimalization)**.** *Given a log surface (X,D) there exists a commutative diagram*

$$\begin{tikzcd}
(X,D) \arrow[r, "\psi_0"] \arrow[dr, "\overline{\psi}_0"'] & (X_1,D_1) \arrow[r, "\psi_1"] \arrow[d, "\alpha_1"] & (X_2,D_2) \arrow[r, "\psi_2"] \arrow[d, "\alpha_2"] & \dots \arrow[r, "\psi_n"] & (X_{n+1},D_{n+1}) \arrow[d, "\alpha_{n+1}"] \\
 & (\overline{X}_1,\overline{D}_1) \arrow[r, "\overline{\psi}_1"] & (\overline{X}_2,\overline{D}_2) \arrow[r, "\overline{\psi}_2"] & \dots \arrow[r, "\overline{\psi}_n"] & (\overline{X}_{n+1},\overline{D}_{n+1})
\end{tikzcd} \tag{3.7}$$

where

(1) $\overline{\psi}_0$ is a partial peeling and $\overline{\psi}_i$ for $i\geqslant 1$ is a composition of a contraction of a single log exceptional curve $\overline{A}_i\not\subseteq\overline{D}_i$ and a partial peeling,

(2) α_i is a pure partial peeling for $i\geqslant 1$,

(3) $\psi_0 = (\overline{\psi}_0)_{\mathrm{am}}$ *and of* $\psi_i = (\overline{\psi}_i \circ \alpha_i)_{\mathrm{am}}$ *for* $i \geqslant 1$*, hence* ψ_i *is a composition of a contraction of an* α_i*-almost log exceptional curve* $A_i := (\alpha_i)_*^{-1}\overline{A}_i \subseteq X_i$ *with a* τ_i*-squeezing, where* $\overline{\psi}_i \circ \alpha_i = \tau_i \circ \mathrm{ctr}_{A_i}$*.*
(4) $\overline{\psi} := \overline{\psi}_n \circ \ldots \circ \overline{\psi}_0$ *is an MMP run and* $(\overline{X}_{n+1}, \overline{D}_{n+1})$ *is a minimal model of* (X, D)*.*
(5) $\psi := \psi_n \circ \ldots \circ \psi_0 = \overline{\psi}_{\mathrm{am}}$ *and* (X_{n+1}, D_{n+1}) *is an almost minimal model of* (X, D)*.*

Every MMP run $\overline{\psi}$ *on* (X, D) *can be obtained this way. Moreover, for every such diagram we have:*
(6) n *is the number of* $\overline{\psi}$*-exceptional curves not contained in* D*.*
(7) $\mathrm{ld}(X_{i+1}) \geqslant \mathrm{ld}(X_i)$ *for* $i \geqslant 0$*, where* $X_0 = X$*,*
(8) $\kappa(K_{X_i} + D_i) = \kappa(K_X + D)$ *for* $i \geqslant 1$*,*

Proof. Clearly, every MMP run $\overline{\psi}$ on (X, D) can be decomposed (non-uniquely) as $\overline{\psi} = \overline{\psi}_n \circ \ldots \circ \overline{\psi}_0$, where $\overline{\psi}_i$ are as in (1). We define inductively ψ_i to be an almost minimalization as in (3) and hence by Corollary 3.8 we get the induced α_i as in (2). By Lemma 3.16 $A_i \cdot K_{X_i} < 0$, so ψ_i is a composition of a contraction of A_i with a τ_i-squeezing. Parts (4) and (6) follow from definition and part (5) follows from equation (3.4). (7) is a consequence of the definition of almost minimalization and of Lemma 2.8. (8) follows from Corollary 3.3. □

Remark 3.18 (Summary)**.** The above lemma shows that an almost minimal model of a given log surface (X, D) can be constructed inductively as follows.
(1) Choose a maximal pure partial peeling morphism α of (X, D) and consider the triple (X, D, α).
(2) If there exists a curve $\ell \subseteq X$ such that $\alpha(\ell)$ is log exceptional then put $D' := D$ in case $\ell \subseteq D$ and $D' := D + \ell$ otherwise, let σ be an $\mathrm{ctr}_\ell \circ\alpha$-squeezing of (X, D') and let α' be a maximal pure partial peeling extending $\mathrm{ctr}_\ell \circ\alpha \circ \sigma^{-1}$. Replace (X, D, α) with $(\sigma(X), \sigma_* D', \alpha')$ and repeat.
(3) The final (X, D) is an almost minimal model and α is a pure peeling morphism onto a minimal model.

In part (2) note that ℓ is necessarily a redundant component of D or an α-almost log exceptional curve, hence a redundant component of D'. Also, σ is the composition of successive contractions of redundant components of D' and its respective images which are contained in $\ell + \operatorname{Exc}\alpha$ and its respective images.

Remark 3.19. In Lemma 3.17 choose $\overline{\psi}_0$ to be a peeling and $\overline{\psi}_i$ for $i \geqslant 1$ to be a composition of a contraction of a single log exceptional curve $\overline{A}_i \not\subseteq \overline{D}_i$ and a peeling. Then in Lemma 3.17 α_i are (pure) peeling morphisms and (X_i, D_i) are squeezed.

Remark 3.20 (Comparison with Miyanishi's construction)**.** Let (X, D) be a log smooth surface with a reduced boundary. In this case there is already a definition of an almost minimal model by Miyanishi [Miy01, 2.3.11] which is very close the one above. The difference is that in our construction ψ (and $\overline{\psi}$) contracts slightly less. To see the difference let $C \subseteq D$ be a superfluous component, that is, a (-1)-curve meeting at most two other components of D, each at most once in the sense of intersection theory. In particular $\beta_D(C) \leqslant 2$. In Miyanishi's construction we contract C, which is natural (as then log smoothness is preserved), but not necessary in general and causes some minor problems. Indeed, if $\beta_D(C) \leqslant 1$ then C is log exceptional, so there is no difference with our definition, but if $\beta_D(C) = 2$ then $C \cdot (K_X + D) = 0$, so C is log exceptional of the second kind, see Definition 2.8. If such C does not meet any maximal admissible (rational) twig or fork of D (see Section 4) then it is not almost log exceptional in our sense and does not need to be contracted. In fact it is not possible to assure in general that the contraction of C is a part of some MMP run. Still, if one wants to contract such curves too, then this can be done after an almost minimal model in the sense of our definition is reached. Lemma 3.24 assures that after such additional contractions the image of (X, D) remains almost minimal.

3E. ***Analogues for curves of the second kind***

We now discuss analogs of the above definitions for curves of the second kind. The following lemma implies that contractions of log exceptional curves of the second kind (if they exist, see Example 2.16) can be delayed until an almost minimal model is reached and then they respect minimality. Recall that given an effective $\mathbb{Q}$-divisor $D = \sum d_i D_i$, where D_i are distinct prime components of D we put $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$, where $\lfloor d_i \rfloor$ is the greatest integer not bigger than d_i.

Lemma 3.21 (Delaying contractions of the second kind)**.** *Let* $\gamma\colon (X, D) \longrightarrow (\overline{X}, \overline{D})$ *be a log crepant birational morphism between log surfaces and let* $\overline{\psi}\colon (\overline{X}, \overline{D}) \longrightarrow (\overline{Y}, \overline{B})$ *be a partial MMP run of the* *first kind or of the second kind with $\lfloor D \rfloor = 0$. Then there exists $\psi\colon (X,D) \longrightarrow (Y,B)$, an MMP run of the first (respectively, second) kind with $\gamma_*^{-1}(\operatorname{Exc}\overline{\psi}) \leqslant \operatorname{Exc}\psi$, such that the induced birational morphism $\overline{\gamma}\colon (Y,B) \longrightarrow (\overline{Y},\overline{B})$ is log crepant. In particular, if (X,D) is minimal then $(\overline{X},\overline{D})$ is minimal.*

Proof. By induction we may assume that $\overline{\psi}$ contracts a single log exceptional curve ℓ of the first (second) kind. Put $\ell' = \gamma_*^{-1}\ell$. We have $\ell' \cdot (K_X + D) = \ell \cdot (K_{\overline{X}} + \overline{D})$, so ℓ' is log exceptional of the first (second) kind. Contract successively ℓ' and log exceptional curves of the first kind contained in the successive images of D. In case ℓ' is log exceptional of the second kind the contraction exists by Lemma 2.15, since the assumption $\lfloor D \rfloor = 0$ implies that D has no isolated reduced components of elliptic type. Denote the resulting morphism by $\psi\colon (X,D) \longrightarrow (Y,B)$ and the induced rational map by $\overline{\gamma}\colon (Y,B) \dashrightarrow (\overline{Y},\overline{B})$. Since Y is normal, $\overline{\gamma}$ is a birational morphism. This gives a commutative diagram

$$\begin{tikzcd} (X,D) \arrow[r, "\psi"] \arrow[d, "\gamma"'] & (Y,B) \arrow[d, "\overline{\gamma}"] \\ (\overline{X},\overline{D}) \arrow[r, "\overline{\psi}"] & (\overline{Y},\overline{B}) \end{tikzcd} \tag{3.8}$$

Since ψ does not decrease log discrepancies, we have $\psi^*(K_Y + B) = K_X + D - E$, where E is an effective divisor, whose support is contained in $\operatorname{Supp}(\operatorname{Exc}\psi)$; cf. (3.1). Let $\overline{A}$ be a component of $\operatorname{Exc}\overline{\gamma}$ and let A be its proper transform on X. We have $A \subseteq \operatorname{Exc}\gamma$, so $\overline{A}\cdot(K_Y+B) = A\cdot(K_X+D) - A\cdot E \leqslant A \cdot (K_X + D) = A \cdot \gamma^*(K_{\overline{X}} + \overline{D}) = 0$. On the other hand, $\overline{A} \cdot (K_Y + B) \geqslant 0$ by the definition of ψ, hence $\overline{A} \cdot (K_Y + B) = 0$. It follows that $\overline{\gamma}$ is log crepant. $\square$

Remark 3.22. Let $\gamma\colon (X,D) \longrightarrow (\overline{X},\overline{D})$ and $\overline{\psi}\colon (\overline{X},\overline{D}) \longrightarrow (\overline{Y},\overline{B})$ be as in Lemma 3.21, with $\ell = \operatorname{Exc}\overline{\psi}$ irreducible of the second kind. As long as the contraction of $\ell' = \gamma_*^{-1}\ell$ exists, for instance ℓ' is not an isolated reduced component of D of elliptic type, cf. Lemma 2.15, then all claims of the above Lemma hold. Indeed, since $\overline{\gamma}\circ\psi = \overline{\psi}\circ\gamma$ is log crepant, $K_Y + B = \overline{\gamma}^*(K_{\overline{Y}} + \overline{B})$ is $\mathbb{Q}$-Cartier.

However, it may happen that ℓ' is not contractible (in the category of schemes), even though ℓ is, see Example 3.23.

Example 3.23 (Non-reversible order of contractions of log exceptional curves of the second kind)**.** Let $\sigma\colon \overline{X} \longrightarrow \mathbb{P}^2$ be a blowup as in Example 2.16(1) and let $\overline{\ell}$ be the proper transform of the cubic. Assume that the centers, say $p_1,\dots,p_n$ of σ are chosen so there exists a contraction of $\overline{\ell}$; denote it by $\overline{\psi}\colon \overline{X} \longrightarrow \overline{Y}$, cf. Remark 2.17. Now pick a point $p \in \ell$ not lying over any $\sigma(p_i)$, $i = 1,\dots n$. Denote by $\gamma\colon X \longrightarrow \overline{X}$ be the blowup at p and put $\ell = \gamma_*^{-1}\overline{\ell}$, where E is the exceptional curve of γ. Clearly, ℓ and E are log exceptional curves of the second kind on (X,ℓ) and by construction we have a log crepant contraction γ of E followed by the contraction of $\gamma(\ell)$. If there exists a contraction of ℓ then the induced morphism contracts the image of E, that is, the order or contractions can be reversed and we get a commutative diagram (3.8). However, this is not always possible. Indeed, the existence of the contraction implies that $\sigma(p), p_1, \dots p_n$ are $\mathbb{Z}$-linearly dependent in the group law of $\sigma(\ell)$. If the field is not an algebraic closure of a finite field then we can always pick p so that this is impossible; cf. Example 2.16.

Lemma 3.24 (Advancing contractions of the second kind)**.** *Let $\alpha\colon (X,D) \longrightarrow (\overline{X},\overline{D})$ be a pure partial peeling morphism and let $\overline{\gamma}\colon (\overline{X},\overline{D}) \longrightarrow (\overline{Y},\overline{B})$ be a log crepant morphism between log surfaces. Then there exists a contraction $\gamma\colon (X,D) \longrightarrow (Y,B)$ onto a log surface of the proper transform of* $\operatorname{Exc}\overline{\gamma}$ *and a commutative diagram of contractions*

$$\begin{tikzcd} (X,D) \arrow[r, "\gamma"] \arrow[d, "\alpha"] & (Y,B) \arrow[d, "\overline{\alpha}"] \\ (\overline{X},\overline{D}) \arrow[r, "\overline{\gamma}"] & (\overline{Y},\overline{B}) \end{tikzcd} \tag{3.9}$$

where $\overline{\alpha}$ is a partial peeling morphism. If α is a peeling then $\overline{\alpha}$ is a peeling.

Proof. By induction we may assume that $\operatorname{Exc}\overline{\gamma}$ consists of a single log exceptional curve $\overline{\ell}$ of the second kind. Let $E = \operatorname{Exc}\alpha$ and let $\overline{D}'$ be the proper transform of $\overline{D}$ on X. We first prove the existence of γ. We may assume that $E \neq 0$ and that each connected component of E meets $\ell := \alpha_*^{-1}\overline{\ell}$. Put $\widetilde{E} = E - \mathrm{ld}_X(\overline{X}, \overline{D})$. We have

$$\widetilde{E} = \alpha^*(K_{\overline{X}} + \overline{D}) - (K_X + \overline{D}'),$$

so for every component E_0 of E we get $E_0 \cdot \widetilde{E} = -E_0 \cdot (K_X + \overline{D}') \leqslant -E_0 \cdot \overline{D}' \leqslant 0$, because α is pure. By Lemma 2.9 we obtain $\widetilde{E} \geqslant 0$. We have $\mathrm{ld}(E_0; \overline{X}, \overline{D}) > \mathrm{ld}(E_0; X, D) = 1 - \mathrm{coeff}_{E_0}(D) \geqslant 0$, so $\mathrm{ld}_X(\overline{X}, \overline{D}) \geqslant 0$. We deduce that ℓ is a log exceptional curve of the second kind on $(X, \overline{D}' + \widetilde{E})$. Note that the latter is a log surface, as each component of E is $\mathbb{Q}$-Cartier by, see Remark 2.4. By Lemma 2.15 it is sufficient to consider the case when ℓ is a connected component of $\mathrm{Supp}(\overline{D}' + \widetilde{E})$. Then $\ell \cdot \widetilde{E} = 0$, so since ℓ meets every connected component of $\mathrm{Supp}\, E$, Lemma 2.9 implies that $\widetilde{E} = 0$. Then ℓ is a connected component of $\mathrm{Supp}\, \overline{D}'$. We have $(\alpha^* K_{\overline{X}} - K_X) + (\alpha^*\overline{D} - \overline{D}') = 0$, so by Corollary 2.10 we get $\alpha^* K_{\overline{X}} = K_X$ and $\alpha^*\overline{D} = \overline{D}'$. It follows that $\overline{D}' \cdot E = 0$, hence $\ell \cdot E = 0$; a contradiction. Thus γ exists.

The induced rational map $\overline{\alpha}$ is regular due to the normality of Y. Since for every prime exceptional component E_0 of E we have $\mathrm{ld}(E_0; \overline{Y}, \overline{B}) > \mathrm{ld}(E_0; X, D)$, by Corollary 3.3 the morphism $\overline{\alpha}$ is a partial peeling. We have $\overline{\gamma}^*(K_{\overline{Y}} + \overline{B}) = K_{\overline{X}} + \overline{D}$, which implies that a proper transform of a log exceptional curve on $(\overline{Y}, \overline{B})$ is log exceptional on $(\overline{X}, \overline{D})$. Therefore, if α is a peeling then $\overline{\alpha}$ is a peeling. □

In an analogy to Definition 3.12 we introduce the following definition.

Definition 3.25 (Redundant and almost log exceptional curves of the second kind)**.** Let α be a pure partial peeling of a log surface (X, D) and let $\ell \subseteq X$ be a curve. We say that:

(1) ℓ is *α-redundant of the second kind* if $\ell \subseteq D$, $\ell \cdot K_X < 0$ and $\alpha(\ell)$ is log exceptional of the second kind on $(\alpha(X), \alpha_* D)$,

(2) ℓ is *α-almost log exceptional of the second kind* if $\ell \not\subseteq D$, $\ell \cdot K_X \neq 0$ and $\alpha(\ell)$ is log exceptional of the second kind on $(\alpha(X), \alpha_* D)$.

A curve is *almost log exceptional of the second kind* (*redundant of the second kind*) if it is α-almost log exceptional (respectively, α-redundant) of the second kind for some pure partial peeling α.

Recall that if ℓ is an almost log exceptional curve of the first or second kind on (X, D) then by Lemma 3.16 we have $\ell \cdot K_X < 0$. In particular, almost log exceptional curves of the second kind and redundant curves of the second kind can be contracted.

The following example shows in particular that in the situation of Lemma 3.24 the morphism $\overline{\alpha}$ needs not to be pure, and hence (Y, B) does not need to be almost minimal, even if $(\overline{Y}, \overline{B})$ is minimal.

Example 3.26 (Non-purity of the induced peeling)**.** Let X be a smooth surface containing a chain $[1, 2]$, that is, two smooth rational curves ℓ and D, such that $\ell^2 = -1$, $D^2 = -2$ and $\ell \cdot D = 1$. The peeling morphism $\alpha\colon (X, D) \longrightarrow (\overline{X}, \overline{D})$ is the contraction of D (here $\overline{D} = 0$), so it is pure and $\alpha^*(K_{\overline{X}} + \overline{D}) \sim K_X$. It follows that ℓ is almost log exceptional of the first kind. After the contraction of ℓ the image of D is a (-1)-curve, so the induced peeling contracting the image of D is not pure.

Similarly, consider a smooth surface X containing a chain $[2, 1, 0]$. Let $D_1 = [2]$, $\ell = [1]$ and $D_2 = [0]$ be its components. Put $D = D_1 + D_2$. The peeling morphism $\alpha\colon (X, D) \longrightarrow (\overline{X}, \overline{D})$ is the contraction of D_1, so it is pure and $\alpha^*(K_{\overline{X}} + \overline{D}) \sim K_X + D_2$. It follows that now ℓ is almost log exceptional of the second kind. Again, after the contraction of ℓ the image of D is $[1, -1]$, so the induced peeling contracts the (-1)-curve, hence is not pure.

Definition 3.27 (Almost minimalization of the second kind)**.** Let $\psi\colon (X, D) \longrightarrow (\overline{X}, \overline{D})$ be a partial MMP run of the second kind.

(1) We call ψ_{am} a *partial almost minimalization of the second kind of* (X, D).

(2) If $\mathrm{Exc}\, \psi \subseteq \mathrm{Supp}\, D$ then ψ is a *partial peeling of* (X, D) *of the second kind* and ψ_{am} a *partial ψ-squeezing of the second kind.*

(3) If ψ is maximal then ψ (respectively, ψ_{am}) is called a *minimalization (respectively, almost minimalization)* of the second kind of (X, D).

(4) If ψ is a maximal partial peeling of the second kind then it is called *peeling of the second kind* and ψ_{am} is called a *squeezing of the second kind.*

Remark 3.28. By Lemma 3.21 contractions of the second kind can be delayed, hence any MMP run of the second kind is a composition of an MMP run with a log crepant morphism. By (3.4) it follows that an almost minimalization of the first kind extends to an almost minimalization of the second kind. On the image of (X, D) under (almost) minimalization of the second kind there can exist log exceptional curves of the second kind which are not contractible in the category of algebraic varieties or which are contractible but the direct image of a log canonical divisor is not $\mathbb{Q}$-Cartier, cf. Example 2.16.

4. Almost minimalization for reduced boundaries

We want to obtain a more explicit description of the process of almost minimalization for log surfaces whose boundary is uniform, that is, all coefficients of prime components are equal to some fixed number $r \in [0, 1]$. Before we do this we need to review the well-known case $r = 1$ in detail.

4A. ***Discriminants and log canonical subdivisors***

Let X be a smooth projective surface and D a reduced divisor on X. We introduce or recall (see [Fuj82], [Miy01]) some notions and notation related to the geometry of divisors and specific subdivisors, which will be later used to discuss log discrepancy divisors and peeling morphisms. We do not assume that D is a simple normal crossing divisor. We denote the number of components of D by $\#D$.

By $p_a(D)$ we denote the arithmetic genus of D, that is, $p_a(D) := \frac{1}{2}D \cdot (K_X + D) + 1$. We have $p_a(D_1 + D_2) = p_a(D_1) + p_a(D) + D_1 \cdot D_2 - 1$ for any divisors D_1, D_2 on X. Given a reduced subdivisor $T \leqslant D$ we call

$$\beta_D(T) = T \cdot (D - T) \tag{4.1}$$

the *branching number of* T *in* D. A *tip of* D is a component with $\beta_D \leqslant 1$ and a *branching component of* D is a component with $\beta_D \geqslant 3$. We say that D is *rational* if all its components are rational. A component of D is *admissible* if it is smooth rational and its self-intersection number is at most (-2).

Let $D_1, \dots, D_n$ be the components of D. We put $Q(D) = [-D_i \cdot D_j]_{i,j \leqslant n}$ and $d(0) = 1$ and we call

$$d(D) := \det(Q(D))$$

the *discriminant of* D; it does not depend on the chosen order of vertices.

Lemma 4.1 (Splitting formula for discriminants)**.** *Assume D_1, D_2 are two reduced divisors on a smooth projective surface which have no common component and which intersect in unique components $T_1 \leqslant D_1$ and $T_2 \leqslant D_2$, respectively. Then*

$$d(D_1 + D_2) = d(D_1)d(D_2) - (T_1 \cdot T_2)d(D_1 - T_1)d(D_2 - T_2). \tag{4.2}$$

Proof. The proof follows from the additivity of the determinant function with respect to column addition and its behavior on block-triangular matrices. □

In particular, if D is a reduced divisor with a tip D_1 and this tip meets a component $D_2 \leqslant D$ then

$$d(D) = (-D_1^2)d(D - D_1) - d(D - D_1 - D_2). \tag{4.3}$$

If D is connected then we call it a *rational tree* if $p_a(D) = 0$ and a we call it a *rational cycle* if $p_a(D) = 1$, D is not a smooth elliptic curve and has no branching component. A rational tree is simply a connected rational snc-divisor with no rational cycle as a subdivisor. Each component of a reducible rational cycle is rational and has $\beta_D = 2$. A rational cycle is *degenerated* if it is not snc. In this case it is a nodal or cuspidal rational curve or a sum of two smooth rational curves intersecting at a unique point with multiplicity two or a triple of smooth rational curves passing through a common point, each two meeting normally. A rational tree with no branching component is a *rational chain*. If D is a rational chain or a rational cycle, and the components are ordered so that D_i meets D_{i+1} for $i \in \{1, \dots n-1\}$, then we write $D = [-D_1^2, \dots, -D_n^2]$ in the first case and $D = ((-D_1^2, \dots, -D_n^2))$ in the second case. A sequence consisting of an integer r repeated k times will be abbreviated by $(r)_k$. A rational chain and a rational cycle are *admissible* if they have admissible components and are negative definite. For chains the second condition is in fact redundant and for a reducible rational cycle it is equivalent to one of the inequalities $-D_i^2 \leqslant 2$ to being strict.

A component T of D is called *superfluous* if it is a (-1)-curve meeting at most two other components of D, each at most once. Equivalently, after the contraction of T the image of T is a simple normal crossing point of the image of D. Note that a log smooth completion of a smooth quasi-projective surface is *minimal* (does not dominate non-trivially some other log smooth completion) if and only if the boundary contains no superfluous component which are not connected components of the boundary.

A rational chain T is *ordered* if it has a fixed ordering of its components $T^{(1)},\dots,T^{(m)}$ such that $T^{(i)}\cdot T^{(i+1)}=1$ for $i\in\{1,\dots,m-1\}$. By $T^{\top}$ we denote the same chain with the opposite order of components. The *tip* of a nonzero ordered chain is by definition $\operatorname{tip}(T)=T^{(1)}$. For $i\geqslant 1$ we put

$$d^{(i)}(T)=d(T^{(i+1)}+\dots+T^{(m)})\quad\text{and}\quad d_{(i)}(T)=d(T^{(1)}+\dots+T^{(i-1)}) \tag{4.4}$$

with $d^{(i)}(0)=d_{(i)}(0)=0$. We put also $d'(T)=d^{(1)}(T)=d(T-\operatorname{tip}(T))$. Let T be an admissible ordered chain. By Lemma 4.1

$$d(T)=(-\operatorname{tip}(T))^2d'(T)-d'(T-\operatorname{tip}(T)). \tag{4.5}$$

We have $0\leqslant d'(T)<d(T)$ and $\gcd(d(T),d'(T))=1$, see [Fuj82, 3.5]. We put

$$\delta(T)=\frac{1}{d(T)}\in\mathbb{Q}\cap(0,1]\quad\text{and}\quad\operatorname{ind}(T)=\frac{d'(T)}{d(T)}\in\mathbb{Q}\cap[0,1) \tag{4.6}$$

and we call $\operatorname{ind}(T)$ the *inductance of* T.

We now define some specific subdivisors $T\leqslant D$ and depict their *extended dual graphs* (see [KM98, Definition 4.6]). White graph vertices correspond to components of T and their weights (if displayed) are negatives of their self-intersection numbers. Black vertices represent $D-T$ (they are not necessarily distinct). The number of edges between two vertices is the intersection number of the represented components, see Fig. 1.

A nonzero (rational) chain $T\leqslant D$ whose all components are non-branching in D, that is $\beta_D(T)\leqslant 2$, is called a *(rational) twig of* D if some component of T is a tip of D ($\beta_D\leqslant 1$) and a *segment of* D otherwise. A segment is *non-degenerate* if T meets D normally (this holds for instance if T has at least three components). A twig is a *maximal twig* of D if it is not properly contained in another twig of D. A twig whose support is a connected component of D is called a *rod* of D. A twig which is not a rod comes (and will be considered) with a unique order in which $\operatorname{tip}(T)$ is a tip of D.

Let F be a rational tree with a unique branching component B and three maximal twigs T_1, T_2, T_3. Then we call F a *(rational) fork*, we write $F=\langle B;T_1,T_2,T_3\rangle$ and we say that F is of type $(-B^2;d(T_1),d(T_2),d(T_3))$. We put

$$\delta(F):=\delta(T_1)+\delta(T_2)+\delta(T_3). \tag{4.7}$$

By a (-2)-*fork (respectively, a* (-2)-*chain)* we mean an fork (respectively, a chain) consisting of (-2)-curves. By a *fork of* D we mean a fork which is a connected components of D. By Lemma 4.1

$$d(F)=d(T_1)d(T_2)d(T_3)(-B^2-\operatorname{ind}(T_1^{\top})-\operatorname{ind}(T_2^{\top})-\operatorname{ind}(T_3^{\top})). \tag{4.8}$$

A fork F with admissible components is called and *admissible fork* if $\delta(F)>1$ and is called a *log canonical fork* if $\delta(F)=1$ and not all components of F are (-2)-curves.

A rational tree $T\leqslant D$ is called a *bench of* D if T is a connected component of D which contains a chain (called *central chain*) $C=C_1+\ldots+C_n$, $n\geqslant 1$ with tips C_1, C_n such that $T-C=T_1+T_2+T_3+T_4$, $T_i=[2]$ for $i=1,2,3,4$, $T_i\cdot C_1=1$ for $i=1,2$ and $T_i\cdot C_n=1$ for $i=3,4$. A bench is *log canonical* if the central chain is admissible and does not consist of (-2)-curves only.

A rational tree $T\leqslant D$ is called a *half-bench of* D if T contains a chain (again called a *central chain*) $C=C_1+\ldots+C_n$, $n\geqslant 1$ with tips C_1, C_n such that $T-C=T_1+T_2$, $T_i=[2]$ and $T_i\cdot C_1=1$ for $i=1,2$ and $T\cdot(D-T)=C_n\cdot(D-T)=1$. In particular, a half-bench of D is a fork of type $(b;2,2,t)$ or a chain $[2,b,2]$ for some $b,t\geqslant 2$. It is *log canonical* if C is admissible.

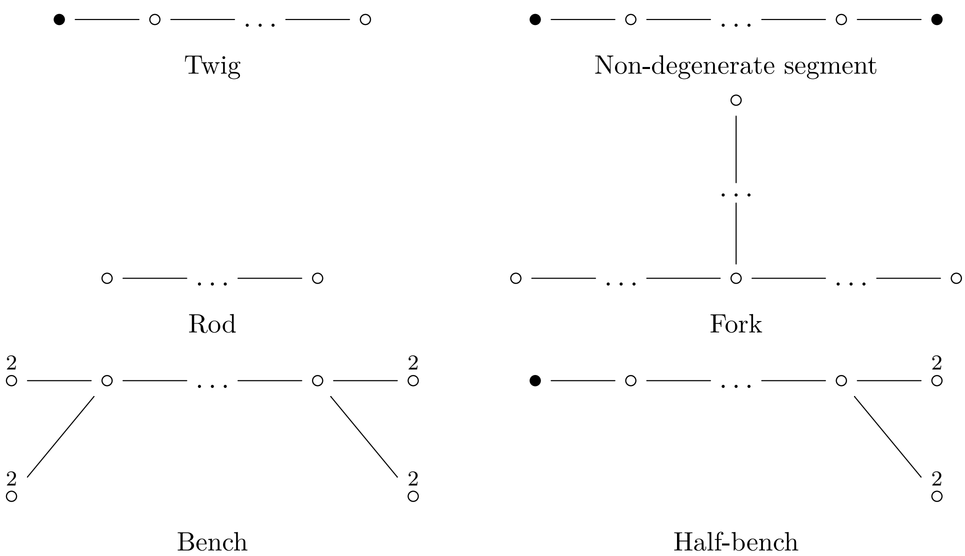

FIGURE 1. Log canonical subdivisors

Remark 4.2.

(1) For every admissible chain T we have $\delta(T)+\operatorname{ind}(T)\leqslant 1$ with the equality for a (-2)-chain only. Hence for a fork F with admissible twigs and $\delta(F)>1$ we infer from (4.8) and from Sylvester's criterion that F is negative if and only if F is admissible (equivalently $-B^2\geqslant 2$).
(2) If a fork with $\delta=1$ or a bench consists of (-2)-curves then its discriminant vanishes. By elementary properties of determinants it follows that a fork with admissible twigs and $\delta=1$ is negative definite if and only if it is log canonical (equivalently, is not a (-2)-fork). Similarly, a bench with an admissible central chain is negative definite if and only if it is admissible (equivalently, its central chain is not a (-2)-chain).

Lemma 4.3 (Log terminal and log canonical subdivisors)**.** *Let $p\in\overline{X}$ be a germ of a normal singular surface and let $\overline{D}$ be a reduced divisor on $\overline{X}$. Let E be the exceptional divisor of a minimal resolution $\pi\colon X\longrightarrow\overline{X}$. Put $D=\pi_*^{-1}\overline{D}+E$. We have $\operatorname{ld}(\mathcal{E}(\pi);\overline{X},\overline{D})>0$ if and only if one of the following holds:*

(1) $\overline{D}=0$ and E is either an admissible fork or an admissible rod of D,
(2) $\overline{D}\neq 0$ and E is a rational admissible twig of D.

We have $\operatorname{ld}(\mathcal{E}(\pi);\overline{X},\overline{D})=0$ if and only if one of the following holds:

(3) $\overline{D}=0$ and E is one of the following:
  (a) a smooth elliptic curve with negative self-intersection number,
  (b) a (possibly degenerated) admissible rational cycle,
  (c) a log canonical fork,
  (d) an admissible bench.
(4) $\overline{D}\neq 0$, $E\cdot(D-E)=1$ and E is a log canonical half-bench of D.
(5) $\overline{D}\neq 0$, $E\cdot(D-E)=2$ and E is a segment of D (possibly degenerated).

Conversely, each nonzero divisor E as above has a negative definite intersection matrix and in cases other than (3a) and (3b) it contracts to a rational singularity.

Proof. A direct arithmetic proof independent of the characteristic of the base field is given in [Kol92, 3.2.7, 3.4.1]; cf. [Kol13, 3.39, 3.40]. Since we do not assume that $p\in(\overline{X},\overline{D})$ is log canonical (we consider a minimal resolution, not a minimal log resolution), in (3b) we allow degenerate cycles - necessary minor corrections of the arguments in [Kol92, (3.1.4), (3.1.5)] can be done easily. Concerning the rationality of log canonical singularities see also [Art62, 2.3] and [Kol13, 2.28]. $\square$

Recall that in case $\operatorname{char}\Bbbk=0$ log terminal singularities are locally analytically isomorphic to quotient singularities, that is, the ones obtained as quotients of $\mathbb{A}^2$ by the actions of finite subgroups of $\mathrm{GL}(2,\Bbbk)$, see [Bri68, Satz 2.10], cf. [Ish14, §7.4]. For an admissible chain T this is the action of the cyclic group $\langle\zeta\rangle\subseteq\Bbbk^*$ of order $d(T)$, given by $\zeta\cdot(x,y)=(\zeta x,\zeta^{d'(T)}y)$, see [BH[+]04, III.5].

4B. ***Barks***

To write down explicit compact formulas for the log discrepancy divisor in case of uniform boundaries we use *barks* (cf. [Miy01, 2.3.5.2]). Let T be an admissible ordered chain. Define the *bark of* T as (see (4.4))

$$\mathrm{Bk}' T = \sum_{i=1}^{m} \frac{d^{(i)}(T)}{d(T)} T^{(i)}. \tag{4.9}$$

In particular, the coefficients of $\mathrm{tip}(T)$ and $T^{(m)}$ in $\mathrm{Bk}' T$ are $\mathrm{ind}(T)$ and $\delta(T)$, respectively. We put

$$\mathrm{Bk}^{\top} T = \mathrm{Bk}'(T^{\top}). \tag{4.10}$$

Denoting by δ_i^j the Kronecker delta, from (4.5) we infer that

$$T^{(i)} \cdot \mathrm{Bk}' T = -\delta_i^1. \tag{4.11}$$

Definition 4.4 (Bark)**.** Assume that X is smooth and D is reduced.

(1) If E is a maximal admissible twig of D but not a rod of D, we put $\mathrm{Bk}_D E = \mathrm{Bk}' E$.
(2) If E is an admissible rod of D then we pick any order which makes it an ordered twig and we put $\mathrm{Bk}_D E = \mathrm{Bk}' E + \mathrm{Bk}^{\top} E$.
(3) If E is an admissible fork of D we denote its maximal twigs by T_1, T_2, T_3 and the central component by E_0 and we put

$$\mathrm{Bk}_D E = u(E_0 + \sum_{i=1}^{3} \mathrm{Bk}^{\top} T_i) + \sum_{i=1}^{3} \mathrm{Bk}' T_i, \quad \text{where} \quad u = \frac{\sum_{i=1}^{3} \delta(T_i) - 1}{-E_0^2 - \sum_{i=1}^{3} \mathrm{ind}(T_i^{\top})}.$$

We extend the definition of Bk_D additively for disjoint sums of admissible twigs, rods and forks. Finally, we define the *bark of* D by $\mathrm{Bk}\, D := \mathrm{Bk}_D(\mathrm{Exc}\, \pi)$, where π is the unique pure peeling morphism for (X, D), see Lemma 4.5.

4C. ***Peeling, redundant and almost log exceptional curves***

We now discuss notions related to almost minimality for reduced boundaries. Many results concerning minimal and almost minimal models of log surfaces with reduced boundary were obtained in the 80s in particular by T. Fujita, M. Miyanishi, S. Tsunoda and F. Sakai, see for instance [Fuj82], [Fuj79], [MT84], [Sak84]. The results we present below are close to [Miy01, 2.3.3-3.5]. We do not claim originality, but the proposed formulation and line of reasoning will be used to obtain analogous results for $(1-r)$-log canonical surfaces with uniform boundary rD. For a definition of a peeling of the first and second kind see Definitions 3.11(1) and 3.27(2).

Lemma 4.5 (Peeling and squeezing for a reduced boundary)**.** *Assume that X is smooth and D is reduced.*

(1) A contraction of some number of admissible forks, admissible rods and admissible twigs of D is a pure partial peeling. Every pure partial peeling is of this type.
(2) A contraction of some number of divisors E as in Lemma 4.3 is a pure partial peeling of the second kind. Every pure partial peeling of the second kind is of this type.
(3) $L \subset D$ is redundant of the first kind if and only if it is a (-1)-curve such that either $\beta_D(L) \leqslant 1$ or $\beta_D(L) = 2$ and L meets some admissible twig of D.
(4) $L \subseteq D$ is redundant of the second kind if and only if it is a (-1)-curve with $\beta_D(L) = 2$ which meets no admissible twig of D.

In particular, if D contains no superfluous (-1)-curve (is snc-minimal) then (X, D) is squeezed and if (X, D) is squeezed then it has a unique peeling.

Proof. (1), (2) Denote the contraction by $\alpha\colon (X, D) \longrightarrow (\overline{X}, \overline{D})$. We may assume that $\mathrm{Exc}\, \alpha$ is connected. By Lemma 4.3 for every component E of $\mathrm{Exc}\, \alpha$ we have $\mathrm{ld}(E; \overline{X}, \overline{D}) \geqslant 0 = \mathrm{ld}(E; X, D)$ and the inequality is strict in case (1). By Lemma 3.1 α factors through a peeling α_0 contracting all components of E for which the inequality is strict. Thus $\alpha = \alpha_0$ and α is a proper peeling. This gives (1). Let $\alpha_1\colon (X, D) \longrightarrow (X', D')$ be a maximal partial peeling of the second kind with $\mathrm{Exc}\, \alpha_1 \subseteq \mathrm{Exc}\, \alpha$ extending α_0. Then $\alpha_2 = \alpha \circ \alpha_1^{-1}\colon (X', D') \longrightarrow (\overline{X}, \overline{D})$ is log crepant. Suppose that $\alpha \neq \alpha_1$. Suppose that $\mathrm{Exc}\, \alpha_2$ is reducible. Let E_0 be its component. Then $\mathrm{Exc}\, \alpha_1 + E_0$ is as in Lemma 4.3, neither a cycle nor elliptic, hence contracts to a rational singularity. But then α_1 is not maximal; a contradiction. Thus $\operatorname{Exc}\alpha_2$ is irreducible.

Conversely, if α is a pure partial peeling of the second kind then it is also a minimal resolution of $\overline{X}$, hence $\operatorname{Exc}\alpha$ is as in Lemma 4.3.

(3), (4) Assume that L is α-redundant of the first or second kind for some pure partial peeling α. Denote by E the sum of connected components of $\operatorname{Exc}\alpha$ meeting L and let $\sigma\colon X \longrightarrow \overline{X}$ be the contraction of $E+L$. Put $\overline{D} = f_*D$ and $q = \sigma(E+L)$. If $E=0$ then we get $0 = L\cdot(K_X+D) = -2+\beta_D(L)$, hence $\beta_D(L)=2$, which gives (4). We may thus assume that $E\neq 0$. By Lemma 2.8 log discrepancies of components of $E+L$ with respect to $(\overline{X},\overline{D})$ are positive, hence by Corollary 3.2 σ is a partial peeling of $D+L$. It follows that log discrepancies of arbitrary exceptional curves over q are non-negative. If $\overline{X}$ is smooth then the log discrepancy of the exceptional curve of the blowup of q equals $2-\operatorname{mult}_q\overline{D}$, hence $\overline{D}$ has normal crossings at q. Assume that $\overline{X}$ is singular. Let $\pi\colon \widetilde{X}\longrightarrow\overline{X}$ be the minimal resolution of singularities and let $\widetilde{D}$ be the total reduced transform of $\overline{D}$. Then π is a pure partial peeling of $\widetilde{D}$ and by (1) $\widetilde{D}$ has normal crossings in a neighborhood of the exceptional divisor. It follows that L contracts to an snc-point of the image of D, which means that L is a superfluous (-1)-curve. This gives (3).

Finally, suppose that (X,D) is squeezed and has more than one peeling morphism. There are connected components E_1 and E_2 of exceptional divisors of these peelings such that $E_1\not\leqslant E_2$, $E_2\not\leqslant E_1$ and E_1 meets E_2. By (1) E_1 and E_2 are admissible twigs of D, hence $E_1\cup E_2$ is an admissible rod of D. It follows that both peelings contract $E_1\cup E_2$; a contradiction. □

Remark 4.6. Assume D as above is connected and squeezed of the second kind (contains no (-1)-curve with $\beta_D\leqslant 2$). Then (X,D) has a unique peeling of the second kind except the following cases:

(1) D is a bench consisting of (-2)-curves.
(2) D is a rational cycle which either consists of (-2)-curves or is negative definite but non-contractible algebraically.

Proof. Suppose that (X,D) has at least two distinct peeling morphisms of the second kind. There are connected components E_1 and E_2 of their exceptional divisors such that $E_1\not\leqslant E_2$, $E_2\not\leqslant E_1$ and $E=E_1\cup E_2$ is connected. It follows that E_1, E_2 are of type (2), (4) or (5) in Lemma 4.3. Assume first that, say, E_2 contains a non-nc point of D. Then it is of type (5) with $\#E_2\leqslant 2$. Then D is a degenerate rational cycle with two or three components. By the maximality of peelings E is not algebraically contractible, hence $D=E$. Since its components have self-intersection numbers at most (-2), D is semi-negative definite, which is a special case of (2). We may thus assume that E contains only nc points of D. Then E is a chain, a fork of type $(b;2,2,n)$, a bench or a non-degenerate rational cycle, and its components have self-intersection numbers at most (-2). By the maximality of peelings E is not algebraically contractible and is a bench or a rational cycle, hence $D=E$. This gives (1) or (2). □

The following lemma is a generalization of [Miy01, Section 2.3.6-8]. The original more computational proof is for D which is snc.

Lemma 4.7 (Almost log exceptional curves for a reduced boundary)**.** *Assume X is smooth and D is reduced. Let A be an α-almost log exceptional curve of the first or second kind, where α is a pure partial peeling with exceptional divisor E. Then A is a (-1)-curve and one of the following holds.*

(1) $A\cdot D\leqslant 1$ and if A is of the first kind then the inequality is strict or the point of intersection belongs to some admissible twig, rod or fork of D,
(2) $A\cdot D=2$ and there is a rod E_1 of D meeting A once, in a tip of D. Moreover, if A meets another connected component E_2 of E then one of the following holds:
(a) E_2 is an admissible twig of D. It meets A in a tip of D or E_1 consists of (-2)-curves.
(b) E_2 is a rod of D.

Proof. Let $\alpha\colon (X,D)\longrightarrow(\overline{X},\overline{D})$ be the partial peeling morphism for which $\alpha(A)$ is log exceptional of the first or second kind. Since α is pure, by Corollary 2.10 we have $\alpha(A)\cdot K_{\overline{X}}\geqslant A\cdot K_X$. We have also $0\geqslant\alpha(A)\cdot(K_{\overline{X}}+\overline{D})\geqslant\alpha(A)\cdot K_{\overline{X}}$. If A is of the first kind, the first inequality is strict, hence $A\cdot K_X<0$. If A is of the second kind then this is so by definition. But $A^2<0$, so in each case A is a (-1)-curve. We may assume that $E+A$ is connected. We may also assume that $E\neq 0$, as otherwise

(1) holds. If $A\cdot D=1$ and A is of the first kind then we have $\alpha(A)\cdot(K_{\overline{X}}+\overline{D})<0=A\cdot(K_X+D)$, so the point of intersection of A and D belongs to E. Let $\overline{\gamma}\colon(\overline{X},\overline{D})\longrightarrow(\overline{Y},\overline{B})$ be the contraction of the log exceptional curve $\alpha(A)$ and let $\alpha'\colon(Y,B)\longrightarrow(\overline{Y},\overline{B})$ be the minimal resolution of singularities. Put $E'=\operatorname{Exc}\alpha'$. Since X is smooth, we have a morphism of log surfaces $\gamma\colon(X,D)\longrightarrow(Y,B)$, such that $\alpha'\circ\gamma=\overline{\gamma}\circ\alpha$. By Lemma 2.11, $(\overline{Y},\overline{B})$ is log terminal, hence by Lemma 4.5, $E'=0$ or E' is an admissible twig, rod or fork of B. In particular, it contains no (-1)-curve, hence γ is a composition of successive contractions of (-1)-curves in $E+A$ and its images. Moreover, since B is a sum of rational trees and is nc in a neighborhood of E', every (-1)-curve contracted by a blowdown being a factor of γ is superfluous in the respective image of D. In particular, $A\cdot D\leqslant 2$.

Assume that $A\cdot D=2$. Since E' is a sum of rational trees, so is $E+A$, hence $E_1\neq E_2$. Since E' is a sum of admissible twigs, rods and forks of B, one of E_i, say E_1, is necessarily a rod of D met by A in a tip. If E_2 is not a connected component of D then it is a twig of D such that $\gamma_*(E_1+A+E_2)$ is a twig of B, which gives (a). We may therefore assume that E_2 is an admissible fork or rod of D. Since $\gamma_*(E_1+A+E_2)$ is an admissible rod or fork of B, we get (a) or (b). □

Remark 4.8 (Almost log exceptional vs log exceptional)**.**

(1) Let A be as in Lemma 4.7. If $A\cdot D=0$ then A is log exceptional, if $A\cdot D=1$ then it is log exceptional of the second kind, if $A\cdot D=2$ we have $A\cdot(K_X+D)>0$.
(2) In Lemma 4.7(2) the fact that $A+\operatorname{Exc}\alpha$ is negative definite and A is α-almost log exceptional (and hence that after the contraction of $\alpha(A)$ all log discrepancies increase) gives additional restrictions on the weights of E_1 and E_2, see Example 4.9.

Example 4.9 (An almost log exceptional curve)**.** Consider a smooth projective surface X with a rational chain $[n,1,m]$ on it with $n\geqslant m\geqslant 2$. Let A be the (-1)-curve and let D be the sum of the other two components. The peeling morphism $\alpha\colon(X,D)\longrightarrow(\overline{X},0)$ is simply the contraction of D. We see that $D+A$ is contractible if and only if $n\geqslant 3$. The contraction of $A+D$ in each case gives a log terminal singularity. We have

$$\alpha(A)\cdot K_{\overline{X}}=A\cdot(K_X+D-\operatorname{Bk}D)=1-A\cdot\operatorname{Bk}D=1-\frac{2}{n}-\frac{2}{m},$$

so we see that A is almost log exceptional of the first kind if and only if $m=2$ and $n\geqslant 3$ or $m=3$ and $n\in\{3,4,5\}$. It is almost log exceptional of the second kind if and only if $(m,n)\in\{(3,6),(4,4)\}$.

5. Almost minimalization for uniform boundaries

As it follows from Example 3.14, to develop a reasonable general description of peeling morphisms and almost minimal models for a log surface for (X,D) one needs to add restrictions on the coefficients of D. We now study general uniform boundaries, that is, boundaries equal to rD, where D is a reduced divisor. We call r the *weight* of the boundary. Weight $r=0$ means that no boundary is considered, so there is no peeling and almost minimal is the same as minimal. Weight $r=1$ was discussed in the previous section. Below we assume that $r\in\mathbb{Q}\cap(0,1]$, $(\overline{X},\overline{D})$ and (X,D) are ($\mathbb{Q}$-factorial) log surfaces.

5A. *Comparing different weights*

Lemma 5.1. *Assume $(\overline{X},r\overline{D})$, where $\overline{D}$ is reduced, is $(1-r)$-lc. Let $\alpha\colon X\longrightarrow\overline{X}$ be a proper birational morphism. Put $D=\alpha_*^{-1}\overline{D}+\operatorname{Exc}\alpha$. Then $\alpha\colon(X,rD)\longrightarrow(\overline{X},r\overline{D})$ is a partial peeling of the second kind. If $(\overline{X},r\overline{D})$ is $(1-r)$-dlt then α is of the first kind, unless $r=1$ and some center of α is a normal crossing point of $(\overline{X},\overline{D})$.*

Proof. Let E be a prime component of $\operatorname{Exc}\alpha$. We have $\operatorname{ld}(E;\overline{X},r\overline{D})\geqslant 1-r=\operatorname{ld}(E;X,rD)$, hence α is a partial peeling of the second kind by Corollary 3.2. Assume $(\overline{X},r\overline{D})$ is $(1-r)$-dlt and either $r<1$ or $r=1$ but no center of α is an nc-point of $(\overline{X},\overline{D})$. By Remark 2.6 $\operatorname{ld}(E;\overline{X},r\overline{D})>1-r$, so the corollary shows that α is of the first kind. □

Lemma 5.2 (Pure peeling varying with r)**.** *Assume $r<1$ and $\alpha\colon(X,rD)\longrightarrow(\overline{X},r\overline{D})$, where D is reduced, is a pure partial peeling of the first (second) kind. Let $r'>r$. Then the following hold.*

(1) $\alpha\colon(X,r'D)\longrightarrow(\overline{X},r'\overline{D})$ is a pure partial peeling of the first (second) kind.
(2) If D is connected and $\operatorname{ld}(\alpha;\overline{X},r\overline{D})\geqslant 1-r$ then $\operatorname{ld}(\alpha;\overline{X},r'\overline{D})\geqslant 1-r'$ and the inequality is strict, unless $\alpha^*K_{\overline{X}}=K$ and $\alpha^*\overline{D}=D$.

(3) If X is smooth then every connected component of $\operatorname{Exc}\alpha$ is a twig, rod or fork of D or it is a (-2)-segment of D (see Lemma 4.3(5)).

Proof. (1) By definition α is a composition of birational contractions $X = X_1 \longrightarrow \ldots \longrightarrow X_{n+1} = \overline{X}$, where $X_i \longrightarrow X_{i+1}$ contracts a log exceptional curve $\ell_i \subseteq X_i$, $i = 1, \ldots, n$ of the first (first or second) kind. Denote the boundary on X_i, which is the direct image of D, by D_i and the composition $X \longrightarrow X_i$ by α_i. By assumption, for the proper transform ℓ_i' of ℓ_i on X we have $\ell_i' \cdot K_X \geqslant 0$. By Corollary 2.10 we have $\alpha_i^* K_{X_i} = K_X + E$ for some effective divisor E supported in $\operatorname{Supp}\operatorname{Exc}\alpha_i$. Hence $\ell_i \cdot K_{X_i} = \ell_i' \cdot \alpha_i^* K_{X_i} \geqslant \ell_i' \cdot K_X \geqslant 0$. Since $r > 0$, it follows that $\ell_i \cdot D_i < 0$ $(\leqslant 0)$, hence

$$\ell_i \cdot (K_{X_i} + r' D_i) = \ell_i \cdot (K_{X_i} + r D_i) + (r' - r)\ell_i \cdot D_i < 0 \ (\leqslant 0).$$

We infer by induction that α_i is a partial peeling of the first (second) kind for $(X, r'D)$.

(2) We have $K + rD \geqslant \alpha^*(K_{\overline{X}} + \overline{D})$, which we write as $(\frac{r'}{r} - 1)K_X + K_X + r'D \geqslant (\frac{r'}{r} - 1)\alpha^* K_{\overline{X}} + \alpha^*(K_{\overline{X}} + r'\overline{D})$, hence $K + r'D \geqslant (\frac{r'}{r} - 1)(\alpha^* K_{\overline{X}} - K) + \alpha^*(K_{\overline{X}} + r'\overline{D})$. Since α is pure, by Corollary 2.10 we have $\alpha^* K_{\overline{X}} - K \geqslant 0$, so $K + r'D \geqslant \alpha^*(K_{\overline{X}} + r'\overline{D})$. Assume that $\operatorname{ld}(\alpha; \overline{X}, r'\overline{D}) = r'$. Then the latter lemma gives $\alpha^* K_{\overline{X}} = K$, as otherwise $\operatorname{Supp}(\alpha^* K_{\overline{X}} - K) = \operatorname{Supp}\operatorname{Exc}\alpha$, which contradicts the assumption. Similarly, Lemma 2.8 gives $\operatorname{ld}_X(\overline{X}, r\overline{D}) = r\operatorname{Exc}\alpha$, hence $\alpha^*\overline{D} = D$. Assume X is smooth.

(3) Take $r' = 1$. By (1) α is a pure peeling of D. By (2) and by Lemma 4.3 we may assume that $\alpha^*_{\overline{X}} = K$ and $\alpha^*\overline{D} = D$. Then $\operatorname{Exc}\alpha$ consists of (-2)-curves with $\beta_D = 2$. Since the latter divisor is negative definite, it is a segment of D.

□

Corollary 5.3 (Resolutions for $(1-r)$-log canonical (X, rD))**.** Let $(\overline{X}, r\overline{D})$, where $\overline{D}$ is reduced, be a germ of a singular $(1-r)$-lc surface and let E be the exceptional divisor of the minimal resolution of $\overline{X}$. Then E is as in Lemma 4.3 and the following hold:

(1) If $r = 1$ then D is snc or $D = E$ is a nodal rational curve.
(2) If $r < 1$ then E is as in case (1), (2) (hence D is snc) or it is as in case (5), with $E = [(2)_k]$, $k \in \{1, 2\}$. In latter case, denoting the unique common point of E and $D - E$ by q, we have:
   (a) $k = 1$, $q \in D - E$ is an ordinary cusp and $r \leqslant \frac{3}{7}$,
   (b) $k = 1$, $q \in D - E$ is smooth and $r \leqslant \frac{2}{3}$,
   (c) $k = 1$, $q \in D - E$ is a point of normal crossings of D of multiplicity 2 and $r \leqslant \frac{1}{2}$,
   (d) $k = 2$, $q \in D - E$ is smooth and $r \leqslant \frac{1}{2}$.
   Moreover, if (X, rD) is $(1-r)$-lt then the inequalities are strict.

Proof. For $r \neq 1$ cases other than (1), (2) and (5) are impossible due to Lemma 5.2. The remaining restrictions come from a straightforward computation of log discrepancies of exceptional divisors of a minimal log resolution (instead of a minimal resolution). □

5B. ***Formulas for the log discrepancy divisor***

Lemma 5.4 (Log discrepancy for a uniform boundary)**.** *Assume that X is smooth and D is reduced. Let $\alpha\colon (X, D) \longrightarrow (\overline{X}, \overline{D})$ be a contraction of (some) admissible twigs, rods and forks of D. Denote by T the sum of connected components of $\operatorname{Exc}\alpha$ which are twigs but not rods of D. Then (see Definition 4.4)*

$$\operatorname{ld}_X(\overline{X}, r\overline{D}) = \operatorname{Bk}_D(\operatorname{Exc}\alpha) + (1-r)\operatorname{Bk}^{\top} T. \tag{5.1}$$

Assume α is a pure partial peeling of (X, rD). Then $\operatorname{ld}_X(\overline{X}, r\overline{D})$ is an effective divisor whose coefficients belong to $(1-r, 1]$. Moreover, $\operatorname{ld}(U; \overline{X}, r\overline{D}) = 1$ for a component U or $\operatorname{Exc}\alpha$ if and only if the connected component of $\operatorname{Exc}\alpha$ containing U is an admissible rod or admissible fork of D which consists of (-2)-curves only.

Proof. Assume first that $r = 1$. The log discrepancy divisor is uniquely determined by the equations $U \cdot (K + D - \operatorname{ld}_X(\overline{X}, \overline{D})) = 0$ where U runs through components of $E := \operatorname{Exc}\alpha$. We may assume that E is connected. By (4.11) in cases (1) and (2) of Definition 4.4 the equations hold for $\operatorname{Bk}' E$ and $\operatorname{Bk}' E + \operatorname{Bk}^{\top} E$, respectively. In case (3) we see that the divisors $E_0 + \sum_{i=1}^{3} \operatorname{Bk}^{\top} T_i$ and $\sum_{i=1}^{3} \operatorname{Bk}' T_i$ intersect trivially with all components of $T_1 + T_2 + T_3$. Then (5.1) follows from the equation $E_0 \cdot \operatorname{ld}_X(\overline{X}, \overline{D}) = E_0 \cdot (K + D) = 1$. For $r \neq 1$ we need to show that

$$K_X + r\alpha_*^{-1}\overline{D} + E = \alpha^*(K_{\overline{X}} + r\overline{D}) + \operatorname{Bk}_D(E) + (1-r)\operatorname{Bk}^{\top} T.$$

Subtracting the equality for $r=1$ and dividing by $1-r$ we see that it is sufficient to prove that

$$\alpha^{*}\overline{D}=\alpha_{*}^{-1}\overline{D}+\operatorname{Bk}^{\top}T, \tag{5.2}$$

and hence that $\alpha_{*}^{-1}\overline{D}+\operatorname{Bk}^{\top}T$ intersects trivially with every component U of T. This is easy to see. Indeed, we may assume that T is a single admissible twig with components $T^{(i)}$, $i=1,\dots,m$ and $U=T^{(i)}$ for some i. Then $U\cdot(\alpha_{*}^{-1}\overline{D}+\operatorname{Bk}^{\top}T)=\delta_{i}^{m}+U\cdot\operatorname{Bk}^{\top}T=0$ by (4.11).

Assume α is a pure partial peeling for (X,rD). Then $\operatorname{ld}(\overline{X},r\overline{D})>1-r$ by Corollary 2.11. Put $G=E-\operatorname{Bk}_{D}(E)+(1-r)\operatorname{Bk}^{\top}T$. For each component E_j of E we have

$$0=E_j\cdot\alpha^{*}(K_{\overline{X}}+r\overline{D})=E_j\cdot(K_X+r\alpha_{*}^{-1}\overline{D}+G)\geqslant E_j\cdot G,$$

because α is pure. By Lemma 2.9 we see that $G=0$ or $G\geqslant 0$ and $\operatorname{Supp}G=\operatorname{Supp}E$. The case $G=0$ happens when $E_j\cdot K_X=E_j\cdot\alpha_{*}^{-1}\overline{D}=0$ for each j, that is, when G is an admissible rod or fork of D consisting of (-2)-curves only. □

Every peeling can be decomposed as a squeezing followed by a pure peeling (see Lemma 3.17). The part consisting of squeezing will be studied in the next section. As for the part consisting of a pure peeling the following corollary together with Lemma 5.4 give an explicit description of the log discrepancy divisor.

Corollary 5.5 (Uniqueness of peeling for squeezed surfaces)**.** Assume that X is smooth, D is reduced and (X,rD) is squeezed. Then (X,D) has a unique pure peeling.

Proof. By Lemma 5.2 the exceptional locus of every pure partial peeling of rD consists of some admissible forks, rods and twigs of D. The contractions start with tips, so we infer that a support of a maximal pure partial peeling morphism is unique. □

Let $(\overline{X},\overline{D})$ be a ($\mathbb{Q}$-factorial) log surface and let $\pi\colon X\longrightarrow\overline{X}$ be a proper birational morphism. For every prime exceptional divisor E we put

$$E^{\flat}_{(X,D)}:=c(E;\overline{X},\overline{D})E=(1-\operatorname{ld}(E;\overline{X},\overline{D}))E \tag{5.3}$$

and we extend this notation additively to reduced divisors supported in $\operatorname{Supp}\operatorname{Exc}\pi$. Usually we skip the lower index (X,D), as it is clear from context. We have $\overline{D}_X=\pi_{*}^{-1}\overline{D}+\operatorname{Exc}^{\flat}\pi$, so the equivalence (2.5) reads as

$$K_Y+\pi_{*}^{-1}\overline{D}+\operatorname{Exc}^{\flat}\pi\sim\pi^{*}(K_X+D). \tag{5.4}$$

Remark 5.6. Assume (X,D) is log smooth, D reduced and snc-minimal. Let $\overline{D}$ denote the direct image of D after peeling of rD. In [Miy01], where the case $r=1$ is discussed, the divisor $\overline{D}_X=D-\operatorname{Bk}D$ is denoted by $D^{\#}$. In [Pal19], where we discussed the case $r=\frac{1}{2}$ for resolutions of planar rational cuspidal curves, we used the notation $D^{\flat}$ for $2\overline{D}_X$ (which is in disagreement with the above notation, so we abandon it).

Corollary 5.7 (Log pullback for a uniform boundary)**.** Assume X is smooth, D is reduced and (X,rD) is squeezed. Let $\alpha\colon(X,rD)\longrightarrow(\overline{X},r\overline{D})$ be a partial peeling of rD. Denote by T the sum of connected components of $\operatorname{Exc}\alpha$ which are twigs but not rods of D. Then

$$\operatorname{Exc}^{\flat}\alpha=E-\operatorname{Bk}_{D}(\operatorname{Exc}\alpha)-(1-r)\operatorname{Bk}^{\top}T. \tag{5.5}$$

The divisor $\operatorname{Exc}^{\flat}\alpha$ has coefficients belonging to $[0,r)$ and $\operatorname{Supp}\operatorname{Exc}\alpha\setminus\operatorname{Supp}\operatorname{Exc}^{\flat}\alpha$ consists of the support of (-2)-rods and (-2)-forks of D.

Corollary 5.8 (Characterization of almost log exceptional curves)**.** Assume X is smooth, D is reduced and let $\alpha\colon(X,rD)\longrightarrow(\overline{X},r\overline{D})$ be a pure partial peeling. Then $A\not\subseteq D$ is α-almost log exceptional of the first (second) kind if and only if A is a (-1)-curve such that the following hold:

(1) $A+\operatorname{Exc}\alpha$ is negative definite,

(2) $A\cdot(r\alpha_{*}^{-1}\overline{D}+\operatorname{Exc}^{\flat}\alpha)<1$ $(=1)$.

Proof. Put $\overline{A}=\alpha(A)$. Clearly, (1) is equivalent to $\overline{A}^2<0$. We may assume (1) holds. We have $\overline{A}\cdot(K_{\overline{X}}+r\overline{D})=A\cdot(K_X+\widetilde{D})$, where $\widetilde{D}=r\alpha_{*}^{-1}\overline{D}+\operatorname{Exc}^{\flat}\alpha$, so we only need to show that if A is almost log exceptional of the first or second kind then it is a (-1)-curve. We have $A\cdot(K_X+\widetilde{D})<0$ $(\leqslant 0)$, so since $\widetilde{D}$ is effective and $A\not\subseteq D$ by definition, we get $A\cdot K_X<0$ (in the case of second kind we have $A\cdot K_X\neq 0$ by definition). Since $A^2<0$, A is a (-1)-curve. □

5C. ***Peeling and squeezing for uniform boundaries***

We discuss the effect of squeezing on log singularities for surfaces of type (X, rD), with X smooth and D reduced. Since partial squeezing is not a partial MMP run, it does not necessarily respect $(1-r)$-(divisorial) log terminality. This is exactly the case in the following example.

Example 5.9 (Partial squeezing does not respect $(1-r)$-log terminality of (X, rD))**.** Let $r \in (0,1] \cap \mathbb{Q}$, $n \in \mathbb{N}_+$ and $N \in \mathbb{N}$ be such that $\frac{1}{r} \leqslant N \leqslant \frac{1}{r} + \frac{1}{n}$. Blowing up over a point $p \in \mathbb{P}^2$ we create a $\mathbb{P}^1$-fibered surface X with a unique degenerate fiber $F = [2, \dots, 2, 1, n]$ and a section U which is a (-1)-curve meeting F in a tip of a maximal (-2)-chain $\Delta \subset F$. Let $\alpha \colon X \longrightarrow \overline{X}$ be the contraction of Δ and let $D = F_1 + \ldots + F_N + U + \Delta$, where $F_1, \dots, F_N$ are distinct non-degenerate fibers. Put $\overline{D} = \alpha_* D$ and $\overline{U} = \alpha(U)$. Then (X, rD) is $(1-r)$-dlt and $\alpha \colon (X, rD) \longrightarrow (\overline{X}, r\overline{D})$ is a pure peeling. We have

$$\overline{U}^2 = -\frac{1}{n} < 0 \quad \text{and} \quad \overline{U} \cdot (K_{\overline{X}} + r\overline{D}) = r(N - \frac{1}{n}) - 1 \leqslant 0,$$

so $\overline{U}$ is log exceptional of the first or second kind and hence U is α-redundant of the first or second kind. Let $\sigma \colon X \longrightarrow X'$ be the contraction of U. Put $D' = \sigma_* D$. Then $\operatorname{ld}(U; X', rD') = 2 - r - rN \leqslant 1 - r$, so while (X, rD) is $(1-r)$-dlt, (X', rD') is not. If $\frac{1}{r} < N < \frac{1}{r} + \frac{1}{n}$ then U is α-redundant of the first kind and (X', rD') is not even $(1-r)$-lc. Moreover, although (X, rD) is $(1-r)$-log terminal and U is almost log exceptional, $(X, r(D+U))$ is not $(1-r)$-log terminal.

We note also that $U \cdot (K_{\overline{X}} + r(\overline{D} - \overline{U})) = rN - 1 \geqslant 0$, so $\overline{U}$ is not log exceptional on $(\overline{X}, r(\overline{D} - \overline{U}))$ and hence U is not almost log exceptional on $(X, r(D-U))$. Still, $\overline{U}$ is log exceptional on $(\overline{X}, r\overline{D})$ and U is redundant on (X, rD).

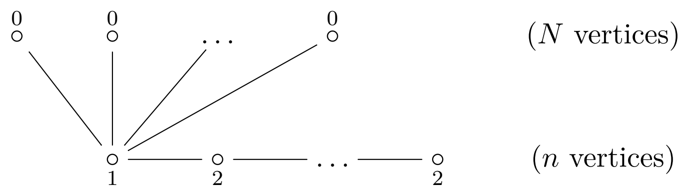

FIGURE 2. The dual graph of D in Example 5.9.

The following characterization plays a key role in the proof of Theorem 1.1.

Lemma 5.10. *Let (X, rD) be a log surface with X smooth, D reduced, $r \in (0,1] \cap \mathbb{Q}$. Let $\alpha \colon (X, rD) \longrightarrow (\overline{X}, r\overline{D})$ be a contraction some admissible twigs $T_1, \dots, T_k$ of D and let $\ell \leqslant D$ be a (-1)-curve. Put $\delta = \sum_{i=1}^k \frac{1}{d(T_i)}$ and $\operatorname{ind}^\top = \sum_{i=1}^k \operatorname{ind}(T_i^\top)$. Then $\alpha(\ell)$ is log exceptional of the first or second kind if and only if the following inequalities hold*

$$\operatorname{ind}^\top < 1, \tag{5.6}$$

$$r\beta_{D-E}(\ell) \leqslant 2 - k + \delta - (1-r)(1 - \operatorname{ind}^\top), \tag{5.7}$$

where the equality in (5.7) holds exactly when $\alpha(\ell)$ is of the second kind.

Proof. Put $\overline{\ell} = \alpha_* \ell$, $R = \alpha_*^{-1}\overline{D} - \ell$, $\delta_i = \delta(T_i) = 1/d(T_i)$ and $\operatorname{ind}_i^\top = \operatorname{ind}(T_i^\top)$. By (4.11) we have $\alpha^* \overline{\ell} = \ell + \sum_i \operatorname{Bk}^\top T_i$. We compute $\overline{\ell}^2 = \ell \cdot \alpha^* \overline{\ell} = \ell \cdot (\ell + \sum_i \operatorname{Bk}^\top T_i) = -1 + \operatorname{ind}^\top$, and

$$\overline{\ell} \cdot (K_{\overline{X}} + r\overline{D}) = \ell \cdot \alpha^*(K_{\overline{X}} + \overline{D}) + (r-1)\ell \cdot \alpha^* \overline{D} = \ell \cdot (K_X + R + \ell + \sum_i (T_i - \operatorname{Bk} T_i)) +$$

$$+ (r-1)\ell \cdot (R + \ell + \sum_i \operatorname{Bk}^\top T_i) = -2 + \ell \cdot R + \sum_i (1 - \delta_i) + (r-1)(\ell \cdot R - 1 + \sum_i \operatorname{ind}_i^\top) =$$

$$= k - \delta - 1 + r(\ell \cdot R - 1) + (r-1)\operatorname{ind}^\top.$$

It follows that $\overline{\ell}$ is log exceptional of the first or second kind if and only if $\operatorname{ind}^\top < 1$ and $r\ell \cdot R \leqslant 2 - k + \delta - (1-r)(1 - \operatorname{ind}^\top)$, with the equality for the second kind only. $\square$

Recall that a divisor on a smooth surface *contracts to a smooth point* if its support is the support of the exceptional divisor of a birational contraction onto a smooth surface. Such a divisor is necessarily a rational tree whose discriminant is equal to 1.

Proposition 5.11 (Description of redundant components)**.** *Let (X, rD) be a log surface with X smooth, D reduced, $r \in (0,1] \cap \mathbb{Q}$. Let α be a pure partial peeling of (X, rD) of the second kind and $\ell \leqslant D$ a (-1)-curve such that $\alpha(\ell)$ is log exceptional of the first (second) kind. Denote by E the sum of connected components of* $\operatorname{Exc}\alpha$ *meeting ℓ and put $R = D - \ell - E$. Then one of the following holds.*

(1) $r = 1$ and $\beta_D(\ell) \leqslant 2$ (so ℓ is log exceptional of the first or second kind). If the intersection of E and $D - E$ is not normal then E is a degenerate segment of D, so $\#E \leqslant 2$.
(2) $r \neq 1$, ℓ is of the first (second) kind and
   (a) $E \cdot R = 0$ and $\ell + E$ is a rational chain which contracts to a smooth point.
   (b) $\ell + E$ is a rod or twig of D.
(3) $\beta_D(\ell) = 3$, $E \cdot R = 0$, $r = \frac{1}{2}$ and $\ell + E$ is one of $[3,1,3]$, $[1,(2)_{m-1},3]$, $m \geqslant 1$, $[3,1,2,3]$. *In particular, all components of $\ell + E$ are of the second kind.*
(4) $\beta_D(\ell) = 3$, $E \cdot R = 0$, $r = \frac{2}{3}$ and $\ell + E$ is one of $[2,1,4]$, $[2,1,3,2]$. *In particular, α is not of the first kind and $\alpha(\ell)$ is of the second kind.*
(5) E is a (-2)-twig such that $0 \leqslant \ell \cdot R - \frac{1}{r} \leqslant \frac{1}{\#E+1}$.
(6) E is a sum of two twigs of D, $[2]$ *and* $[3]$, *$\ell \cdot R = 1$ and $\frac{1}{2} \leqslant r \leqslant \frac{4}{5}$.*

In cases (5) and (6) the inequalities on the right (respectively, left) become equalities if and only if $\alpha(\ell)$ (respectively, ℓ) is of the second kind.

Proof. If $E = 0$ then ℓ is log exceptional of the first (second) kind, which is a part of (1) or (2a). We may therefore assume that $E \neq 0$. To avoid confusion below we refer to the cases of Lemma 4.3 as (L1)-(L5). By Lemma 5.2 α is a pure partial peeling of the second kind of rD, and E is as in (L2), (L4) or (L5). Moreover, since $E + \ell$ is negative definite, in cases (L4) and (L5) E does not consist of (-2)-curves, so $r = 1$. Consider the case $r = 1$. By Lemma 3.21 we have $\alpha = \alpha_2 \circ \alpha_1$, where α_1 is a peeling of the first kind and α_2 contracts some number of log exceptional curves of the second kind. It follows that ℓ is α_1-redundant, so $\beta_D(\ell) \leqslant 2$ by Lemma 4.5. If the intersection of E and $D - E$ is not normal then E is a degenerate segment of D, which gives (1).

We may further assume that $r < 1$. Then $E = T_1 + \ldots + T_k$, where $T_1, \ldots, T_k$ are admissible twigs of D and $k \geqslant 1$. It follows that $\ell \cdot E = k$ and $E \cdot R = 0$. Put $\delta = \sum_{i=1}^k \frac{1}{d(T_i)}$ and $\operatorname{ind}^\top = \sum_{i=1}^k \operatorname{ind}(T_i^\top)$. Denote the contraction of $\alpha(\ell)$ by $\sigma\colon \alpha(X) \longrightarrow \overline{X}$. Put $\overline{D} = \sigma_*\alpha_*D$ and $q = \sigma(\alpha(\ell)) \in \overline{X}$.

Assume that $q \in \overline{X}$ is singular. Let $\pi\colon \widetilde{X} \longrightarrow \overline{X}$ be the minimal resolution of singularities. Since X is smooth, we get a factorization $\sigma = \pi \circ \varphi$. Put $\widetilde{D} = \varphi_*D$ and $\widetilde{E} = \varphi_*E$.

$$\begin{tikzcd} (X, rD) \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & (\widetilde{X}, r\widetilde{D}) \arrow[d, "\pi"] \\ (\alpha(X), r\alpha_*D) \arrow[r, "\sigma"] & (\overline{X}, \overline{D}) \end{tikzcd} \tag{5.8}$$

Since α is a partial peeling of rD, Corollary 3.2 implies that π is a partial peeling of $r\widetilde{D}$. Since π is minimal, it is pure, hence $\widetilde{E}$ is as in Lemma 4.3. By Lemma 5.2 $\widetilde{E}$ is a sum of admissible twigs, rods, forks and (-2)-segments of $\widetilde{D}$. If $\beta_D(\ell) \leqslant 2$ then, since $E \neq 0$, ℓ is superfluous in D (hence log exceptional of the first kind on (X, rD)), which gives (2a) or (2b). We are left with case $\beta_D(\ell) \geqslant 3$. Then $\varphi(\ell)$ is not a point of simple normal crossings of $\widetilde{D}$, so $\widetilde{E}$ is a degenerate (-2)-segment of $\widetilde{D}$. The fact that ℓ is not superfluous in D restricts possible types of blowing ups constituting φ^{-1}. Let G be a component of E not contracted by φ. Since $\pi^*K_{\overline{X}} = K_{\widetilde{X}}$ and $\pi^*\overline{D} = D'$, we have $\operatorname{ld}(G; \overline{X}, r\overline{D}) = 1 - r = \operatorname{ld}(G; X, rD)$. By Lemma 2.8 it follows that $\alpha(\ell)$ is of the second kind. For $\#\widetilde{E} = 2$ we obtain $\ell + E = [3,1,3]$ and hence $r = \frac{1}{2}$. Assume $\#\widetilde{E} = 1$. Then there exists $m \geqslant 1$ such that $\ell + E = [1,(2)_{m-1},3]$ or $\ell + E = [2,1,3,(2)_{m-2},3]$, where $[3,(2)_{-1},3] := [4]$. Since $\alpha(\ell) \cdot (K_{\alpha(X)} + r\alpha_*D) = 0$, we obtain $r = \frac{1}{2}$ and $r = \frac{2m}{4m-1}$, respectively. In the second case the log discrepancy of the component of $T_1 = [3,(2)_{m-2},3]$ meeting ℓ equals $u = \frac{1}{4m}(1 + (1-r)(2m-1))$. Since the contraction of T_1 is a peeling of the second kind, we have $u \geqslant 1 - r$, hence $\frac{2m}{2m+1} \leqslant r$, which gives $m = 1$ and $r = \frac{2}{3}$. This gives part of (3) and (4).

We may therefore assume that $q \in \overline{X}$ is smooth. It follows that ℓ is superfluous in $E + \ell$, hence $k \in \{1, 2\}$. Assume that $k = 1$. Then $E = [(2)_m]$ for some $m \geqslant 1$. In particular, $\operatorname{ind}(E^\top) = 1 - \frac{1}{m+1}$. The inequality (5.7) gives $\beta_{D-E}(\ell) \leqslant \frac{1}{r} + \frac{1}{m+1}$, where the equality holds if $\alpha(\ell)$ is log exceptional of the second kind. Note that by Lemma 5.10 ℓ is log exceptional of the first kind if and only if $\beta_D(\ell) < \frac{1}{r}$, that is, if $\beta_{D-E}(\ell) < \frac{1}{r}$. If the latter inequality holds we get (2a), otherwise we get (5).

We are left with the case $k = 2$. Put $d_i = d(T_i)$. We may and shall assume that $d_1 \geqslant d_2$. Since $q \in \overline{X}$ is smooth, we have $d(\ell + E) = 1$. By Lemma 4.1 we obtain $1 = d(\ell + E) = d_1 d_2(1 - \operatorname{ind}^{\top})$. Hence $\gcd(d_1, d_2) = 1$ and $1 - \operatorname{ind}^{\top} = \frac{1}{d_1 d_2} > 0$. The former implies that $d_1 - 1 \geqslant d_2 \geqslant 2$. By Lemma 5.10 we obtain

$$d_1 d_2 r \ell \cdot R \geqslant d_1 + d_2 - 1 + r.$$

We may assume that ℓ is not log exceptional of the first kind, as otherwise we get (2a). This gives $0 \leqslant \ell \cdot (K + rD) = -1 + r(\ell \cdot R - 1)$. We obtain

$$\frac{1}{\ell \cdot R + 1} \leqslant r \leqslant \frac{d_1 + d_2 - 1}{d_1 d_2 \ell \cdot R - 1}. \tag{5.9}$$

It follows that $(d_1 d_2 \ell \cdot R - 1) \leqslant (d_1 + d_2 - 1)(\ell \cdot R + 1)$, hence $\ell \cdot R(d_1 - 1)(d_2 - 1) \leqslant d_1 + d_2$. Since $d_1 - 1 \geqslant d_2$, we obtain $\ell \cdot R(d_2 - 1) \leqslant 2$. Suppose that $\ell \cdot R = 2$. Then $d_2 = 2$ and $d_1 \leqslant 4$, hence $d_1 = 3$. The inequality 5.9 gives $r \leqslant \frac{4}{11}$. But then $T_1 \cdot (K + rD) = 1 - 2r > 0$, which contradicts the fact the α is a peeling of the second kind.

Thus $\ell \cdot R = 1$ and the above inequality reads as

$$\frac{1}{2} \leqslant r \leqslant \frac{d_1 + d_2 - 1}{d_1 d_2 - 1}. \tag{5.10}$$

It follows that $(d_1 - 2)(d_2 - 2) \leqslant 4$, hence $d_2 \leqslant 3$ and $d_1 \leqslant 5$. Assume that $d_2 = 3$. Since $\ell + E$ contracts to a smooth point, we get $T_2 = [3]$ and $T_1 = [3, 2]$ or $T_2 = [2, 2]$ and $T_1 = [4]$. But in the second case we get $r \geqslant \frac{6}{11}$, so the contraction of T_1 is not a partial peeling of the second kind of rD. In the first case we have $r = \frac{1}{2}$, which gives the remaining part of (3).

Finally, assume that $d_2 = 2$. Since $\ell + E$ contracts to a smooth point, we get $T_1 = [(2)_{m-1}, 3]$ for some $m \geqslant 1$. The log discrepancy of the (-3)-curve equals $u = \frac{1}{2m+1}(1 + (1 - r)m)$. Since the contraction of T_1 is a peeling of the second kind, we get $u \geqslant 1 - r$, hence $1 - \frac{1}{m+1} \leqslant r$. Then (5.10) gives $\frac{m}{m+1} \leqslant \frac{2m+2}{4m+1}$, so $m \in \{1, 2\}$. For $m = 2$ we get $r = \frac{2}{3}$, which gives the remaining part of (4). For $m = 1$ we get $\frac{1}{2} \leqslant r \leqslant \frac{4}{5}$, which gives (6). $\square$

Lemma 5.12. *Assume X is smooth, D is reduced, $r \in (0, 1] \cap \mathbb{Q}$ and (X, rD) is $(1-r)$-dlt ($(1-r)$-lc). Denote by $\pi \colon \widetilde{X} \longrightarrow X$ the minimal resolution of singularities and put $\widetilde{D} = \pi^{-1}(D) + \operatorname{Exc} \pi$. If α is a pure partial peeling of (X, rD) of the first (second) kind then $\alpha \circ \pi$ is a partial peeling of $(\widetilde{X}, r\widetilde{D})$ of the first (respectively, second) kind. It is pure, unless D is irreducible and $\pi_*^{-1} D$ is a (-1)-curve with $\beta_{\widetilde{D}} \leqslant 2$.*

Proof. By Corollary 3.2 and Lemma 3.21 π is a partial peeling of the first (second) kind, hence so is $\tilde{\alpha} = \alpha \circ \pi$. Assume that $\tilde{\alpha}$ is not pure. It sufficient to prove that D is irreducible, because then in Lemma 5.11 we have $R = 0$, so $\beta_{\widetilde{D}}(D) \leqslant 2$. We may assume that $\alpha = \operatorname{ctr}_{\gamma(\ell)} \circ \gamma$, where $\gamma \circ \pi$ is pure and $\ell \subseteq D$. By assumption $\widetilde{\ell} \colon = \pi_*^{-1} \ell$ is a (-1)-curve. Let $E_1 = \operatorname{Exc} \pi$ and $E_2 = \pi_*^{-1} \operatorname{Exc} \gamma$. Then $E = E_1 + E_2$ is as in Lemma 5.11. In particular, $\ell \cdot E \leqslant 2$ and each connected component of E is an admissible chain or an admissible fork. It follows that the same holds for E_1, hence $\operatorname{Supp} \operatorname{Bk} E = \operatorname{Supp} E$. Since α is pure, we have $0 \leqslant \ell \cdot K_X = \ell \cdot (K_{\widetilde{X}} + E_1 - \operatorname{Bk} E_1) < -1 + \ell \cdot E_1$, hence $\ell \cdot E \geqslant \ell \cdot E_1 \geqslant 2$. We obtain $E = E_1$ and $\ell \cdot \operatorname{Bk} E \leqslant 1$. This fails in cases (3)-(6). Assume we have case (2a) and let T_1, T_2 be the maximal twigs of $\widetilde{D}$ constituting E. With the notation as in the proof of the lemma we get $1 - \ell \cdot \operatorname{Bk} = 1 - \operatorname{ind}^{\top} - \delta = \frac{1}{d_1 d_2} - \frac{1}{d_1} - \frac{1}{d_2} < 0$; a contradiction. Thus we have case (1) or (2b) with $R = 0$. Hence D is irreducible. $\square$

Proposition 5.13. *Assume that (X, rD), where D is reduced, is $(1 - r)$-dlt ($(1 - r)$-lc) or that X is smooth. If ψ is a pure partial peeling of (X, rD) of the first (second) kind then ψ_{am} is a partial (X, rD)-MMP of the first (respectively, second) kind.*

Proof. Let $\overline{X} = \psi(X)$ and $\overline{D} = \psi_* D$. By Lemma 3.7 we may assume that $\psi = \operatorname{ctr}_{\alpha(\ell)} \circ \alpha$, where α is a pure partial peeling of the second kind of rD and $\ell \subseteq X$. Let $\pi \colon \widetilde{X} \longrightarrow X$ be the minimal resolution of singularities and let $\tilde{\alpha} = \alpha \circ \pi$. Put $\widetilde{\ell} = \pi_*^{-1} \ell$, $\widetilde{D} = \pi_*^{-1} D + \operatorname{Exc} \pi$, $\widetilde{\psi} := \psi \circ \pi$ and $E = \operatorname{Exc} \tilde{\alpha}$. By Lemma 5.12 $\tilde{\alpha}$ is a partial peeling of $r\widetilde{D}$ of the first or second kind.

Since ψ is a peeling, we have $\ell \subseteq D$. Then $\alpha(D) \neq 0$, so Lemma 5.12 implies that $\widetilde{\alpha}$ is pure. Let $\sigma = \mathrm{ctr}_\ell$. Then $\psi \circ \sigma^{-1} \colon (\sigma(X), \sigma_* D) \longrightarrow (\overline{X}, r\overline{D})$ is a partial peeling of the first (second) kind. Since π is a peeling of the second kind, $\ell \cdot (K_X + rD) \leqslant \widetilde{\ell} \cdot (K_{\widetilde{X}} + r\widetilde{D})$. We infer that if $\widetilde{\ell}$ is log exceptional of the first (second) kind then ℓ is log exceptional of the first (second) and hence $(\sigma(X), r\sigma_* D)$ is $(1-r)$-lt ($(1-r)$-lc) or, respectively, X is smooth. But due to (3.4) in the latter situation we are done by induction with respect to $\#\operatorname{Exc}\psi_{\mathrm{am}}$. We may therefore assume that $\widetilde{l}$ is not log exceptional of the first kind, hence E is as in Lemma 5.11(4)-(6). Let $E_1 \leqslant E$ be the maximal (-2)-twig meeting $\widetilde{\ell}$. The contraction of $\widetilde{\ell} + E_1$ is a peeling of $r\widetilde{D}$ of the first (second) kind and at the same time $\widetilde{\psi}_{\mathrm{am}}$ factors through it. By Lemma 3.3 the contraction of $\pi(\widetilde{\ell} + E_1)$ is a peeling of the first (second) kind of rD and we check easily that ψ_{am} factors through it. But the image of (X, rD) under this contraction is $(1-r)$-lt ($(1-r)$-lc) or, respectively, $\sigma(X)$ is smooth, so again we are done by induction. □

Lemma 5.14. *Assume that X is smooth, D is reduced and $r \in [0,1] \cap \mathbb{Q}$. Let $\ell \not\subseteq D$ be an α-almost log exceptional curve of the first (second) kind for some pure partial peeling α of rD of the first (respectively, second) kind. Let E be the sum of connected components of $\operatorname{Exc}\alpha$ meeting ℓ. Put $R = D - E$. Then one of the following holds:*

(1) ℓ is superfluous in $D + \ell$,

(2) $\ell + E = [1, (2)_{k-1}]$ for some $k \geqslant 1$ and E is a twig or rod of D.

(3) $\ell \cdot D = 2$, $r = \frac{1}{2}$, $\ell + E = [1, (2)_{m-1}, 3]$, $m \geqslant 1$ and R meets $E + \ell$ at the point $\ell \cap E$. In particular, all components of $\ell + E$ are of the second kind.

(4) $\ell \cdot D = 3$, $r = \frac{m}{2m+1}$, $\ell + E = [1, (2)_{m-1}, 3]$ for some $m \geqslant 1$, and $R \cdot E = 0$. In particular, α is not of the first kind and $\alpha(\ell)$ is of the second kind.

(5) $\ell \cdot D = 3$, $E \cdot R = 0$, $r = \frac{1}{2}$ and $\ell + E$ is one of $[3,1,3]$, $[2,1,4]$ or $[2,1,3,(2)_{m-2},3]$ for some $m \geqslant 2$. In particular, α is not of the first kind and $\alpha(\ell)$ is of the second kind.

(6) ℓ is log exceptional of the first (second) kind and $\ell + E$ contracts to a smooth point.

Proof. Let $\sigma \colon \alpha(X) \longrightarrow \overline{X}$ be the contraction of $\alpha(\ell)$ and let $\pi \colon \widetilde{X} \longrightarrow \overline{X}$ be the minimal resolution of singularities. Denote by $\varphi \colon X \longrightarrow \widetilde{X}$ the induced morphism. Denote by $q \in \overline{X}$ the image of $\alpha(\ell)$. Put $\overline{D} = \sigma_* \alpha_* D$, $\widetilde{E} = \operatorname{Exc}\pi$, and $\widetilde{D} = \varphi_* D$; see the diagram (5.8).

Consider the case $r = 1$. By Lemma 3.21 we have $\alpha = \alpha_2 \circ \alpha_1$, where α_1 is of the first kind and α_2 is log crepant. Then ℓ is α_1-almost log exceptional, so by Lemma 4.7 it is superfluous in $D + \ell$. Therefore, we may and will assume that $r < 1$ and that ℓ is not superfluous in $D + \ell$.

In the smooth case the proof is an elementary computation. We may assume that $q \in \overline{X}$ is singular. By Lemma 5.2(3) every connected component of E is a twig, rod or fork of D or it is a (-2)-segment of D. Since ℓ is not superfluous in $D + \ell$, the divisor $\widetilde{D}$ is not snc in (every) neighborhood of $\widetilde{E}$, which implies that $\widetilde{E}$ is a degenerated (-2)-segment of $\widetilde{D}$. Let G be a component of E not contracted by φ. Since $\pi^* K_{\overline{X}} = K_{\widetilde{X}}$ and $\pi^* \overline{D} = \widetilde{D}$, we have $\operatorname{ld}(G; \overline{X}, r\overline{D}) = 1 - r = \operatorname{ld}(G; X, rD)$. By Lemma 2.8 it follows that $\alpha(\ell)$ is log exceptional of the second kind. Since ℓ is not superfluous in $D + \ell$, the log surface $(X, D + \ell)$ is dominated by the minimal log resolution over the point $\varphi(\ell)$. If $\#\widetilde{E} = 2$ we get $\ell + E = [3,1,3]$ and hence $r = \frac{1}{3}$. Assume $\#\widetilde{E} = 2$. Then $\ell + E = [1, (2)_{m-1}, 3]$ or $\ell + E = [2,1,3,(2)_{m-2},3]$ for some $m \geqslant 1$, where $[3,(2)_{-1},3] := [4]$. In the first case we get $r = \frac{1}{2}$ if E is a twig of D and $r = \frac{m}{2m+1}$ if E is a rod of D. In the second case we get $r = \frac{1}{2}$. This gives (3), (4) and (5). □

Example 5.15. Let $a \geqslant 2$ be an integer and let $r \in (1 - \frac{1}{3a-4}, 1] \cap \mathbb{Q}$. Consider a log smooth surface $(\widetilde{X}, B)$, where B is reduced and has a twig $T = [3,1,3,a]$, $T \neq B$. Denote by $T_1, \dots, T_4$ subsequent components of T and by $\sigma \colon \widetilde{X} \longrightarrow X$ the contraction of $T_3 + T_4$. Put $D = \sigma_*(T - T_2)$, $\ell = \sigma(T_2)$ and $E = \sigma(T_1)$. Let α be the contraction of E and $\psi \colon X \longrightarrow \overline{X}$ the contraction of $E + \ell$. We have $E \cdot (K_X + D) = 1 - 3r < 0$. By (5.5) we have $c(T_3; X, rD) = \frac{2a-2+r}{3a-1}$, and $c(T_4; X, rD) = \frac{3a-5+3r}{3a-1}$. Since $r > 1 - \frac{1}{3a-4}$, we infer that (X, rD) is $(1-r)$-lt for and σ is a pure partial peeling. Since $r > \frac{1}{2}$, we get $\ell \cdot (K_X + rD) = \frac{3ar-a-1}{3a-1} > 0$. We compute $\alpha(\ell) \cdot (K_{\alpha(X)} + r\alpha_* D) = \frac{r-4/3}{3a-1} < 0$. Since T_1 is not a (-1)-curve, ψ_{am} contracts only ℓ. We conclude that:

(a) (X, rD) is $(1-r)$-dlt,

(b) ℓ is α-almost log exceptional (and not log exceptional) on (X, rD),

(c) $(X', rD') := \psi_{\mathrm{am}}(X, rD)$, the almost minimal model of (X, rD), is not $(1-r)$-lt,

(d) ψ_{am} is not an MMP run for (X,rD).

Part (d) is a consequence of (a) and (c). For (c) note that the exceptional divisor of the minimal resolution is a segment of the reduced total transform of D', contrary to Lemma 5.2(c). Another way to see (c) is a direct computation: we have $\operatorname{ld}(T_3;X',rD')=\frac{3}{2a-1}(1-r)\leqslant 1-r$ and $\operatorname{ld}(T_4;X',rD')=\frac{(a+1)}{2a-1}(1-r)\leqslant 1-r$, so (X,rD) is not even $(1-r)$-lc, unless $r=1$ or $a=2$.

Theorem 5.16 (The effect of squeezing on log singularities)**.** *Assume X is smooth, D is reduced, $r\in(0,1]\cap\mathbb{Q}$ and (X,rD) is $(1-r)$-dlt ($(1-r)$-lc). Let $\sigma\colon(X,rD)\longrightarrow(X',rD')$ be a partial squeezing of the second kind. Then X' is smooth and the following hold:*

(1) If (X',rD') contains no redundant curves of the first kind passing through $\sigma(\operatorname{Exc}\sigma)$ (in particular, if σ is a squeezing) then (X,rD) is $(1-r)$-dlt (respectively, $(1-r)$-lc).

(2) If $\frac{1}{r}$ is an integer then (X',rD') is $(1-r)$-lc.

Proof. Since X is smooth, redundant curves of the first and second kind are in particular (-1)-curves, so the smoothness of X' is clear. Let $\widetilde{E}$ be the exceptional divisor of some partial pure peeling morphism α and let $U\subset D$ be an α-redundant curve of the first or second kind. Let E be the sum of connected components of $\widetilde{E}$ meeting U. Denote by $\sigma'\colon X\longrightarrow Y$ the contraction of U and by $\overline{\sigma}$ the contraction of $\alpha(U)$. Put $B=\sigma'_*D$. We may assume that U is not log exceptional of the first (respectively, first or second) kind, as otherwise we are done by Corollary 2.11 and induction. If $r=1$ then we are in case (3) of Lemma 5.11 with $E\neq 0$, which implies that U meets two distinct components of D and hence we are done by induction. We may thus assume that $r<1$. We are therefore in case (1) or (2) of Lemma 5.11.

(1) We argue that σ is a peeling of rD, and hence (X',rD') is $(1-r)$-dlt (respectively, $(1-r)$-lc) by Corollary 2.11. By Corollary 3.3 and (3.4) we may assume that $\widetilde{E}=E$, so we have $U\subseteq\operatorname{Exc}\sigma\subseteq U+E$. Consider case (1) of Lemma 5.11. Let U' be the component of the (-2)-twig E meeting U. We have $\beta_{D-E}(U)-\frac{1}{r}\leqslant\frac{1}{\#E+1}$. The contraction of $E'=\sigma'_*(E-U')$ is a peeling of (Y,rB). We compute $\beta_{B-E'}(U')=(U+U')\cdot(D-E)=U\cdot(D-E)=\beta_{D-E}(U)$, hence $\beta_{B-E'}(U')-\frac{1}{r}\leqslant\frac{1}{\#E+1}<\frac{1}{\#E'+1}$. This means that U' is redundant of the first kind. By assumption there are no redundant curves of the first kind passing through $\sigma(\operatorname{Exc}\sigma)$, so by induction we get $\operatorname{Exc}\sigma=U+E$, which shows that $\sigma=\overline{\sigma}\circ\alpha$ is a peeling. Consider case (2) of Lemma 5.11. Write $E=U_1+U_2$, with $U_1=[2]$ and $U_2=[3]$. The curve $\sigma'(U_1)$ is redundant of the first kind and after its contraction the image of $\sigma'(U_2)$ is redundant of the first kind. The composition of these three contractions equals $\sigma=\overline{\sigma}\circ\alpha$, which is a peeling.

(2) Since $\frac{1}{r}$ is an integer, after the contraction of U we are in case (1) of Lemma 5.11 with $\beta_{D-E}(U)=\frac{1}{r}$. We have $U\cdot(K_X+rD)=-1+r(-1+\beta_{D-E}(U)+U\cdot E)=r(U\cdot E-1)=0$, so U is log exceptional of the second kind. Then (X',rD') is $(1-r)$-lc by Corollary 2.11. ☐

Proof of Theorem 1.1. Let $\overline{\psi}\colon(X,rD)\longrightarrow(\overline{X},r\overline{D})$ be an MMP run and let $\psi=\overline{\psi}_{\mathrm{am}}$. Let α be a pure partial peeling morphism with $\operatorname{Exc}\alpha\subseteq\operatorname{Exc}\overline{\psi}$ and let $U\subseteq X$ be α-redundant or α-almost log exceptional. We may assume that $E+U$ is connected. We may also assume that the image of (X,rD) after the contraction of U is not $(1-r)$-dlt (respectively, not $(1-r)$-lc), because otherwise we are done by induction using (3.4). Since (X,rD) is $(1-r)$-dlt (respectively, $(1-r)$-lc), $(\overline{X},r\overline{D})$ is $(1-r)$-dlt (respectively, $(1-r)$-lc). By Lemma 3.16 $\overline{\psi}$ is a peeling of $(X,r(D+U))$ and U is α-redundant (note that $(X,r(D+U))$ is not necessarily $(1-r)$-dlt). By Lemma 2.11 U is not log exceptional on (X,rD), so we are in case (1) or (2) of Lemma 5.11. Let $\sigma\colon(X,rD)\longrightarrow(X',rD')$ be the contraction of $U+E$. As noticed in the proof of Theorem 5.16 σ is a peeling of $(X,r(D+U))$, and hence (X',rD') is $(1-r)$-dlt (respectively, $(1-r)$-lc). In particular, $\sigma=\sigma_{\mathrm{am}}$. Then again we are done by induction using (3.4). ☐

Remark 5.17. Assume D is uniform, that is, $D=rD_{\mathrm{red}}$ for some $r\in(0,1]\cap\mathbb{Q}$, and (X,D) is $(1-r)$-lt ($(1-r)$-lc). Choose an MMP run $\overline{\psi}\colon(X,D)\longrightarrow(\overline{X},\overline{D})$ as in Remark 3.19, that is, each time give priority to contractions of rays supported in D and its images. Then the proof of Theorem 1.1 shows that each of the intermediate log surfaces (X_i,D_i), $i\geqslant 1$ in Lemma 3.17 is $(1-r)$-dlt (respectively, $(1-r)$-lc).

5D. *Weighted Kodaira dimension of open varieties*

We now make a digression to define the weighted Kodaira dimension of a (not necessarily complete) algebraic variety S. We assume that $\dim S$ and the base field $\Bbbk$ are such that each variety of dimension $\dim S$ over $\Bbbk$ has a log resolution of singularities which is an isomorphism over the smooth locus. This is true for instance if $\dim S \leqslant 3$ or $\operatorname{char} \Bbbk = 0$.

By a *completion* of S we mean a pair (X, D) consisting of a normal complete variety X and a reduced Weil divisor D. Given a $\mathbb{Q}$-Cartier divisor F on X we denote by $\kappa(X, F) \in \{-\infty, 0, 1, \dots, \dim X\}$ the Iitaka dimension of F, that is, the supremum of dimensions of images of rational maps determined by linear systems $|mF|$.

Definition 5.18. Let S be a variety as above and let $r \in [0, 1] \cap \mathbb{Q}$. If S is smooth then we define the *weighted Kodaira dimension of S of weight r* as

$$\kappa_r(S) := \kappa(X, K_X + rD) \tag{5.11}$$

where (X, D) is some log smooth completion of S. If S is singular we put $\kappa_r(S) := \kappa_r(\widetilde{S})$, where $\widetilde{S}$ is a resolution of singularities of S.

Lemma 5.19. *The weighted Kodaira dimension is well defined, that is, it does not depend on the choice of (X, D).*

Proof. Let (X', D') and (X, D) be two log smooth completions of S. We have a rational map $f\colon X' \dashrightarrow X$ which restricts to an isomorphism on S. Taking the closure of the graph of f, $\Gamma \subseteq X \times X'$, we obtain completion of S dominating (X, D) and (X', D'). There exists a resolution of singularities $\pi\colon \widetilde{\Gamma} \longrightarrow \Gamma$ which is an isomorphism over S and such that $\widetilde{D} := \pi^{-1}(\Gamma \setminus S) \cup \operatorname{Exc}(\pi)$ is snc. Replacing (X, D) with $(\Gamma, \widetilde{D})$ we may therefore assume that f is a morphism of completions. We have $D' = f^{-1}_{*}D + \operatorname{Exc} f$. Since (X, D) is log smooth, (X, rD) is $(1-r)$-log canonical, so $\operatorname{ld}_{X'}(X, rD)$ is effective. Then

$$f^{*}(K_X + rD) \sim_{\mathbb{Q}} K_{X'} + r f^{-1}_{*}D + \operatorname{Exc} f - \operatorname{ld}_{X'}(X, rD) \leqslant K_{X'} + rD', \tag{5.12}$$

so $h^0(m(K_X + rD)) \leqslant h^0(m(K_{X'} + rD'))$ for each m. Since $f_{*}(K_{X'} + rD') = K_X + rD$, the equality holds. ☐

Clearly, we have $\kappa_{r'}(S) \leqslant \kappa_r(S)$ for $r' \leqslant r$, hence $\kappa(X) \leqslant \kappa_r(S) \leqslant \kappa(S)$.

Remark 5.20. The proof shows that if S has a completion (X, D) which is $(1-r)$-log canonical then $\kappa_r(S) = \kappa(K_X + rD)$.

Definition 5.21. For a smooth variety S we define the *Kodaira positivity threshold* of S as

$$\kappa t(S) := \inf\{r : \kappa_r(S) \geqslant 0\} \in \{-\infty\} \cup [0, 1] \tag{5.13}$$

We note that if S is a surface then the infimum can be replaced with a minimum, see [Miy01, 2.2.6.1] (the proof works for snc $\mathbb{Q}$-divisors) or [Fuj84]; for related more general results the reader may consult [Kol92, 11.2.1], [Fuj12, Corollary 1.2] and [Tan14, Theorem 1.2].

The number $\kappa t(S)$ is an interesting invariant for open varieties with completion of negative Kodaira dimension ($\kappa_0(S) = -\infty$). For instance, it is an open problem whether for every $\mathbb{Q}$-acyclic affine variety S of arbitrary dimension one has $\kappa t(S) > 0$. So far this is known only for curves and surfaces as a consequence of rationality of S, see [GP99]. As another example we note that it is an open problem whether for every smooth $\mathbb{Q}$-acyclic surface S one has $\kappa t(S) > \frac{1}{2}$, see [Pal19, Conjecture 4.7] and [Peł21].

6. Uniform boundaries with weight $r \leqslant \frac{1}{2}$

Recall that exceptional divisors of minimal resolutions of canonical (du Val) singularities are (-2)-chains or admissible (-2)-forks. Their Dynkin diagrams are A_k for $k \geqslant 1$, D_k for $k \geqslant 1$ and E_k for $r = 6, 7, 8$. We say that a singular point is of *type* $\mathrm{A}_k^\star$, $k \geqslant 1$ if the exceptional divisor of the minimal resolution is

$$E = [(2)_{k-1}, 3].$$

If E is a connected component of D we say that it is an $\mathrm{A}_k^\star$-rod of D. If the index k is irrelevant we will speak about singularities of *type* $\mathrm{A}^\star$. By convention, we choose an order of components on E such that the (-3)-tip is the last component. If the base field has characteristic zero then an $\mathrm{A}_k^\star$-singularity is exactly a Hirzebruch–Jung surface singularity $\frac{1}{2k+1}(1,k)$, that is, a germ of $\{z^{2k+1} = xy^k\}$, see [BH+04, III.5].

As before, let $r \in (0,1] \cap \mathbb{Q}$, let X be a projective surface and D a divisor on X. Assume X is smooth, D is reduced and (X, rD) is squeezed; see Corollary 3.13. Denote by $\alpha\colon (X, rD) \longrightarrow (\overline{X}, r\overline{D})$ a partial pure peeling morphism and put $E = \operatorname{Exc}\alpha$. Note that by Corollary 5.5 α can be extended to a unique pure peeling morphism. By Lemma 5.2, E is a sum of admissible forks, admissible rods and admissible twig of D. Denote the sum of the connected components of E which are twigs (but not rods) of D by T. By the definition of $E^\flat$ (5.3) we have $K_Y + r\pi_*^{-1}\overline{D} + E^\flat \sim \pi^*(K_{\overline{X}} + r\overline{D})$, where by Corollary 5.7

$$E^\flat = E - \operatorname{Bk}_D(E) - (1-r)\operatorname{Bk}^\top T. \tag{6.1}$$

Moreover, $0 \leqslant E_i^\flat < rE_i$ for every exceptional prime divisor E_i and $E_i^\flat = 0$ if and only if the connected component of $\operatorname{Exc}\alpha$ containing E_i is a (-2)-fork or a (-2)-rod of D.

Corollary 6.1. Assume X is smooth and D is reduced. Let $\alpha\colon X \longrightarrow \overline{X}$ be a birational contraction. Put $E = \operatorname{Exc}\alpha$ and $k = \#E$.

(1) If E is a (-2)-rod or a (-2)-fork of D then $\operatorname{coeff}_U(E^\flat) = 0$ for every component U of E.

(2) If E is a (-2)-twig of D then $\operatorname{coeff}_{E^{(i)}}(E^\flat) = \dfrac{ir}{k+1}$.

(3) If E is an $\mathrm{A}_k^\star$-rod of D (so $E = [(2)_{k-1}, 3]$) then $\operatorname{coeff}_{E^{(i)}}(E^\flat) = \dfrac{i}{2k+1}$.

Proof. This is a consequence of (5.7) and (4.9). ☐

Lemma 6.2 (Peeling and squeezing for $r \leqslant \frac{1}{2}$)**.** *Assume that X is smooth and D is reduced. If $0 < r \leqslant \frac{1}{2}$ then the following hold.*

(1) A contraction of some of (-2)-twigs, admissible (-2)-forks and $\mathrm{A}_k^\star$-rods of D with

$$k < \frac{r}{1-2r} \in (0, \infty] \tag{6.2}$$

is a pure partial peeling. Every pure partial peeling of (X, rD) is of this type.

(2) $L \subseteq D$ is redundant of the first or second kind if and only if it is a (-1)-curve meeting at most one (-2)-twig Δ contracted by the pure partial peeling and such that

$$\beta_{D-\Delta}(L) \leqslant \frac{1}{r} + \frac{1}{d(\Delta)}, \tag{6.3}$$

where the equality holds exactly when L is of the second kind.

Proof. Let $\alpha\colon (X, rD) \longrightarrow (\overline{X}, r\overline{D})$ be a pure partial peeling morphism and let L be a curve not contracted by α. Let $\Delta_1, \ldots, \Delta_n$ be all (-2)-twigs of D meeting L contracted by α. Put $\Delta = \Delta_1 + \ldots + \Delta_n$, $\overline{D} = \alpha_*(D)$, $\overline{L} = \alpha(L)$, $\beta = \beta_{D-\Delta}(L)$ and $\delta = \sum_{i=1}^n \frac{1}{d(\Delta_i)}$. We compute

$$\overline{L} \cdot (K_{\overline{X}} + r\overline{D}) = L \cdot K_X + r(L^2 + \beta) + L \cdot E^\flat = L \cdot K_X(1-r) + r(2p_a(L) - 2 + \beta) + r(L \cdot \Delta - \delta),$$

hence

$$\overline{L} \cdot (K_{\overline{X}} + r\overline{D}) = L \cdot K_X(1-r) + r(2p_a(L) + \beta - 2 + L \cdot \Delta - \delta) \tag{6.4}$$

Since $d(\Delta_i) \geqslant 2$, we have $L \cdot \Delta - \delta \geqslant \frac{n}{2}$.

(1) From (6.4) we see that if L is a (-2)-curve in $D - \Delta$ then $\overline{L} \cdot (K_{\overline{X}} + r\overline{D}) = r(\beta - 2 + L \cdot \Delta - \delta) = r(\beta_D(L) - 2 - \delta)$. This implies that a contraction of some (any) (-2)-twigs, (-2)-rods and admissible

(-2)-forks is a partial pure peeling morphism and every pure partial peeling morphism contracting only (-2)-curves is of this type.

Assume that $\alpha(L)$ is log exceptional and $L\cdot K_X\geqslant 0$. By (6.4), $p_a(L)=0$ and by the above observations we may assume that $L\cdot K_X\geqslant 1$. We obtain $0>(1-r)L\cdot K_X+r(\beta+L\cdot\Delta-\delta-2)\geqslant(1-r)L\cdot K_X-2r$, hence $L\cdot K_X=1$ and $\frac{n}{2}\leqslant\beta+L\cdot\Delta-\delta<3-\frac{1}{r}\leqslant 1$. It follows that $\beta=0$ and $n\leqslant 1$, so $\Delta+L$ is a rod $[(2)_{k-1},3]$ with $L=[3]$ and $k\geqslant 1$ such that $\frac{1}{r}<2+\frac{1}{k}$. Conversely, for such a rod $L\cdot K_X\geqslant 0$ and after the contraction of Δ the curve $\overline{L}$ is log exceptional.

(2) Let $\alpha\colon X\longrightarrow\overline{X}$ be a pure partial peeling and $L\subset D$ a (-1)-curve such that $\overline{L}:=\alpha(L)$ is log exceptional of the first or second kind. By (1) we may assume that $\Delta:=\operatorname{Exc}\alpha$ is a sum of (-2)-twigs meeting L. Since $\Delta+L$ is negative definite, we may in fact assume that Δ is a single (-2)-twig meeting L, possibly zero. Since $\overline{L}$ is log exceptional of the first or second kind, (6.4) gives $\beta\leqslant\frac{1}{r}+\frac{1}{d(\Delta)}$. ☐

Remark 6.3. We note that the condition (6.2) is empty for $r=\frac{1}{2}$ and gives $k=0$ for $r\leqslant\frac{1}{3}$. Also, while for $r=1$ peeling contracts all admissible twigs, for $r\leqslant\frac{1}{2}$ only (-2)-twigs are contracted. On the other hand, while an snc-minimal divisor D is automatically squeezed, it is not so for $r\leqslant\frac{1}{2}$. In fact, the smaller r is the more (-1)-curves in D are contracted by a squeezing morphism for (X,rD).

Notation 6.4. Assume that X is smooth, D is reduced and $0<r\leqslant\frac{1}{2}$. We define:

(1) Γ as the sum of all (-2)-rods and admissible (-2)-forks of D,

(2) Λ as the sum of all $\mathrm{A}_k^{\star}$-rods of D with $k<\frac{r}{1-2r}$. (Recall that we order each such chain so that the (-3)-curve is the last component.)

(3) Δ as the sum of maximal (-2)-twigs in $D-\Gamma-\Lambda$,

For a connected component of Λ, say $\Lambda_0=[(2)_{k-1},3]$, with the above fixed ordering convention definition (4.9) gives:

$$\operatorname{Bk}'\Lambda_0=\sum_{i=1}^{k}\tfrac{2(k-i)+1}{2k+1}\Lambda_0^{(i)}\quad\text{and}\quad\operatorname{Bk}^{\top}\Lambda_0=\sum_{i=1}^{k}\tfrac{i}{2k+1}\Lambda_0^{(i)}\tag{6.5}$$

We define $\operatorname{Bk}'\Lambda$ by extending additively the above formula for connected components of Λ. It follows that

$$\operatorname{Bk}'\Lambda+2\operatorname{Bk}^{\top}\Lambda=\Lambda.\tag{6.6}$$

Analogously, for Δ we have $\operatorname{Bk}_D\Delta+\operatorname{Bk}^{\top}\Delta=\Delta$ and for Γ we have $\operatorname{Bk}_D\Gamma=\Gamma$.

Corollary 6.5. Assume that X is smooth, D is reduced, $0<r\leqslant\frac{1}{2}$ and (X,rD) is squeezed. Let $\alpha\colon(X,rD)\longrightarrow(\overline{X},r\overline{D})$ denote be the unique peeling morphism. Then:

(1) $\operatorname{Exc}\alpha=\Gamma+\Lambda+\Delta$,

(2) $\operatorname{Exc}^{\flat}\alpha=\operatorname{Bk}^{\top}\Lambda+r\operatorname{Bk}^{\top}\Delta$.

Proof. Part (1) is proved in Lemma 6.2 and part (2) follows from Corollary 6.1. ☐

As in case $r=1$, the above computations give a description of $(1-r)$-dlt singularities.

Lemma 6.6 ($(1-r)$-dlt singularities for $r\leqslant\frac{1}{2}$)**.** *Let $0<r\leqslant\frac{1}{2}$ and let $(\overline{X},r\overline{D})$, with $\overline{D}$ reduced, be a germ of a log surface at a point $p\in\overline{X}$. Let E be the exceptional divisor of the minimal log resolution of singularity $\pi\colon X\longrightarrow\overline{X}$ and let $D=\pi_*^{-1}\overline{D}+E$. If $(\overline{X},r\overline{D})$ is $(1-r)$-dlt then one of the following holds:*

(1) E is a nonzero admissible (-2)-rod or admissible (-2)-fork or $\mathrm{A}_k^{\star}$-rod of D with $k<\frac{r}{1-2r}$ (hence $\overline{D}=0$ and $p\in\overline{X}$ is singular - canonical or of type $\mathrm{A}^{\star}$),

(2) E is a nonzero (-2)-twig of D (hence $p\in\overline{X}$ is singular canonical and $p\in\overline{D}$),

(3) $p\in\overline{X}$ is smooth and π is a partial squeezing of rD, hence is a composition of contractions of (-1)-curves as in Lemma 6.2(2).

Conversely, if one of the above holds and $(\overline{X},r\overline{D})$ is squeezed in case (3) then $(\overline{X},r\overline{D})$ is $(1-r)$-dlt.

Proof. Assume $(\overline{X},r\overline{D})$ is $(1-r)$-dlt. By Lemma 5.1 we see that $\pi\colon(X,rD)\longrightarrow(\overline{X},r\overline{D})$ is a partial peeling. We can decompose it as $\pi=\pi''\circ\pi'$, where π'' is a minimal resolution, hence a pure peeling by the same lemma. If $p\in\overline{X}$ is singular then by Lemma 6.2(1) we get (1) and (2) for the exceptional divisor of π''. But in this case a minimal resolution is at the same time a log resolution, so $\pi'=\mathrm{id}$.

Conversely, the divisor D is snc, so (X, rD) is $(1-r)$-dlt. In case (1) and (2) π is a partial peeling by Lemma 6.2, so $(\overline{X}, r\overline{D})$ is $(1-r)$-dlt. In case (3) we get the same claim by Theorem 5.16. □

Remark 6.7 (Multiplicity for $(1-r)$-dlt singularities)**.** Assume $(\overline{X}, r\overline{D})$, with $\overline{D}$ reduced, is $(1-r)$-dlt and $p \in \overline{X}$ is smooth. Then

$$\operatorname{mult}_p(\overline{D}) < 1 + \frac{1}{r}. \tag{6.7}$$

If $\frac{1}{r}$ is an integer then for every infinitely near point q of p we have

$$\operatorname{mult}_q(D) \leqslant 1 + \frac{1}{r}, \tag{6.8}$$

where D is the total reduced transform of $\overline{D}$.

Proof. Let σ be a blowup at p with exceptional divisor E. Then

$$K_X + rD - \sigma^*(K_{\overline{X}} + r\overline{D}) = (1 - rE\cdot D)E = -r(\operatorname{mult}_p(\overline{D}) - 1 - \tfrac{1}{r})E,$$

which gives (6.7). Assume $\frac{1}{r}$ is an integer. By Lemma 6.6(3) the minimal log resolution, and hence σ, is a squeezing. Then the above formula shows that (6.8) holds by Theorem 5.16. □

Example 6.8. If $\frac{1}{r}$ is not an integer then the condition (6.8) may fail for infinitely near points of p. This happens for instance for an ordinary cusp when $r \in (\frac{1}{2}, \frac{4}{5}]$, as the multiplicity of one of the infinitely near points equals 3; cf. Lemma 6.2(2).

We now describe almost log exceptional curves for $r = \frac{1}{2}$. A similar characterization is possible for smaller r, but the number of cases grows as r decreases. Since on squeezed log surfaces of type (X, rD) the peeling morphism is unique, we may and will speak about *almost log exceptional curves* meaning that they are almost log exceptional with respect to this unique peeling.

Lemma 6.9 (Almost log exceptional curves for $r = \frac{1}{2}$)**.** *Let X be a smooth projective surface and D a reduced divisor such that $(X, \frac{1}{2}D)$ is squeezed. Put $E = \Gamma + \Lambda + \Delta$ (see Notation 6.4 and Corollary 6.5). A curve $A \not\subseteq D$ is almost log exceptional of the first kind on $(X, \frac{1}{2}D)$ if and only if it is a (-1)-curve such that one of the following holds:*

(1) $A\cdot D \leqslant 1$. If $A \cdot T = 1$ for some component $T \subseteq E$ then T is a tip of Δ (not necessarily of D) or of a rod of $\Gamma + \Lambda$ or it is the middle component of $[2, 2, 3]$ - a connected component of Λ.

(2) $A \cdot D = 2$, and A meets two different components T_1, T_2 of D, such that

(a) $T_1 \subseteq D - E$ and T_2 is a tip of Δ, of Λ, or of a (-2)-rod of Γ,

(b) $T_1 \subseteq D - E$ and T_2 is the middle curve of a connected component $[2, 2, 3]$ of Λ,

(c) T_1, T_2 are (-3)-curves in Λ,

(d) T_1 is a (-3)-curve in Λ, and $T_2 = [2]$.

(3) $A \cdot D = 3$ and $A \cdot (D - E) = 1$, A meets a connected component $[2]$ of Γ and a (-3)-curve in Λ.

A curve $A \not\subseteq D$ is almost log exceptional of the second kind on $(X, \frac{1}{2}D)$ if and only if it is a (-1)-curve such that

(4) $A\cdot D = 2$ and $A \cdot E = 0$ or

(5) $A \cdot D = 3$ and A meets E once, in a tip of Γ.

Proof. Let $\alpha\colon (X, \frac{1}{2}D) \longrightarrow (\overline{X}, \frac{1}{2}\overline{D})$ be the unique peeling morphism. We have $E = \operatorname{Exc}\alpha = \Gamma + \Lambda + \Delta$. Put $R = D - E$. Let A be almost log exceptional of the first or second kind on $(X, \frac{1}{2}D)$. By Lemma 3.16, A is α-redundant of the first kind on $(X, \frac{1}{2}(D + A))$. By Lemma 6.2 it is a (-1)-curve meeting at most one (-2)-twig Δ_A of $D + A$ and such that

$$A \cdot (D - \Delta_A) < 2 + \frac{1}{d(\Delta_A)}.$$

It follows that $A \cdot D \leqslant 2 + A \cdot \Delta_A \leqslant 3$. Moreover, since Δ_A is a twig of $D + A$ met by A, it is a (-2)-rod of D (in particular a connected component of Γ) met by A in a tip.

The negative definiteness of $E + A$ implies that A does not meet a (-2)-fork in Γ and if it meets $\Delta + \Gamma$ then only once, in a tip. Similarly, if it meets Λ then each connected component at most once, either in a tip or in the middle component of $[2, 2, 3]$. For $A \cdot D \leqslant 1$ we get (1). We may thus assume that $A \cdot D \in \{2, 3\}$. If $A \cdot E = 0$ then $A \cdot (K_X + \frac{1}{2}R + E^\flat) = -1 + \frac{1}{2}A \cdot D \geqslant 0$, which gives (4). We may thus assume that $A \cdot E \neq 0$, too.

Consider the case $A \cdot D = 2$. Then A is of the first kind, because otherwise $A \cdot (R + 2E^\flat) = -2A \cdot K_X = 2$, hence $A \cdot E > 2A \cdot E^\flat = -A \cdot (2K_X + R) = 2 - A \cdot R = A \cdot E$, which is impossible. By the negative definiteness of $E + A$, we see that $T_1 \neq T_2$ and that in case $A \cdot R = 1$ we have (2a) or (2b). Similarly, in case $A \cdot R = 0$ we get (2c) or (2d).

Consider the case $A \cdot D = 3$. Then $A \cdot \Delta_A = 1$ and $A \cdot (D - \Delta_A) = 2$. In particular, $A \cdot R \leqslant 2$. The negative definiteness of $E + A$ implies that $A \cdot (\Gamma - \Delta_A) = A \cdot \Delta = 0$ and $A \cdot R \neq 0$, hence we get (3) or (5). If A is of the second kind then $A \cdot E^\flat = 1 - \frac{1}{2} A \cdot R$, which holds for (5) and fails for (3). ☐

The characterization in Lemma 6.9 gives the following corollary.

Corollary 6.10. Let X be a smooth projective surface and D a reduced divisor which contains no superfluous (-1)-curve (for instance D is snc-minimal or $(X, \frac{1}{2}D)$ is squeezed). If a curve $A \not\subseteq D$ is almost log exceptional of the first kind on $(X, \frac{1}{2}D)$ then $A \cap (X \setminus D)$ is isomorphic to $\mathbb{P}^1$, $\mathbb{A}^1$, $\mathbb{A}^*$ or $\mathbb{A}^{**} = \mathbb{A}^1 \setminus \{0, 1\}$. In the last case A meets three connected components of D (in particular $X \setminus D$ is not affine), each once in the sense of intersection theory; one of them is $[2]$ and the second $[3, 2, \ldots, 2]$ and A meets the (-3)-curve.

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