Nonlinear eigenvalue problems and \textit{PT}-symmetric quantum mechanics
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$PT$-symmetric quantum theory is an extension of QM into the complex plane

$$H = H^\dagger$$

($^\dagger$ means transpose + complex conjugate)

- guarantees real energy and probability-conserving time evolution

- but … is a **mathematical** axiom and not a **physical** axiom of quantum mechanics

Dirac Hermiticity can be generalized…
The idea: Replace Dirac Hermiticity by the *physical* and *weaker* condition of \( PT \) symmetry

\[ P = \text{parity} \]
\[ T = \text{time reversal} \]

(physical because \( P \) and \( T \) are elements of the Lorentz group)

**Example:**

\[
H = p^2 + ix^3
\]

This Hamiltonian has \( PT \) symmetry!
Class of $\textit{PT}$-symmetric Hamiltonians discovered in 1998:

$$H = p^2 + x^2 (ix)^\varepsilon \quad (\varepsilon \text{ real})$$

Transition at $\varepsilon = 0$
Some of my work on $PT$ symmetry

PT papers (2008-2010)

PT papers (2011-2012)

• Y. Chong, L. Ge, and A. Stone, Physical Review Letters 106, 093902 (2011)
• L. Feng, M. Ayache, J. Huang, Y. Xu, M. Lu, Y. Chen, Y. Fainman, A. Scherer, Science 333, 729 (2011)
• A. Zezyulin and V. V. Konotop, Physical Review Letters 108, 213906 (2012)
PT papers (2013)

• C. Yidong, Nature Physics 10, 336 (2014)
Developments in *PT* Quantum Mechanics
(Since its ‘official’ beginning in 1998)

- Nearly 20 international conferences – *FOUR this summer!*
- Nearly 2000 published papers
- Website: “*PT symmeter*” <http://ptsymmetry.net>
- Many many *many* experimental results in last four years!
Rigorous proof of real eigenvalues:

“ODE/IM Correspondence”
P. Dorey, C. Dunning, and R. Tateo,
Upside-down potential with real positive eigenvalues?!

\[ V(x) = -x^4 \]

Z. Ahmed, CMB, and M. V. Berry, 

CMB, D. C. Brody, J.-H. Chen, H. F. Jones, K. A. Milton, and M. C. Ogilvie,  
[arXiv: hep-th/0605066]
Hermitian Hamiltonians:  
BORING!

Eigenvalues are always real – nothing interesting happens
Transition between parametric regions of broken and unbroken $PT$ symmetry...
Can be observed experimentally!
A recent experiment…


At a physical level, $PT$-symmetric quantum systems are intermediate between closed and open systems.

Hermitian $H$  $PT$-symmetric $H$  Non-Hermitian $H$
**PT** quantum mechanics is fun! You can re-visit things you already know about traditional Hermitian quantum mechanics.
Three examples:

1. “Ghost Busting: PT-Symmetric Interpretation of the Lee Model,”

2. “No-ghost Theorem for the Fourth-Order Derivative Pais-Uhlenbeck Oscillator Model,”

3. “PT-Symmetric Interpretation of Double-Scaling”
and
“Double-Scaling Limit of the O(N)-Symmetric Anharmonic Oscillator”
Three current PT research problems:

(1) Conformal Liouville quantum field theory
   Interaction term: $\exp(i\phi)$, $S$ duality

(2) Electromagnetic back-reaction force

(3) Nonlinear systems and nucleotide (DNA) chemical simulations
And now for something completely different...

Nonlinear eigenvalue problems...
Outline of talk

(1) Beginning

(2) Middle

(3) End
Linear eigenvalue problems...

\[-\psi''(x) + V(x)\psi(x) = E\psi(x)\]
\[\psi(\pm\infty) = 0\]

Difficult because this is a *global* (not a local) problem with widely separated boundary conditions!

Example of a difficult global problem...
Difficult problem with widely separated boundary conditions
Problem even with not so distant boundary conditions
For linear problems WKB gives a good approximation for large eigenvalues.

\[ \int_{x_1}^{x_2} dx \sqrt{E_n - V(x)} \sim (n + 1/2)\pi \quad (n \to \infty) \]

Example 1: harmonic oscillator

\[ V(x) = x^2 \]

\[ E_n \sim n \quad (n \to \infty) \]

Example 2: anharmonic oscillator

\[ V(x) = x^4 \]

\[ E_n \sim Bn^{4/3} \quad (n \to \infty) \]

\[ B = \left[ \frac{3\Gamma(3/4)\sqrt{\pi}}{\Gamma(1/4)} \right]^{4/3} \]
WKB approximation works for $PT$ as well:

$$E_n \sim \left[ \frac{\Gamma \left( \frac{3}{2} + \frac{1}{\varepsilon + 2} \right) \sqrt{\pi} n}{\sin \left( \frac{\pi}{\varepsilon + 2} \right) \Gamma \left( 1 + \frac{1}{\varepsilon + 2} \right)} \right]^{\frac{2\varepsilon + 4}{\varepsilon + 4}} \quad (n \to \infty)$$
Hyberasymptotics

Leading asymptotic behavior for large positive $x$

$$\psi(x) \sim C[V(x) - E]^{-1/4} \exp \left[ \int_{x}^{\infty} ds \sqrt{V(s) - E} \right] \quad (x \to \infty)$$

NOTE: Only ONE arbitrary constant!

Second arbitrary constant is invisible because it is contained in the subdominant solution:

$$\psi(x) \sim D[V(x) - E]^{-1/4} \exp \left[ - \int_{x}^{\infty} ds \sqrt{V(s) - E} \right] \quad (x \to \infty)$$

This is the physical solution. Unstable under small changes in $E$. 
Three characteristic properties of solutions

(1) **Oscillatory** in *classically allowed* region (*nth* eigenfunction has *n* nodes)

(2) **Monotone decay** in *classically forbidden* region

(3) **Transition** at the boundary (*turning point*)
Nonlinear toy eigenvalue problem

\[ y'(x) = \cos[\pi xy(x)], \quad y(0) = a \]

Some references:

Solutions for 50 initial conditions

Note: (1) oscillation    (2) monotone decay    (3) transition
Asymptotic behavior for large $x$

Solution behaves like:

$$y(x) \sim \frac{m + 1/2}{x}$$

where $m = 0, 1, 2, 3, \ldots$ is an integer
There’s a **big** problem here...

Where are the odd-$m$ solutions?!?
Furthermore, no arbitrary constant appears in the asymptotic behavior!!
Where is the arbitrary constant?!?
Higher-order asymptotic behavior for large $x$ still contains no arbitrary constant!

\[ y(x) \sim \frac{m + 1/2}{x} + \sum_{k=1}^{\infty} \frac{c_k}{x^{2k+1}} \quad (x \to \infty) \]

\[
\begin{align*}
  c_1 &= \frac{(-1)^m}{\pi} (m + 1/2), \\
  c_2 &= \frac{3}{\pi^2} (m + 1/2), \\
  c_3 &= (-1)^m \left[ \frac{(m + 1/2)^3}{6\pi} + \frac{15(m + 1/2)}{\pi^3} \right], \\
  c_4 &= \frac{8(m + 1/2)^3}{3\pi^2} + \frac{105(m + 1/2)}{\pi^4}, \\
  c_5 &= (-1)^m \left[ \frac{3(m + 1/2)^5}{40\pi} + \frac{36(m + 1/2)^3}{\pi^3} + \frac{945(m + 1/2)}{\pi^5} \right], \\
  c_6 &= \frac{38(m + 1/2)^5}{15\pi^2} + \frac{498(m + 1/2)^3}{\pi^4} + \frac{10395(m + 1/2)}{\pi^6}.
\end{align*}
\]
Hyperasymptotic analysis

\[ Y(x) \equiv y_1(x) - y_2(x) \]

\[ Y'(x) = \cos[\pi xy_1(x)] - \cos[\pi xy_2(x)] \]
\[ = -2 \sin \left[ \frac{1}{2} \pi xy_1(x) + \frac{1}{2} \pi xy_2(x) \right] \sin \left[ \frac{1}{2} \pi xy_1(x) - \frac{1}{2} \pi xy_2(x) \right] \]
\[ \sim -2 \sin \left[ \pi (m + \frac{1}{2}) \right] \sin \left[ \frac{1}{2} \pi x Y(x) \right] \quad (x \to \infty) \]
\[ \sim -(-1)^m \pi x Y(x) \quad (x \to \infty). \]

\[ Y(x) \sim K \exp \left[ -(-1)^m \pi x^2 \right] \quad (x \to \infty) \]

Aha! \( K \) is the arbitrary constant!
Odd \( m \) unstable, even \( m \) stable
$y(0) = a \in \{1.6026, 2.3884, 2.9767, 3.4675, 3.8975, 4.2847, \ldots \}$

Eigenvalues correspond to odd $m$ ...

*Separatrices* (unstable) begin at eigenvalues
We calculated up to $m=500,001$

Let $m = 2n - 1$

We determined that for large $n$ the $n$th eigenvalue grows like the square root of $n$ times a constant $A$, and we used Richardson extrapolation to show that

$$A = 1.7817974363...$$

and then we guessed $A$!!!
A surprising result:

\[
a_n \sim A \sqrt{n} \quad (n \to \infty)
\]

\[
A = 2^{5/6}
\]

This is a nontrivial problem...
Another nontrivial problem
...and we found the analytic solution!
Some scaling changes of variable:

\[ m = 2n - 1 \]
\[ x = \sqrt{2n - 1/2} t, \quad y(x) = \sqrt{2n - 1/2} z(t) \]
\[ \lambda = (2n - 1/2)\pi \]
\[ z'(t) = \cos[\lambda tz(t)] \]

For large \( \lambda \), the eigenfunctions (separatrix curves) approach a limiting curve, which we call \( Z(t) \)...
First four separatrix curves
$m = 500,001$ separatrix curve
Convergence to $Z$ is like convergence of Fourier series

\[ f(x) = 1 \]

\[ S_{2N+1}(x) = \frac{4}{\pi} \sum_{n=0}^{N} \frac{\sin[(2n+1)x]}{2n+1} \]
Analytic calculation of the constant $A$

Multiply $z'(t) = \cos[\lambda tz(t)]$ by $z(t) + t z'(t)$

Integrate from 0 to $t$ and use double-angle formula for cosines:

$$[z(t)]^2 - [z(0)]^2 + t^2/2 + \eta(t) = O(1/\lambda) \quad (\lambda \to \infty),$$

$$\eta(t) = \int_0^t ds \ s \cos[2\lambda sz(s)]$$
Problem is to calculate $\eta(t)$

$\eta(t)$ is just one of a doubly-infinite set of moments defined as:

$$A_{n,k}(t) \equiv \int_0^t ds \cos[n\lambda s z(s)] \frac{s^{k+1}}{[z(s)]^k}$$

Note that $\eta(t) = A_{2,0}(t)$
For large $\lambda$ these moments satisfy the linear difference equation

$$A_{n,k}(t) = -\frac{1}{2}A_{n-1,k+1}(t) - \frac{1}{2}A_{n+1,k+1}(t) \quad (n \geq 2)$$

To get this result we multiply the integrand in $\eta$ by 1:

$$\frac{z(s) + sz'(s)}{z(s)} - \frac{sz'(s)}{z(s)}$$

The moments are associated with a semi-infinite linear one-dimensional random-walk in which random walkers become static as they reach $n=1$. 
The random-walk analysis goes as follows: We let \( \alpha_{n,k} \) be the numerical coefficient of the integrals in \( A_{n,k} \). The initial condition is \( \alpha_{n,0} = 0 \) if \( n \neq 2 \) and \( \alpha_{2,0} = 1 \). Integration by parts gives the relations between the coefficients:

\[
2\alpha_{1,k} + \alpha_{2,k-1} = 0,
\]

\[
2\alpha_{2,k} + \alpha_{3,k-1} = 0,
\]

\[
2\alpha_{n,k} + \alpha_{n-1,k-1} + \alpha_{n+1,k-1} = 0 \quad (n \geq 3).
\]

\[
\alpha_{n,k} = \frac{(-1)^n(n - 1)k!}{2^k(k/2 + n/2)! (k/2 - n/2 + 1)!}.
\]
\[
\eta(t) = - \int_0^t ds \ z(s) z'(s) - \frac{1}{2\sqrt{\pi}} \sum_{p=1}^{\infty} \frac{\Gamma(p + 1/2)}{(p + 1)!} \int_0^t ds \ z'(s) \frac{s^{2p+2}}{[z(s)]^{2p+1}}
\]

No explicit reference to \(\lambda\), so we pass to limit of large \(\lambda\). In this limit the \(z(t)\) oscillates rapidly and approaches the smooth and non-oscillatory function \(Z(t)\).

We get an integral equation satisfied by \(Z(t)\):

\[
[Z(t)]^2 - [Z(0)]^2 + \frac{1}{2} t^2 - \int_0^t ds \ Z(s) Z'(s) + \int_0^t ds \ Z(s) Z'(s) \sqrt{1 - s^2/[Z(s)]^2} = 0.
\]
Differentiate integral equation with respect to $t$:

$$Z(t)Z'(t) + t + Z'(t)\sqrt{[Z(t)]^2 - t^2} = 0$$

Let $Z(t) = tG(t)$

$$\frac{K}{t^3} = (1 + 3[G(t)]^2) \left( G(t) + \sqrt{[G(t)]^2 - 1} \right) \frac{\sqrt{[G(t)]^2 - 1 - 2G(t)}}{\sqrt{[G(t)]^2 - 1 + 2G(t)}}$$

$G(1) = 1$ gives $K = -4$

We thus get $Z(0) = 2^{1/3}$

and from this we get $A = 2^{5/6}$

CMB, A. Fring, and J. Komijani


Possible connection with the *power series constant* \( P \)???

W. K. Hayman, *Research Problems in Function theory*  
[Athlone Press (University of London), London, 1967]


\[
1 \leq P \leq 2
\]

\[
\sqrt{2} \leq P \leq 2
\]

\[
1.7 \leq P \leq 12^{1/4}
\]

\[
1.7818 \leq P \leq 1.82
\]
\[ f_\tau(z) = \sum_{k=0}^{\infty} \exp[i\pi \tau (k^2 + k)] z^k \] 

The maximum values are \( \rho_{50}(f_{0.3780}) = \rho_{50}(f_{0.8780}) \approx 1.7818 \), which coincide with the best known lower bound for \( P \) up to the precision of the computation.
Two *nontrivial* second-order nonlinear eigenvalue problems
Solution $y(x)$ must \textit{choose} between two possible asymptotic behaviors as $x$ gets large and negative:

$$y(x) \sim \pm \sqrt{-x} \quad (x \to -\infty)$$
Example of a difficult choice ...
Two possible asymptotic behaviors

Lower branch is \textit{stable}:

\[ y(x) \sim -\sqrt{-x} + c(-x)^{-1/8} \cos \left[ \frac{4}{5} \sqrt{2}(-x)^{5/4} + d \right] \quad (x \to -\infty) \]

Upper branch is \textit{unstable}:

\[ y(x) \sim \sqrt{-x} + c_{\pm}(-x)^{-1/8} \exp \left[ \pm \frac{4}{5} \sqrt{2}(-x)^{5/4} \right] \quad (x \to -\infty) \]
Two possible kinds of solutions:

Stable

Unstable
FIG. 6: Eigencurve solutions to the first Painlevé transcendent. The eigencurves pass through $y(0) = 1$ and the slopes of the curves at $x = 0$ are the eigenvalues $a_n$. As $x \to -\infty$, the eigencurves approach $+\sqrt{-x}$ exponentially rapidly. Left panel: first two eigencurves corresponding to the eigenvalues $a_1 = 0.231955$ and $a_2 = 3.980669$. The $a_1$ curve approaches $+\sqrt{-x}$ from above and the $a_2$ curve approaches $+\sqrt{-x}$ from below. Right panel: The second two eigencurves for the Painlevé transcendent corresponding to the eigenvalues $a_3 = 6.257998$ and $a_4 = 8.075911$. Note that the second pair of eigenvalues passes through one double pole before approaching the curve $+\sqrt{-x}$. 

Unstable branch

Stable branch
First four eigenfunctions (separatrices)
Numerical calculation of eigenvalues

\[ y(0) = 1 \text{ and } y'(0) = a. \] There is a discrete set of eigencurves whose initial positive slopes are
\[ a_1 = 0.231955, a_2 = 3.980669, a_3 = 6.257998, a_4 = 8.075911, a_5 = 9.654843, a_6 = 11.078201, a_7 = 12.389217, a_8 = 13.613878, a_9 = 14.769304, a_{10} = 15.867511, a_{11} = 16.917331, a_{12} = 17.925488. \]

\[ a_n \sim C n^{3/5} \quad (n \to \infty), \quad C \approx 4.284031379 \]
Analytical calculation of eigenvalues

\[ a_n \sim C n^{3/5} \quad (n \to \infty), \]

\[ C = \frac{1}{\sqrt{3}} \left[ \frac{12 \sqrt{\pi} \Gamma(11/6) 2^{1/3}}{\Gamma(4/3)} \right]^{3/5} \]
Obtained by using WKB to calculate the large eigenvalues of the cubic \textit{PT}-symmetric Hamiltonian

\[ H = \frac{1}{2}p^2 + \frac{1}{3}ix^3 \]

(Do you remember the cubic \textit{PT}-symmetric Hamiltonian?!)
(2) Second Painleve transcendent

\[ y''(x) = [y(x)]^3 + xy(x) \]

Now, both solutions

\[ y(x) \sim \pm \sqrt{-x} \quad (x \to -\infty) \]

are unstable and \( y(x) = 0 \) is stable.
Numerical and analytical calculation of eigenvalues

\[ a_n \sim D n^{2/3} \quad \quad D \approx 1.659221145 \]

\[ D = \frac{1}{\sqrt{2}} \left[ \frac{2 \sqrt{\pi} \Gamma(7/4)}{\Gamma(5/4)} \right]^{2/3} \]
Obtained by using WKB to calculate the large eigenvalues of the **quartic PT-symmetric Hamiltonian**

\[ H = \frac{1}{2}p^2 - \frac{1}{4}x^4 \]

(Do you remember the quartic upside-down **PT**-symmetric Hamiltonian?!)
We hope we have opened a window to a new area of asymptotic analysis.

Thanks for listening!