Eremenko's Conjecture and the Eremenko-Lyubich class

An epic poem in three parts (sadly unfinished)

Lasse Rempe-Gillen

Department of Mathematical Sciences,
University of Liverpool

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A prologue in hexameters

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Corollary 1 (Eremenko)

Every connected component of the closure $\overline{I(f)}$ is unbounded.

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Conjecture (Eremenko's Conjecture)

Let f be any transcendental entire function.
Then every connected component of the escaping set $I(f)$ is unbounded.

Variants of Eremenko's Conjecture

$f : \mathbb{C} \to \mathbb{C}$ transcendental entire; $z_0 \in I(f)$

Eremenko's Conjecture asks:
Is there an unbounded and connected set $A \ni z_0$ with $A \subset I(f)$?

Can A be chosen furthermore such that

- A is an arc connecting z_0 to ∞? (Strong version)
- $f^n|_A \to \infty$ uniformly? (Uniform version)

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Theorem (Rottenfußer, R-G, Rückert, Schleicher; Ann. of Math. 2011)

There is a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ such that $I(f)$ contains no non-degenerate curves.

The Eremenko-Lyubich class

The set $\text{sing}(f^{-1})$ consists of all:

- Critical values: $c = f(z)$, where $f'(z) = 0$; and
- Asymptotic values: $a = \lim f(\gamma(t))$, where $\gamma(t) \to \infty$.

Eremenko-Lyubich class:

$$\mathcal{B} := \{f : \mathbb{C} \to \mathbb{C} \text{ transcendental, entire} : \text{sing}(f^{-1}) \text{ is bounded}\}.$$

Speiser class:

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Why do we care about these classes?

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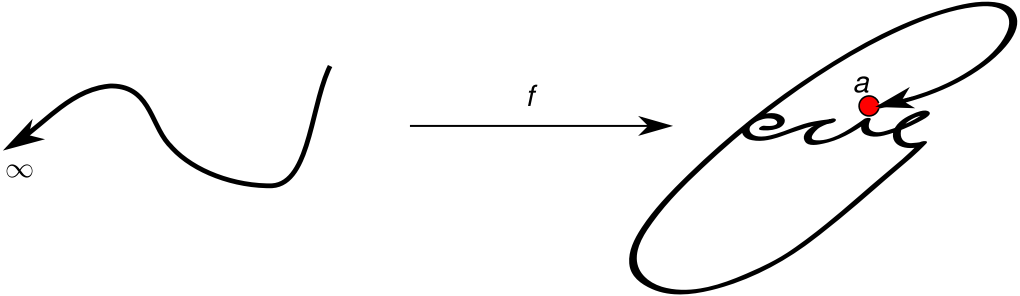

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A soliloquy

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- Strong expansion near infinity (Eremenko-Lyubich).
- A natural notion of *hyperbolic functions* in $\mathcal{B}$ (implying expansion and structural stability).
- Universal structure near infinity within a given parameter space.
- In particular, the strong and uniform variants of the conjecture depend only on the "geometry" of the function.
- (That is, they simultaneously hold or fail for all maps in the same parameter space.)

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What if the conjecture is false?

The importance of being bounded

It can be very hard to control the components of the escaping set. For example,

- $I(f) \cup \{\infty\}$ is always connected (Rippon-Stallard).
- For $f = \exp$, every path-connected components of $I(f)$ is nowhere dense, but $I(f)$ is connected.

In $\mathcal{B}$: may gain control via a geometric-combinatorial principle.

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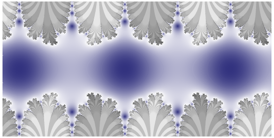
Geometric-Combinatorial Principle I

(R-G, Acta Math. 2010)

Geometry



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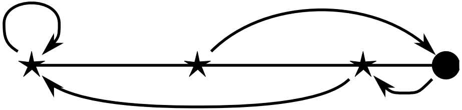
Combinatorics



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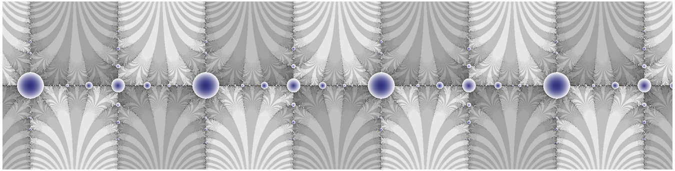
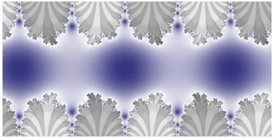


Dynamics

Geometric-Combinatorial Principle II

(Mihaljević-Brandt TAMS 2012)

Geometry



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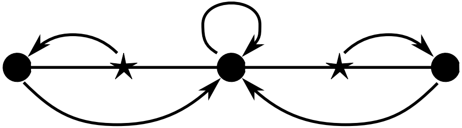

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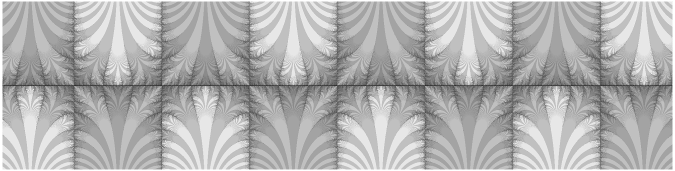

Dynamics

Disjoint-type functions

Maps with trivial combinatorics

Definition (Disjoint-type function)

A transcendental entire function f is of *disjoint type* if there is a compact and connected set K such that

- $\text{sing}(f^{-1}) \subset K$, and
- $f(K) \subset \text{interior}(K)$.

Equivalently, f is hyperbolic with connected Fatou set.

- If f is of disjoint type, then $J(f)$ has uncountably many connected components, each of which is an unbounded, closed, connected set.
- The example from [RRRS] (containing no nondegenerate curves) is of disjoint type.
- What topology can these components have?

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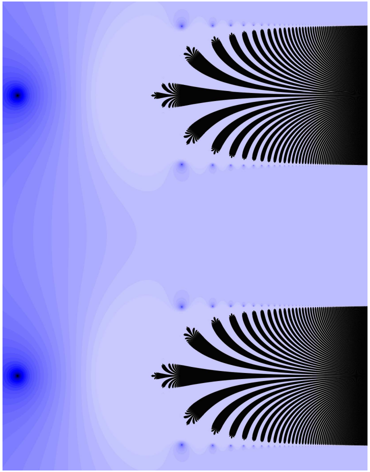

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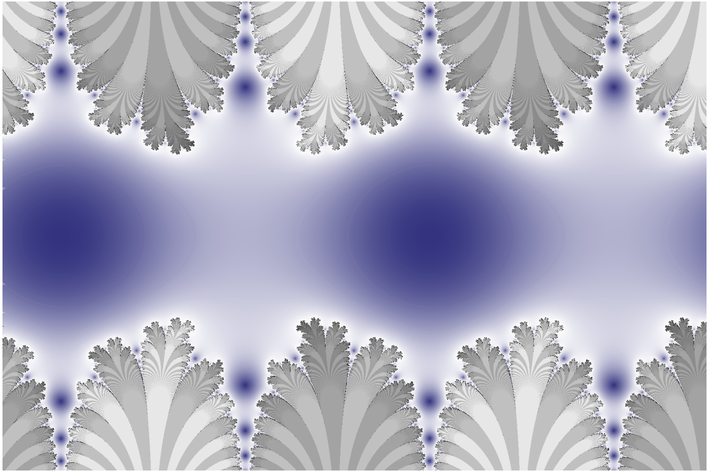

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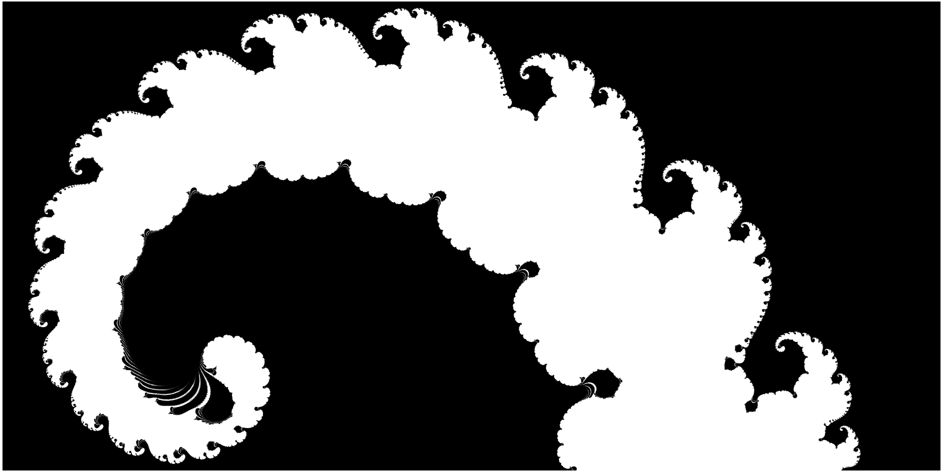

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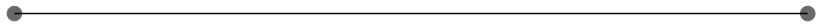

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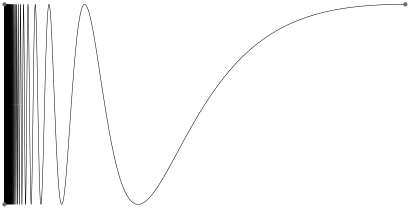

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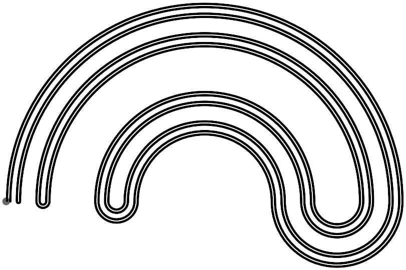

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We give an almost complete answer using the notion of arc-like continua.
(R-G, 2014)

The uniform Eremenko conjecture

Theorem 1

Let f be a transcendental entire function of disjoint type, and let C be an invariant component of $J(f)$. Then the following are equivalent:

- *$C \cup \{\infty\}$ is an indecomposable continuum;*
- *the uniform version of Eremenko's conjecture fails for some $z_0 \in C \cap I(f)$.*

Theorem 2

There is a disjoint-type entire function f such that, for every component C of $J(f)$, the set $C \cup \{\infty\}$ is a pseudo-arc.
(The pseudo-arc is a certain hereditarily indecomposable continuum.)

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More non-uniform escape

Theorem 3

There is a disjoint-type entire function f and a component C of $J(f)$ such that

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Back to Eremenko's Conjecture

- Trivial/simple geometry (e.g. exponential maps / finite order) implies Eremenko's Property for f.
- Trivial/simple combinatorics (e.g.disjoint type / bounded postsingular sets) implies Eremenko's Property for f.
- We now know what geometry is necessary for a (potential) counterexample; the next step would be to control the combinatorics.

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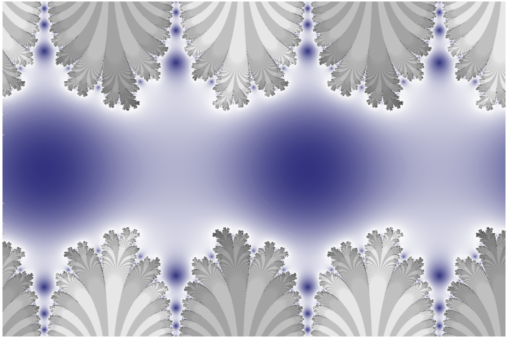
Happy Birthday, Alex!



Special thanks to Samuel Taylor Coleridge, Publius Vergilius Maro, the German national football team, Henri Poincaré, William Shakespeare, Oscar Wilde and Giacomo da Lentini.