Hyperbolic equations with random boundary conditions

Zdzisław Brzeźniak and Szymon Peszat

Department of Mathematics, University of York, York, YO10 5DD, UK,
zb500@york.ac.uk
and
Institute of Mathematics, Polish Academy of Sciences, Św. Tomasza 30/7, 31-027 Kraków, Poland, e-mail: napeszat@cyf-kr.edu.pl

Following Lasiecka and Triggiani an abstract hyperbolic equation with random boundary conditions is formulated. As examples wave and transport equations are studied.

0.1. Introduction

Assume that $\mathcal{A}$ is the generator of a C_0-group $U = (U(t))_{t\in\mathbb{R}}$ of bounded linear operators on a Hilbert space $\mathcal{H}$. Let $\mathcal{U}$ be another Hilbert space and let $u \in L^2_{\rm loc}(0,+\infty;\mathcal{U})$. Typical examples of $\mathcal{H}$ and $\mathcal{U}$ will be spaces $L^2(\mathcal{O})$ and $L^2(\partial\mathcal{O})$. Lasiecka and Triggiani, see[14–17] and,[4,18] discovered that for a large class of boundary operators τ, the hyperbolic initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \mathcal{A}X(t), \qquad t \geq 0, \qquad X(0) = X_0,$$

considered with a non-homogeneous boundary condition

$$\tau(X(t)) = u(t), \qquad t \geq 0,$$

can be written in the form of the homogeneous boundary problem

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \mathcal{A}X(t) + (\lambda - \mathcal{A})Eu(t), \qquad X(0) = X_0, \tag{0.1.1}$$

by choosing properly the space $\mathcal{U}$, an operator $E \in L(\mathcal{U},\mathcal{H})$ and a scalar λ from the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$.

In this paper u will the time derivative of a $\mathcal{U}$-valued càdlàg process ξ. We will discuss the existence and regularity of a solution to the problem (0.1.1). The abstract framework will be illustrated by the wave and the transport equations.

Let us describe briefly the history of the problem studied in this paper. To our (very limited) knowledge, the first paper which studied evolution problems with boundary noise was a paper[3] by Balakrishnan. The equation studied in that paper was first order in time and fourth order in space with Dirichlet boundary noise. Later Sowers[28] investigated general reaction diffusion equation with Neumann type

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boundary noise. Da Prato and Zabczyk in their second monograph,[12] see also,[11] have explained the difference between the problems with Dirichlet and Neumann boundary noises. In particular, the solution to the former is less regular in space than the solution to the latter. Maslowski[23] studied some basic questions such as exponential stability in the mean of the solutions and the existence and uniqueness of an invariant measure. Other related works for parabolic problems with boundary noise are E. Alòs and S. Bonaccorsi[1,2] and Brzeźniak et. al.[5] Similar question have also been investigated in the case of hyperbolic SPDEs with the Neumann boundary conditions, see for instance Mao and Markus,[22] Dalang and Lévêque.[7–9,19] Moreover, some authours, see for example Chueshov, Duan and Schmalfuss[6,13] have studied problems in which deterministic partial differential equations are coupled to stochastic by some sort of boundary conditions. To our best knowledge, our paper is the first one in which the hyperbolic SPDEs with Dirichlet boundary conditions are studied.

0.1.1. *The Wave equation*

Let Δ denote the Laplace operator, τ be a boundary operator, $\mathcal{O}$ be a domain in $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$, and let $\mathcal{S}'(\partial\mathcal{O})$ denote the space of distributions on $\partial\mathcal{O}$. We assume that ξ take values in $\mathcal{S}'(\partial\mathcal{O})$. In the first part of the paper, see Sections from 0.3 to 0.9 we will be concerned with the following initial value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = \Delta u & \text{on } (0,\infty)\times\mathcal{O}, \\ \tau u = \dfrac{\mathrm{d}\xi}{\mathrm{d}t} & \text{on } (0,\infty)\times\partial\mathcal{O}, \\ u(0,\cdot) = u_0 & \text{on } \mathcal{O}, \\ \dfrac{\partial u}{\partial t}(0,\cdot) = u_{0,1} & \text{on } \mathcal{O}. \end{cases} \tag{0.1.2}$$

In fact, we will only consider the Dirichlet and the Neumann boundary conditions. In the case of the Dirichlet boundary conditions we put $\tau = \tau_D$, where $\tau_D\psi(x) = \psi(x)$ for $x \in \partial\mathcal{O}$, whereas in case of the Neumann boundary conditions we put $\tau = \tau_N$, where $\tau_N\psi(x) = \frac{\partial\psi}{\partial\mathbf{n}}(x)$, $x \in \partial\mathcal{O}$ and $\mathbf{n}$ is the exterior unit normal vector field on $\partial\mathcal{O}$.

0.1.2. *The Transport equation*

In Section 0.10 we will consider the following stochastic generalization of the boundary value problem associated to the following simple transport equation introduced in[4] Example 4.1, p. 466,

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x} & \text{on } (0,\infty)\times(0,2\pi), \\ u(0,\cdot) = u_0 & \text{on } (0,2\pi), \\ \tau(u(t,\cdot)) = \dfrac{\mathrm{d}\xi}{\mathrm{d}t}(t) & \text{for } t\in(0,\infty), \end{cases} \tag{0.1.3}$$

where $\tau(\psi) = \psi(2\pi) - \psi(0)$.

This paper is organized as follows. The next section is devoted to an abstract framework. This framework is adapted from works by Lasiecka and Triggiani and also the book by Bensoussan at al.[4] In the following section we study the wave equation with the Dirichlet and the Neumann boundary conditions. In particular we will investigate the concepts of weak and mild solutions, their equivalence, their relation with the abstract framework and their regularity. The final section (Section 0.10) is devoted to the transport equation.

0.2. Abstract formulation

We will identify the Hilbert space $\mathcal{H}$ with its dual $\mathcal{H}'$. Therefore the adjoint operator $\mathcal{A}^*$ is bounded from its domain $\mathrm{D}(\mathcal{A}^*)$, equipped with the graph norm, into $\mathcal{H}$ and hence it has a bounded dual operator $(\mathcal{A}^*)'\colon \mathcal{H} \mapsto (\mathrm{D}(\mathcal{A}^*))'$. It is easy to see that the latter operator is a bounded linear extension of a linear map $\mathcal{A}\colon \mathrm{D}(\mathcal{A}) \mapsto \mathcal{H}$. Since $\mathcal{A}$ is a closed operator, $\mathrm{D}(\mathcal{A})$ endowed with a graph norm is a Hilbert space. Alternatively, we can take $\kappa \in \rho(\mathcal{A})$ and endow $\mathrm{D}(\mathcal{A})$ with the norm $\|(\kappa - \mathcal{A})f\|_{\mathcal{H}}$. Clearly these two norms are equivalent.

We assume that for any $T > 0$ there exists a constant $K > 0$ such that

$$\int_0^T \|E^*\mathcal{A}^*U(t)^*f\|^2_{\mathcal{H}}\mathrm{d}t \le K\,\|f\|^2_{\mathcal{H}}, \qquad f \in \mathrm{D}(\mathcal{A}^*). \tag{0.2.1}$$

Assume that X is a mild solution of (0.1.1) with u being the weak time derivative of ξ, i.e.

$$X(t) = U(t)X_0 + \int_0^t U(t-r)(\lambda - \mathcal{A})E\frac{\mathrm{d}\xi(r)}{dr}, \qquad t \ge 0. \tag{0.2.2}$$

Then, integrating by parts we see that the mild form of (0.1.1) is

$$\begin{aligned} X(t) &= U(t)X_0 + (\lambda - (\mathcal{A}^*)')\left[E\xi(t) - U(t)E\xi(0)\right] \\ &\quad + \int_0^t (\mathcal{A}^*)'U(t-r)(\lambda - \mathcal{A})E\xi(r)\mathrm{d}r, \qquad t \ge 0. \end{aligned} \tag{0.2.3}$$

In other words,

$$X(t) = U(t)x_0 + (\lambda - (\mathcal{A}^*)')\left[E\xi(t) - U(t)E\xi(0)\right] + (\mathcal{A}^*)'\,Y(t), \qquad t \ge 0,$$

where

$$Y(t) := \int_0^t U(t-r)(\lambda - \mathcal{A})E\xi(r)\mathrm{d}r, \qquad t \geq 0.$$

Since the trajectories of the process ξ are càdlàg, they are also locally bounded and hence locally square integrable. Hence, by[4] Proposition 4.1, the process Y has trajectories in $C(\mathbb{R}_+;\mathcal{H})$. Since $(\mathcal{A}^*)'$ is a bounded operator from $\mathcal{H}$ to $(\mathrm{D}(\mathcal{A}^*))'$ we have the following result.

Theorem 0.1. *Assume* (0.2.1). *If ξ is an $\mathcal{U}$-valued càdlàg process, then the process X defined by* (0.2.3) *has càdlàg trajectories in* $(\mathrm{D}(\mathcal{A}^*))'$.

0.3. The Wave equation - introduction

In the next section we will introduce a concept of weak, in the PDE sense, solution to the boundary problem for wave equation (0.1.2). Next, denoting by τ the appropriate boundary operator, by D_τ the boundary map with a certain parameter κ and by Δ_τ the Laplace operator with homogeneous boundary conditions, in Section 0.5 we will show in a heuristic way that problem (0.1.2) can be written as follows

$$\begin{aligned}\mathrm{d}X =&\mathcal{A}_\tau X\mathrm{d}t + (\kappa - \mathcal{A}_\tau^2)\mathbf{D}_\tau \mathrm{d}\xi,\\ =&\mathcal{A}_\tau X\mathrm{d}t + \left((\kappa - \Delta_\tau)\, D_\tau\right)^\dagger \mathrm{d}\xi,\end{aligned} \tag{0.3.1}$$

where

$$\begin{gathered} X = \begin{pmatrix} u \\ \frac{\partial u}{\partial t}\end{pmatrix}, \quad \mathcal{A}_\tau = \begin{bmatrix} 0 & I \\ \Delta_\tau & 0 \end{bmatrix}, \quad \mathbf{D}_\tau = \begin{pmatrix} 0 \\ D_\tau \end{pmatrix}, \\ \left((\kappa - \Delta_\tau)\, D_\tau\right)^\dagger = \begin{pmatrix} 0 \\ (\kappa - \Delta_\tau)\, D_\tau \end{pmatrix}. \end{gathered} \tag{0.3.2}$$

Note that

$$\begin{aligned}\left(\kappa - \mathcal{A}_\tau^2\right)\mathbf{D}_\tau &= \left(\begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix} - \begin{bmatrix} 0 & I \\ \Delta_\tau & 0 \end{bmatrix}\begin{bmatrix} 0 & I \\ \Delta_\tau & 0 \end{bmatrix}\right)\begin{pmatrix} 0 \\ D_\tau \end{pmatrix} \\ &= \begin{bmatrix} \kappa - \Delta_\tau & 0 \\ 0 & \kappa - \Delta_\tau \end{bmatrix}\begin{pmatrix} 0 \\ D_\tau \end{pmatrix} = \begin{pmatrix} 0 \\ (\kappa - \Delta_\tau)\, D_\tau \end{pmatrix} \\ &= \left((\kappa - \Delta_\tau)\, D_\tau\right)^\dagger. \end{aligned} \tag{0.3.3}$$

Let U_τ be the semigroup generated by the operator $\mathcal{A}_\tau$. In Section 0.7 we will show that a weak solution to problem (0.1.2) exists and moreover, it is given by the following formula

$$X(t) = U_\tau(t)X(0) + \int_0^t U_\tau(t-r)\left(\kappa - \mathcal{A}_\tau^2\right)\mathbf{D}_\tau \mathrm{d}\xi(r), \qquad t \geq 0. \tag{0.3.4}$$

In other words, a weak solution to problem (0.1.2) is the mild solution to problem (0.3.1). In (0.3.4), the integrals are defined by integration by parts. Thus

$$\begin{aligned} &\int_0^t U_\tau(t-r)(\kappa-\mathcal{A}_\tau^2)\mathbf{D}_\tau \mathrm{d}\xi(r) \\ &= (\kappa-\mathcal{A}_\tau^2)\mathbf{D}_\tau\xi(t) - U_\tau(t)(\kappa-\mathcal{A}_\tau^2)\mathbf{D}_\tau\xi(0) \\ &\quad + \int_0^t \mathcal{A}_\tau U_\tau(t-r)\left(\kappa-\mathcal{A}_\tau^2\right)\mathbf{D}_\tau\xi(r)\mathrm{d}r, \qquad t\geq 0. \end{aligned} \tag{0.3.5}$$

From now on we will assume that $\kappa>0$ is such that both κ and $\sqrt{\kappa}$ belong to the resolvent set of $\mathcal{A}_\tau$. Then

$$\begin{aligned} \left(\sqrt{\kappa}-\mathcal{A}_\tau\right)\begin{pmatrix} D_\tau \\ \sqrt{\kappa}D_\tau \end{pmatrix} &= \begin{bmatrix} \sqrt{\kappa} & -I \\ -\Delta_\tau & \sqrt{\kappa} \end{bmatrix}\begin{pmatrix} D_\tau \\ \sqrt{\kappa}D_\tau \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \left(\kappa-\Delta_\tau\right)D_\tau \end{pmatrix} = \left(\left(\kappa-\Delta_\tau\right)D_\tau\right)^\dagger, \end{aligned}$$

and hence we see that our case fits into the abstract framework with $\lambda=\sqrt{\kappa}$, and

$$E = \begin{pmatrix} D_\tau \\ \sqrt{\kappa}D_\tau \end{pmatrix}.$$

Since condition (0.2.1) is satisfied,[4] we get the following corollary to Theorem 0.1.

Proposition 0.1. *The process X defined by formula* (0.3.4)*, is* $(\mathrm{D}(\mathcal{A}^*))'$*-valued càdlàg.*

Alternatively, one can give a proper meaning of the term $(\kappa-\mathcal{A}_\tau^2)\mathbf{D}_\tau$ by using the scales of Hilbert spaces

$$H_s^\tau = \mathrm{D}\big((\kappa-\Delta_\tau)^{s/2}\big), \qquad \mathcal{H}_s^\tau := \begin{matrix} H_{s+1}^\tau \\ \times \\ H_s^\tau \end{matrix}, \qquad s\in\mathbb{R},$$

where κ belongs to the resolvent set of Δ_τ, see Section 0.6 for more details. It turns out that if ξ is sufficiently regular in space variable, then $D_\tau\xi$ takes values in the domain of Δ_τ considered on $H_{\tau,-s}$ for s large enough, so the term appearing in the utmost right hand side of (0.3.1) is well defined.

0.4. Weak solution to the wave equation

We will introduce a notion of weak solution to the wave equation and we will discuss their uniqueness.

By $\mathcal{S}(\overline{\mathcal{O}})$ we will denote the class of restrictions of all test function $\varphi\in\mathcal{S}(\mathbb{R}^d)$ to $\overline{\mathcal{O}}$ and we will denote by $(\cdot,\cdot)$ the duality forms on both $\mathcal{S}'(\overline{\mathcal{O}})\times\mathcal{S}(\overline{\mathcal{O}})$ and $\mathcal{S}'(\partial\mathcal{O})\times\mathcal{S}(\partial\mathcal{O})$. We will always assume that any $\mathcal{S}'(\overline{\mathcal{O}})$-valued process v is weakly measurable, that is

$$\Omega\times[0,\infty)\ni(t,\omega)\mapsto(v(t)(\omega),\varphi)\in\mathbb{R}$$

is measurable for any $\varphi \in \mathcal{S}(\overline{\mathcal{O}})$.

Assume now that ξ is an $\mathcal{S}'(\partial\mathcal{O})$-valued process. Taking into account the Green formula, see e.g. the monograph[20] by Lions and Magenes, we arrive at the following definitions of a weak solution.

Definition 0.1. We will say that an $\mathcal{S}'(\overline{\mathcal{O}}) \times \mathcal{S}'(\overline{\mathcal{O}})$-valued process (u, v) is a *weak solution* to (0.1.2) considered with the Dirichlet boundary condition, i.e. $\tau = \tau_D$, iff (aD) for all $t > 0$,

$$(u(t), \varphi) = (u_0, \varphi) + \int_0^t (v(r), \varphi) \mathrm{d}r, \qquad \mathbb{P} - a.s. \qquad \forall\, \varphi \in \mathcal{S}(\overline{\mathcal{O}}), \tag{0.4.1}$$

and (bD) for all $t \geq 0$, $\mathbb{P}$-a.s. for all $\psi \in \mathcal{S}(\overline{\mathcal{O}})$ satisfying $\psi = 0$ on $\partial\mathcal{O}$,

$$(v(t), \psi) = (u_{0,1}, \psi) + \int_0^t (u(r), \Delta\psi) \mathrm{d}r + \left(\xi(t) - \xi(0), \frac{\partial \psi}{\partial \mathbf{n}}\right). \tag{0.4.2}$$

We will call an $\mathcal{S}'(\overline{\mathcal{O}}) \times \mathcal{S}'(\overline{\mathcal{O}})$-valued process (u, v) a *weak solution* to (0.1.2) considered with the Neumann boundary condition, i.e. $\tau = \tau_N$, iff (aN) equality (0.4.1) holds and (bN) for all $t > 0$ and $\psi \in \mathcal{S}(\overline{\mathcal{O}})$ satisfying $\frac{\partial \psi}{\partial \mathbf{n}} = 0$ on $\partial\mathcal{O}$, $\mathbb{P}$-a.s.

$$(u(t), \psi) = (u_0, \psi) + \int_0^t (u(r), \Delta\psi) \mathrm{d}r - (\xi(t) - \xi(0), \psi). \tag{0.4.3}$$

Let $\kappa \geq 0$. For the future consideration we will need also a concept of a weak solution to the (deterministic) elliptic problem

$$\begin{aligned} \Delta u(x) &= \kappa u(x), \quad x \in \mathcal{O}, \\ \tau u(x) &= \gamma(x), \quad x \in \partial\mathcal{O}. \end{aligned} \tag{0.4.4}$$

Definition 0.2. Let $\gamma \in \mathcal{S}'(\partial\mathcal{O})$. We call $u \in \mathcal{S}'(\overline{\mathcal{O}})$ the *weak solution* to (0.4.4) considered with the Dirichlet boundary condition $\tau = \tau_D$ iff

$$(u, \Delta\psi) + \left(\gamma, \frac{\partial \psi}{\partial \mathbf{n}}\right) = \kappa(u, \psi), \qquad \forall\, \psi \in \mathcal{S}(\overline{\mathcal{O}})\colon\; \psi = 0 \text{ on } \partial\mathcal{O}. \tag{0.4.5}$$

We will call $u \in \mathcal{S}'(\overline{\mathcal{O}})$ a *weak solution* to (0.4.4) considered with the Neumann boundary condition $(\tau = \tau_N)$ iff

$$(u, \Delta\psi) - (\gamma, \psi) = \kappa\,(u, \psi), \qquad \forall\, \psi \in \mathcal{S}(\overline{\mathcal{O}})\colon\; \frac{\partial \psi}{\partial \mathbf{n}} = 0 \text{ on } \partial\mathcal{O}. \tag{0.4.6}$$

For the completeness of our presentation we present the following result on the uniqueness of solutions.[5]

Proposition 0.2. (i) *For any* u_0, $u_{0,1}$ *and* γ *problem* (0.1.2) *considered with Dirichlet or Neumann boundary conditions has at most one solution.*

(ii) For any $\gamma \in \mathcal{S}'(\partial\mathcal{O})$ and $\kappa \geq 0$ problem (0.4.4) *with Dirichlet boundary condition has at most one solution.*
(iii) For any $\gamma \in \mathcal{S}'(\partial\mathcal{O})$ and $\kappa > 0$ problem (0.4.4) *with Neumann boundary condition has at most one solution.*

We will denote by $D_D\gamma$ and $D_N\gamma$ the solution to (0.4.4) with Dirichlet and Neumann boundary conditions. We call D_D and D_N the *Dirichlet and Neumann boundary maps*. Note that both these maps depend on the parameter κ and hence should be denoted by D_D^κ and D_N^κ. However, we have decided to use less cumbersome notation.

0.5. Mild formulations

In this section we will heuristically derive a mild formulation of the solution to the stochastic nonhomogeneous boundary value problems to the wave equation. In Section 0.7 we will show that a mild solution is in fact a weak solution.

Assume now that a process u solves wave problem (0.1.2). As in[11] we consider a new process $y := u - D_\tau \frac{\partial \xi}{\partial t}$. Clearly $\tau y(t) = 0$ for $t > 0$ and

$$y(0) = u_0 - D_\tau \frac{\partial \xi}{\partial t}(0), \quad \frac{\partial y}{\partial t}(0) = u_{0,1} - \frac{\partial}{\partial t} D_\tau \frac{\partial \xi}{\partial t}(0). \tag{0.5.1}$$

Next, by the definition of the map D_τ, we have

$$\frac{\partial^2 y}{\partial t^2} = \Delta y + \kappa D_\tau \frac{\partial \xi}{\partial t} - \frac{\partial^2}{\partial t^2} D_\tau \frac{\partial \xi}{\partial t}.$$

Let $(U_\tau(t))_{t\in\mathbb{R}}$ be the group generated by the operator $\mathcal{A}_\tau$ defined by equality (0.3.2). Let us put $z = \frac{\partial y}{\partial t}$. Then, for $t \geq 0$,

$$\begin{aligned} \begin{pmatrix} y \\ z \end{pmatrix}(t) &= U_\tau(t) \begin{pmatrix} y(0) \\ z(0) \end{pmatrix} \\ &\quad + \int_0^t U_\tau(t-r) \begin{pmatrix} 0 \\ \kappa D_\tau \frac{\partial \xi}{\partial r}(r) - \frac{\partial^2}{\partial r^2} D_\tau \frac{\partial \xi}{\partial r}(r) \end{pmatrix} \mathrm{d}r. \end{aligned}$$

On the other hand, by the integration by parts formula, we have

$$\begin{aligned} -\int_0^t U_\tau(t-r) \begin{pmatrix} 0 \\ \frac{\partial^2}{\partial r^2} D_\tau \frac{\partial \xi}{\partial r}(r) \end{pmatrix} \mathrm{d}r &= -\begin{pmatrix} 0 \\ \frac{\partial}{\partial t} D_\tau \frac{\partial \xi}{\partial t}(t) \end{pmatrix} \\ + U_\tau(t) \begin{pmatrix} 0 \\ \frac{\partial}{\partial t} D_\tau \frac{\partial \xi}{\partial t}(0) \end{pmatrix} &- \int_0^t \mathcal{A}_\tau U_\tau(t-r) \begin{pmatrix} 0 \\ \frac{\partial}{\partial r} D_\tau \frac{\partial \xi}{\partial r}(r) \end{pmatrix} \mathrm{d}r. \end{aligned}$$

Set

$$Z(t) := -U_\tau(t) \begin{pmatrix} D_\tau \frac{\partial \xi}{\partial t}(0) \\ 0 \end{pmatrix} + \begin{pmatrix} D_\tau \frac{\partial \xi}{\partial t}(t) \\ 0 \end{pmatrix}.$$

Then, for $t \geq 0$,

$$\int_0^t \mathcal{A}_\tau U(t-r) \begin{pmatrix} 0 \\ \frac{\partial}{\partial r} D_\tau \frac{\partial \xi}{\partial r}(r) \end{pmatrix} \mathrm{d}r = \int_0^t U_\tau(t-r) \begin{pmatrix} \frac{\partial}{\partial r} D_\tau \frac{\partial \xi}{\partial r}(r) \\ 0 \end{pmatrix} \mathrm{d}r$$
$$= Z(t) + \int_0^t \mathcal{A}_\tau U_\tau(t-r) \begin{pmatrix} D_\tau \frac{\partial \xi}{\partial r}(r) \\ 0 \end{pmatrix} \mathrm{d}r$$
$$= Z(t) + \int_0^t \mathcal{A}_\tau^2 U_\tau(t-r) \begin{pmatrix} 0 \\ D_\tau \frac{\partial \xi}{\partial r}(r) \end{pmatrix} \mathrm{d}r.$$

Hence, in view of the equalities appearing in (0.5.1), we arrive at the following identity

$$\begin{pmatrix} y \\ z \end{pmatrix}(t) = U_\tau(t) \begin{pmatrix} u_0 \\ u_{0,1} \end{pmatrix} - \begin{pmatrix} D_t \frac{\partial \xi}{\partial t}(t) \\ \frac{\partial}{\partial t} D_\tau \frac{\partial \xi}{\partial t}(t) \end{pmatrix}$$
$$+ \int_0^t (\kappa - \mathcal{A}_\tau^2) U_\tau(t-r) \begin{pmatrix} 0 \\ D_\tau \end{pmatrix} \frac{\partial \xi}{\partial r}(r) \mathrm{d}r, \qquad t \geq 0.$$

Putting $v(t) := \frac{\partial u(t)}{\partial t}$ we observe that

$$v(t) = \frac{\partial y(t)}{\partial t} + \frac{\partial}{\partial t} D_\tau \frac{\partial \xi}{\partial t}(t) = z(t) + \frac{\partial}{\partial t} D_\tau \frac{\partial \xi}{\partial t}(t),$$

and hence we obtain (0.3.4).

0.6. Scales of Hilbert spaces

Let $(A, \mathrm{D}(A))$ be the infinitesimal generator of an analytic semigroup S on a real separable Hilbert space H. Let $\tilde{\kappa}$ belongs to the resolvent set of A. Then, see e.g.,[21] the fractional power operators $(\tilde{\kappa} - A)^s$, $s \in \mathbb{R}$, are well defined. In particular, for $s < 0$, $(\tilde{\kappa} - A)^s$ is a bounded linear operator and for $s > 0$, $(\tilde{\kappa} - A)^s := \left((\tilde{\kappa} - A)^{-s}\right)^{-1}$. For any $s \geq 0$ we set $H_s := \mathrm{D}\left((\tilde{\kappa} - A)^{s/2}\right) = \mathrm{R}\left((\tilde{\kappa} - A)^{-s/2}\right)$. We equip the space H_s with the norm

$$\|f\|_{H_s} := \|(\tilde{\kappa} - A)^{s/2} f\|_H, \qquad f \in H_s.$$

Note that for all $s, r \geq 0$, $(\tilde{\kappa} - A)^{r/2} \colon H_{s+r} \mapsto H_s$ is an isometric isomorphism.

We also introduce the spaces H_s for $s < 0$. To do this let us fix $s < 0$. Note that the operator $(\tilde{\kappa} - A)^{s/2} \colon H \mapsto H_s$ is an isometric isomorphism. Hence we can define H_s as the completion of H with respect to the norm $\|f\|_{H_s} := \|(\tilde{\kappa} - A)^{s/2} f\|_H$. We have

$$\| (\tilde{\kappa} - A)^{-s/2} f \|_{H_s} = \|f\|_H \qquad \text{for } f \in H_{-s}.$$

Thus, since H_{-s} is dense in H, $(\tilde{\kappa} - A)^{-s/2}$ can be uniquely extended to the linear isometry denoted also by $(\tilde{\kappa} - A)^{-s/2}$ between H_s and H.

Assume now that A is a self-adjoint non-positive definite linear operator in H. It is well known (and easy to see) that A considered on any H_s with $s < 0$ is essentially self-adjoint. We denote by A_s its unique self-adjoint extension. Note that $\mathrm{D}\,(A_s) = H_{s+2}$. Finally, for any $s \geq 0$, the restriction A_s of A to H_{s+2} is a self-adjoint operator on H_s. The spaces H_s, $s < 0$, can be chosen in such a way that

$$H_s \hookrightarrow H_r \hookrightarrow H \hookrightarrow H_{-r} \hookrightarrow H_{-s}, \qquad \forall\, s \geq r \geq 0,$$

with all embedding dense and continuous. Identifying H with its dual space H' we obtain

$$H_s \hookrightarrow H \equiv H' \hookrightarrow H'_s, \qquad s \geq 0.$$

Remark 0.1. Under the identification above we have $H'_s = H_{-s}$. Moreover,

$$\langle A_s f, g\rangle = \langle f, A_{-s} g\rangle, \qquad \forall\, f \in \mathrm{D}\,(A_s),\ g \in \mathrm{D}\,(A_{-s}),$$

where $\langle\cdot,\cdot\rangle$ is the bilinear form on $H_s \times H_{-s}$ whose restriction to $(H_s \cap H) \times H$ is the scalar product on H.

Given $s \in \mathbb{R}$, define

$$\mathcal{H}_s := \begin{array}{c} H_{s+1} \\ \times \\ H_s \end{array}.$$

On $\mathcal{H}_s$ we consider an operator $\mathcal{A}_s$ defined by the following formulas

$$\mathcal{A}_s := \begin{bmatrix} 0 & I \\ A_s & 0 \end{bmatrix}, \qquad \mathrm{D}\,(\mathcal{A}_\mathrm{s}) = \mathcal{H}_{\mathrm{s}+1}.$$

By the Lumer–Philips theorem, see e.g.,[24] $\mathcal{A}_s$ generates an unitary group U_s on $\mathcal{H}_s$.

0.6.1. *Application to the boundary value problem*

Let $-\Delta_D$ and $-\Delta_N$ be the Laplace operators on $H = L^2(\mathcal{O})$ with the homogeneous Dirichlet and Neumann boundary conditions, respectively. The corresponding scales of Hilbert spaces will be denoted by (H^D_s) and (H^N_s) and the restriction (or, if $s < 0$, the unique self-adjoint extension) of Δ_D and Δ_N to H^D_s and H^N_s by Δ^D_s and Δ^N_s. Finally, we write

$$\mathcal{H}^D_s := \begin{array}{c} H^D_{s+1} \\ \times \\ H^D_s \end{array} \qquad \text{and} \qquad \mathcal{H}^N_s := \begin{array}{c} H^N_{s+1} \\ \times \\ H^N_s \end{array}$$

and

$$\mathcal{A}^D_s := \begin{bmatrix} 0 & I \\ \Delta^D_s & 0 \end{bmatrix}, \qquad \mathrm{D}\,(\mathcal{A}^\mathrm{D}_\mathrm{s}) = \mathcal{H}^\mathrm{D}_{\mathrm{s}+1},$$

$$\mathcal{A}^N_s := \begin{bmatrix} 0 & I \\ \Delta^N_s & 0 \end{bmatrix}, \qquad \mathrm{D}\,(\mathcal{A}^\mathrm{N}_\mathrm{s}) = \mathcal{H}^\mathrm{N}_{\mathrm{s}+1}.$$

Example 0.1. (*i*) Let $\mathcal{O} = (0,1)$ and $\kappa = 0$. Define functions ψ_i, $i = 1,2$ by $\psi_1(x) = 1$ and $\psi_2(x) = x$ for $x \in [0,1]$. Then, see e.g.,[5] $D_D\colon \partial\mathcal{O} \equiv \mathbb{R}^2 \mapsto C([0,1])$ is given by

$$D_D \begin{pmatrix} a \\ b \end{pmatrix} = a\psi_1 + (b-a)\psi_2,$$

and, by Remark 0.1, see also,[5] $\Delta_s^D \psi_1 = \frac{\mathrm{d}}{\mathrm{d}x}\delta_0 - \frac{\mathrm{d}}{\mathrm{d}x}\delta_1$ and $\Delta_s^D \psi_2 = -\frac{\mathrm{d}}{\mathrm{d}x}\delta_1$ for $s \le -1$. Taking into account (0.3.3), for $s \ge 1$ we have

$$-(\mathcal{A}_{-s}^D)^2 \mathbf{D}_D \begin{pmatrix} a \\ b \end{pmatrix} = \left(-\Delta_{-s}^D D_D\right)^\dagger \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ -a\frac{\mathrm{d}}{\mathrm{d}x}\delta_0 + b\frac{\mathrm{d}}{\mathrm{d}x}\delta_1 \end{pmatrix}.$$

(*ii*) For the Neumann boundary conditions on $\mathcal{O} = (0,1)$ we take $\kappa = 1$. Then

$$D_N \begin{pmatrix} a \\ b \end{pmatrix} = \frac{b - a\mathrm{e}}{\mathrm{e} - \mathrm{e}^{-1}}\psi_1 + \left(a + \frac{b - a\mathrm{e}}{\mathrm{e} - \mathrm{e}^{-1}}\right)\psi_2, \quad \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \equiv \partial\mathcal{O},$$

where $\psi_1(x) = \mathrm{e}^{-x}$ and $\psi_2(x) = \mathrm{e}^x$. Then, by Remark 0.1, for any $\phi \in \mathrm{D}\,(\Delta_{\mathrm{N}}^1)$, and $s \ge 1$,

$$\left(\Delta_{-s}^N \psi_i, \phi\right) = \left(\Delta_{-1}^N \psi_i, \phi\right) = \int_0^1 \psi_i(x) \frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x)\mathrm{d}x.$$

Since

$$\begin{aligned}\int_0^1 \psi_1(x)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x)\mathrm{d}x &= \int_0^1 \mathrm{e}^{-x}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x)\mathrm{d}x \\ &= \mathrm{e}^{-1}\frac{\mathrm{d}\phi}{\mathrm{d}x}(1) - \frac{d\phi}{\mathrm{d}x}(0) + \mathrm{e}^{-1}\phi(1) - \phi(0) + \int_0^1 \mathrm{e}^{-x}\phi(x)\mathrm{d}x\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \psi_2(x)\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x)\mathrm{d}x &= \int_0^1 \mathrm{e}^{x}\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}(x)\mathrm{d}x \\ &= \mathrm{e}\frac{\mathrm{d}\phi}{\mathrm{d}x}(1) - \frac{\mathrm{d}\phi}{\mathrm{d}x}(0) - \mathrm{e}\phi(1) + \phi(0) + \int_0^1 \mathrm{e}^{x}\phi(x)\mathrm{d}x,\end{aligned}$$

and since $\frac{\mathrm{d}\phi}{\mathrm{d}x}(0) = \frac{\mathrm{d}\phi}{\mathrm{d}x}(1) = 0$, it follows

$$\Delta_{-s}^N \psi_1 = \mathrm{e}^{-1}\delta_1 - \delta_0 + \psi_1,$$

$$\Delta_{-s}^N \psi_2 = -\mathrm{e}\delta_1 + \delta_0 + \psi_2.$$

Consequently,

$$\begin{aligned}&\left(1 - \Delta_{-s}^N\right) D_N \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \frac{b - a\mathrm{e}}{\mathrm{e} - \mathrm{e}^{-1}}\left(\delta_0 - \mathrm{e}^{-1}\delta_1\right) + \left(a + \frac{b - a\mathrm{e}}{\mathrm{e} - \mathrm{e}^{-1}}\right)(\mathrm{e}\delta_1 - \delta_0) \\ &= -a\delta_0 + (b - a\mathrm{e})\,\delta_1,\end{aligned}$$

and hence

$$\left(1-(\mathcal{A}^N_{-s})^2\right)\mathbf{D}_N\begin{pmatrix}a\\ b\end{pmatrix}=\left(\left(1-\Delta^N_{-s}\right)D_N\right)^\dagger\begin{pmatrix}a\\ b\end{pmatrix}$$
$$=\begin{pmatrix}0\\ -a\delta_0+(b-a\mathrm{e})\,\delta_1,\end{pmatrix}.$$

(iii) Let $\mathcal{O}=(0,\infty)$ and $\kappa=1$. Then, see e.g.,[5] $D_D\colon\partial\mathcal{O}\equiv\mathbb{R}\mapsto C([0,\infty))$ is given by $D_Da=a\psi$, where $\psi(x)=\mathrm{e}^{-x}$. Next, again by Remark 0.1, see also,[5] $\Delta^D_{-s}\psi=\frac{\mathrm{d}}{\mathrm{d}x}\delta_0+\psi$, $s\geq 1$, and consequently for any $s\geq 1$,

$$\left(1-(\mathcal{A}^D_{-s})^2\right)\mathbf{D}_D(a)=\left(\left(1-\Delta^D_{-s}\right)D_D\right)^\dagger(a)=\begin{pmatrix}0\\ -a\frac{\mathrm{d}}{\mathrm{d}x}\delta_0\end{pmatrix}.$$

(iv) For the Neumann boundary problem on $(0,\infty)$ with $\kappa=1$ we have $D_Na=-a\psi$, where $\psi(x)=\mathrm{e}^{-x}$. Then, $\Delta^N_{-s}\psi=\delta_0+\psi$. Consequently,

$$\left(1-(\mathcal{A}^N_{-s})^2\right)\mathbf{D}_N(a)=\left(\left(1-\Delta^N_{-s}\right)D_N\right)^\dagger(a)=\begin{pmatrix}0\\ -a\delta_0\end{pmatrix}.$$

0.7. Equivalence of weak and mild solutions

We denote by τ either the Dirichlet or the Neumann boundary condition. We assume that D_τ is a corresponding boundary map, i.e. it satisfies

$$\Delta D_\tau\psi=\kappa D_\tau\psi,\qquad \tau D_\tau\psi=\psi\ \text{ on }\partial\mathcal{O}. \tag{0.7.1}$$

Recall that H^τ_s is the domain of $(\tilde\kappa-\Delta_\tau)^s$, hence in particular, H^τ_{s+1} is the domain of the Laplace operator Δ^τ_s considered on H^τ_s. We denote by $\left(U^\tau_s(t)\right)_{t\in\mathbb{R}}$ the C_0-group on $\mathcal{H}^\tau_s$ generated by $\mathcal{A}^\tau_s$.

The following existence theorem is the main result of this section.

Theorem 0.2. *Assume that $D_\tau\xi$ is a càdlàg process in a space H^τ_s for some $s\in\mathbb{R}$. Then for any $X_0:=(u_0,u_{0,1})^{\mathrm{T}}\in\mathcal{H}^\tau_{s-4}$ there exists a unique weak solution u to problem* (0.1.2) *whose trajectories are H_{s-3}-valued càdlàg, with the boundary condition $\tau u=\dot\xi$. Moreover, the process $X=\left(u,\frac{\partial u}{\partial t}\right)^{\mathrm{T}}$ is given by*

$$\begin{aligned}&U^\tau_{s-3}(t)X_0+\int_0^t U^\tau_{s-3}(t-r)\left((\kappa-\Delta^\tau_{s-2})D_\tau\right)^\dagger\mathrm{d}\xi(r)\\&=U^\tau_{s-3}(t)X_0+\int_0^t\mathcal{A}^\tau_{s-3}U^\tau_{s-2}(t-r)\left((\kappa-\Delta^\tau_{s-2})D_\tau\right)^\dagger\xi(r)\mathrm{d}r\\&+\left((\kappa-\Delta^\tau_{s-2})D_\tau\right)^\dagger\xi(t)-U^\tau_{s-2}(t)\left((\kappa-\Delta^\tau_{s-2})D_\tau\right)^\dagger\xi(0),\qquad t\geq 0.\end{aligned}\tag{0.7.2}$$

Proof. Clearly the process X defined by formula (0.7.2) is càdlàg in $\mathcal{H}^\tau_{s-3}$. We will show that it is a weak solution to problem (0.1.2). By the well-known equivalence result, see e.g.[10] or,[25] for any $h \in \mathrm{D}\left(\left(\mathcal{A}^{\tau}_{s-3}\right)^{*}\right)$,

$$\begin{aligned}\langle X(t), h\rangle_{\mathcal{H}^{\tau}_{s-3}} = \langle X(0), h\rangle_{\mathcal{H}^{\tau}_{s-3}} + \int_0^t \langle X(r), \left(\mathcal{A}^{\tau}_{s-3}\right)^{*} h\rangle_{\mathcal{H}^{\tau}_{s-3}} \mathrm{d}r \\ + \langle h, \left(\left(\kappa - \Delta^{\tau}_{s-2}\right) D_{\tau}\right)^{\dagger} (\xi(t) - \xi(0))\rangle_{\mathcal{H}^{\tau}_{s-3}}, \qquad t \geq 0.\end{aligned} \tag{0.7.3}$$

Clearly,

$$\left(\mathcal{A}^{\tau}_{s-3}\right)^{*} = \begin{pmatrix} 0 \; \Delta^{\tau}_{s-3} \\ I \; 0 \end{pmatrix}.$$

Let $X = (u, v)^T$, and let $\varphi \in \mathcal{S}(\overline{\mathcal{O}})$ and $\psi \in \mathcal{S}(\overline{\mathcal{O}})$ be such that $\tau\psi = 0$ on $\partial\mathcal{O}$. Note that $h_1 := (\varphi, 0)^T$, $h_2 := (0, \psi)^T \in \mathrm{D}\left(\left(\mathcal{A}^{\tau}_{s-3}\right)^{*}\right)$. Applying equality (0.7.3) to $h = h_1$ we obtain

$$\langle u(t), \varphi\rangle_{H_{s-2}} = \langle u_0, \varphi\rangle_{H_{s-2}} + \int_0^t \langle v(r), \varphi\rangle_{H_{s-2}} \mathrm{d}r, \qquad t \geq 0.$$

Consequently, for $\tilde{\varphi} := (\kappa - \Delta)^{s-2} \varphi$,

$$(u(t), \tilde{\varphi}) = (u_0, \tilde{\varphi}) + \int_0^t (v(r), \tilde{\varphi}) \mathrm{d}r, \qquad t \geq 0.$$

Next, for $h = h_2$,

$$\langle v(t), \psi\rangle_{H_{s-3}} = \langle u_{0,1}, \psi\rangle_{H_{s-3}} + \int_0^t \langle u(r), \Delta\psi\rangle_{H_{s-3}} \mathrm{d}r + R_{\tau}(t), \ t \geq 0,$$

where, for $t \geq 0$,

$$R_{\tau}(t) := \langle \psi, \left(\kappa - \Delta^{\tau}_{s-2}\right) D_{\tau}\xi(t)\rangle_{H_{s-3}} - \langle \psi, \left(\kappa - \Delta^{\tau}_{s-2}\right) D_{\tau}\xi(0)\rangle_{H_{s-3}}.$$

Let $\tilde{\psi} := (\tilde{\kappa} - \Delta)^{s-3} \psi$. It remains to show that in the case of the Dirichlet boundary operator,

$$R_{\tau}(t) = \left(\xi(t) - \xi(0), \frac{\partial \tilde{\psi}}{\partial \mathbf{n}}\right)$$

and, in the case of the Neumann boundary operator,

$$R_{\tau}(t) = -\left(\xi(t) - \xi(0), \tilde{\psi}\right).$$

These two identities follow from an observation that by Definition 0.2, if z is such that $D_{\tau}z \in H_s$, then for any $\psi \in \mathcal{S}(\overline{\mathcal{O}})$ satisfying $\tau\psi = 0$,

$$\left(\psi, \left(\lambda - \Delta^{\tau}_{s-2}\right) D_{\tau} z(t)\right) = \begin{cases} \left(z(t), \frac{\partial \psi}{\partial \mathbf{n}}\right) & \text{if } \tau \text{ is Dirichlet,} \\ -\left(\xi(t) - \xi(0), \tilde{\psi}\right) & \text{if } \tau \text{ is Neumann.} \end{cases}$$

□

0.8. The Fundamental Solution

Let G_τ be the fundamental solution to the Cauchy problem for $\frac{\partial^2 u}{\partial t^2} = \Delta u$ associated with the boundary operator τ. In other words, $G_\tau\colon (0,\infty)\times\mathcal{O}\times\mathcal{O}\mapsto\mathbb{R}$ satisfies $\tau G_\tau(t,x,y) = 0$ with respect to x and y variables, $G_\tau(0,x,y) = 0$ and $\frac{\partial G_\tau}{\partial t}(0,x,y) = \delta_x(y)$,

$$\frac{\partial^2 G_\tau}{\partial t^2}(t,x,y) = \Delta_x G_\tau(t,x,y) = \Delta_y G_\tau(t,x,y), \quad t>0,\ x,u\in\mathcal{O}.$$

Then the wave semigroup is given by, for $x\in\mathcal{O}$,

$$\begin{aligned} &U_\tau(t)\begin{pmatrix} u_0 \\ u_{0,1}\end{pmatrix}(x) \\ &= \begin{pmatrix} \int_{\mathcal{O}} \left(\frac{\partial}{\partial t}G_\tau(t,x,y)u_0(y) + G_\tau(t,x,y)u_{0,1}(y)\right)\mathrm{d}y \\ \int_{\mathcal{O}} \left(\frac{\partial^2}{\partial t^2}G_\tau(t,x,y)u_0(y) + \frac{\partial}{\partial t}G_\tau(t,x,y)u_{0,1}(y)\right)\mathrm{d}y \end{pmatrix}. \end{aligned}$$

Hence, for $x\in\mathcal{O}$,

$$\begin{aligned} &U_\tau(t)\left((\kappa-\Delta_\tau)D_\tau\right)^\dagger v(x) \\ &= \begin{pmatrix} \int_{\mathcal{O}} G_\tau(t,x,y)(\kappa-\Delta_\tau)D_\tau v(y)\mathrm{d}y \\ \int_{\mathcal{O}} \frac{\partial}{\partial t}G_\tau(t,x,y)(\kappa-\Delta_\tau)D_\tau v(y)\mathrm{d}y \end{pmatrix}. \end{aligned} \tag{0.8.1}$$

Let us now denote by σ the surface measure on $\partial\mathcal{O}$. The following result gives the formula for the solution to wave problem (0.1.2) in terms of the fundamental solution.

Theorem 0.3. *Assume that u is a solution to wave problem* (0.1.2)*, where for simplicity $u_0 = 0 = u_{0,1}$.*
(i) If τ is the Dirichlet boundary operator, then the solution is given by

$$u(t,x) = -\int_0^t\int_{\partial\mathcal{O}} \frac{\partial G_\tau}{\partial \mathbf{n}_y}(t-s,x,y)v(y)\mathrm{d}\xi(s)(y)\sigma(\mathrm{d}y), \qquad t\ge 0,\ x\in\mathcal{O}.$$

(ii) If τ is the Neumann boundary operator, then

$$u(t,x) = \int_0^t\int_{\partial\mathcal{O}} G_\tau(t-s,x,y)\mathrm{d}\xi(s)(y)\sigma(\mathrm{d}y), \qquad t\ge 0,\ x\in\mathcal{O}.$$

Proof. Let us assume that $u_0 = u_{0,1} = 0$. Then by (0.8.1), the solution to wave problem (0.1.2) is given by

$$u(t,x) = \int_0^t \mathcal{T}_\tau(t-s)\mathrm{d}\xi(s), \qquad t\ge 0,\ x\in\mathcal{O},$$

where

$$\mathcal{T}_\tau(t)v(x) := \int_{\mathcal{O}} G_\tau(t,x,y)(\kappa-\Delta_\tau)D_\tau v(y)\mathrm{d}y, \qquad t\ge 0,\ x\in\mathcal{O}.$$

Note that, first by Remark 0.1, and then by the fact that G_τ satisfies the boundary condition $\tau G_\tau(t,x,y)=0$ with respect to y-variable, we obtain, for $t\ge 0$, $x\in\mathcal{O}$,

$$\begin{aligned}\mathcal{T}_\tau(t)v(x) &:= \int_{\mathcal{O}} G_\tau(t,x,y)(\kappa-\Delta_\tau)D_\tau v(y)\mathrm{d}y\\ &= \int_{\mathcal{O}} (\kappa-\Delta)_y G_\tau(t,x,y)D_\tau v(y)\mathrm{d}y.\end{aligned}$$

The Green formula and the fact that $(\kappa-\Delta)D_\tau v=0$, yield then

$$\mathcal{T}_\tau(t)v(x) = \int_{\partial\mathcal{O}} \left(-\frac{\partial G}{\partial \mathbf{n}_y}(t,x,y)D_\tau v(y) + G(t,x,y)\frac{\partial D_\tau v}{\partial \mathbf{n}_y}(y)\right)\sigma(\mathrm{d}y)$$

for all $t\ge 0$ and $x\in\mathcal{O}$. Hence if τ is the Dirichlet boundary operator, then

$$\mathcal{T}_\tau(t)v(x) = -\int_{\partial\mathcal{O}} \frac{\partial G_\tau}{\partial \mathbf{n}_y}(t,x,y)v(y)\sigma(\mathrm{d}y), \qquad t\ge 0,\ x\in\mathcal{O}.$$

If τ is the Neumann boundary operator, then repeating the previous argument we obtain

$$\mathcal{T}_\tau(t)v(x) = \int_{\partial\mathcal{O}} G_\tau(t,x,y)v(y)\sigma(\mathrm{d}y), \qquad t\ge 0,\ x\in\mathcal{O}.$$

□

0.9. Applications

Assume that the process $\xi(t)(x)$, $t\ge 0$, $x\in\partial\mathcal{O}$, is of one of the following two forms

$$\xi(t)(x) = \sum_k \lambda_k W_k(t)e_k(x), \qquad t\ge 0,\ x\in\mathcal{O}, \tag{0.9.1}$$

or

$$\xi(t)(x) = \sum_k Z_k(t)e_k(x), \qquad t\ge 0,\ x\in\mathcal{O}, \tag{0.9.2}$$

where (λ_k) is a sequence of real numbers, (e_k) is a sequence of measurable functions on $\partial\mathcal{O}$, (W_k) is a sequence of independent real-valued standard Wiener processes, and (Z_k) is a sequence of uncorrelated real-valued pure jump Lévy processes, that is

$$Z_k(t) = a_k t + \int_0^t\int_{\{|z|\le 1\}} z\widehat{\pi}_k(\mathrm{d}s,\mathrm{d}z) + \int_0^t\int_{\{|z|>1\}} \pi(\mathrm{d}s,\mathrm{d}z), \qquad t\ge 0,$$

where π_k are Poisson random measures each with the jump measure ν_k.

In the jump case, let us set

$$\lambda_{k,R} := a_k + \nu_k\{1<|z|\le R\}, \qquad \tilde\lambda_{k,R} := \lambda_{k,R} + \nu_k\{0<|z|\le R\}.$$

Then, for all k and R,

$$Z_k(t) = \lambda_{k,R}t + M_{k,R}(t) + \int_0^t\int_{\{|z|>R\}} \pi(\mathrm{d}s,\mathrm{d}z), \qquad t\ge 0,$$

where

$$M_{k,R}(t) := \int_0^t \int_{\{|z|\le R\}} z\widehat{\pi}_k(\mathrm{d}s,\mathrm{d}z), \qquad t \ge 0.$$

Let

$$\tau_{k,R} := \inf\{t \ge 0\colon |Z_k(t) - Z_k(t-)| \ge R\}.$$

Then $\tau_{k,R} \uparrow +\infty$ as $R \uparrow +\infty$. Moreover,

$$Z_k(t) = \lambda_{k,R} t + M_{k,R}(t) \quad \text{on } \{\tau_{k,R} \ge t\}.$$

Recall, see e.g.[25] Lemma 8.22, or,[26] that $M_{k,R}$ are square integrable martingales and that for any predictable process f and any $T > 0$,

$$\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t f(s)\mathrm{d}M_{k,R}(s) \right| \le C\,\nu_k\{0 < |z| \le R\} \int_0^T \mathbb{E}\,|f(s)|\mathrm{d}s,$$

and

$$\mathbb{E} \sup_{t\in[0,T]} \left| \int_0^t f(s)\mathrm{d}M_{k,R}(s) \right|^2 \le C \int_{\{0<|z|\le R\}} z^2 \nu_k(\mathrm{d}z) \int_0^T \mathbb{E}\,|f(s)|^2\mathrm{d}s,$$

where C is a certain (independent of f, T, R and π_k) universal constant.

Recall that G_τ is the fundamental solution to the Cauchy problem for the wave equation with boundary operator τ. It turns out that the Dirichlet problem has a function valued solution if ξ is absolutely continuous. Therefore we present below the result on the Neumann problem. For specific examples see e.g.[7–9,19]

Theorem 0.4. *Assume that τ is the Neumann boundary operator. (i) If ξ is given by (0.9.1), then the solution u to (0.1.2) is a square integrable random field on $[0,+\infty) \times \mathcal{O}$ if and only if*

$$I(t,x) := \sum_k \lambda_k^2 \int_0^t \left| \int_{\partial\mathcal{O}} G_\tau(s,x,y) e_k(y)\sigma(\mathrm{d}y) \right|^2 \mathrm{d}s < \infty, \quad t \ge 0,\ x \in \mathcal{O}.$$

Moreover,

$$u(t,x) = \sum_k \lambda_k \int_0^t \int_{\partial\mathcal{O}} G_\tau(t-s,x,y) e_k(y)\mathrm{d}\sigma(y)\mathrm{d}W_k(s), \quad t \ge 0,\, x \in \mathcal{O}$$

and $\mathbb{E}\,|u(t,x)|^2 \le I(t,x)$ for all $t \ge 0$ and $x \in \mathcal{O}$.
(ii) If ξ is given by (0.9.2), then for all $R > 0$,

$$J_R(t,x) := \sum_k \tilde{\lambda}_{k,R} \int_0^t \left| \int_{\partial\mathcal{O}} G_\tau(s,x,y) e_k(y)\sigma(\mathrm{d}y) \right| \mathrm{d}s < \infty.$$

and the solution u to (0.1.2) is a random field if $\tau_R \to +\infty$ as $R \to +\infty$. Moreover,

$$u(t,x) = -\sum_k \int_0^t \int_{\partial\mathcal{O}} G_\tau(t-s,x,y) e_k(y)\sigma(\mathrm{d}y)\mathrm{d}Z_k(s),\ t \ge 0,\ x \in \mathcal{O}$$

and $\mathbb{E}\,|u(t,x)|\chi_{\{t\le\tau_R\}} \le C J_R(t,x)$ for all $t > 0$, $x \in \mathcal{O}$, and $R > 0$.

Example 0.2. As an example of the wave equation with stochastic Dirichlet boundary condition, consider $d = 1$ and $\mathcal{O} = (0, +\infty)$. Then the fundamental solution G is given by

$$G(t,x,y) = \frac{1}{2}\left(\chi_{\{|x-y|<t\}} - \chi_{\{|x+y|<t\}}\right), \qquad t \geq 0,\ x, y \geq 0.$$

Then

$$-\frac{\partial G}{\partial \mathbf{n}_y}(t,x,0) = \delta_t(x), \qquad t, x \geq 0,$$

and, by Theorem 0.3, see also Example 0.1(iii), the solution to (0.1.2) with $u_0 = u_{0,1} = 0$, is given by

$$u(t,x) = \int_0^t \delta_{t-s}(x)\mathrm{d}\xi(s), \qquad t, x \geq 0.$$

In general, process $u = \big(u(t)\big)_{t\geq 0}$ takes values in a proper space of distributions. In fact, for a test function φ,

$$(u(t), \varphi) = \int_0^t \varphi(t-s)\mathrm{d}\xi(s), \qquad t \geq 0.$$

For similar results in the case of the transport equation see Examples 0.3 and 0.4.

0.10. The Transport equation

Let us describe an abstract framework in which we will study problem (0.1.3). Namely, in this case we put $\mathcal{H} = L^2(0, 2\pi)$ and $\mathcal{U} = \mathbb{R}$. We will identify $\mathcal{H}$ with the space of all locally square integrable 2π periodic functions $f\colon \mathbb{R} \mapsto \mathbb{R}$. Alternatively, we can identify $\mathcal{H}$ with the space $L^2(S^1)$, where S^1 is the standard unit circle equipped with the Haar measure (multiplied by 2π). In the space $\mathcal{H}$ we consider an operator $\mathcal{A}$ defined by

$$\mathrm{D}(\mathcal{A}) = H^{1,2}_{\mathrm{per}}(0, 2\pi), \qquad \mathcal{A}u = \frac{\mathrm{d}u}{\mathrm{d}x}, \tag{0.10.1}$$

where $H^{1,2}(0, 2\pi)$ is the Sobolev space of functions $u \in L^2(0, 2\pi)$ with the weak derivative $\frac{\mathrm{d}u}{\mathrm{d}x} \in L^2(0, 2\pi)$, and

$$H^{1,2}_{\mathrm{per}}(0, 2\pi) = \{u \in H^{1,2}(0, 2\pi)\colon\ u(0+) = u(2\pi-)\}.$$

It is easy to see that $\mathcal{A}$ generates a C_0-group $(U(t))_{t\in\mathbb{R}}$ in $\mathcal{H}$. In fact, this group is the standard translation group defined by

$$U(t)u(x) = u(t \dot{+} x), \qquad t \in \mathbb{R},\ x \in (0, 2\pi), \tag{0.10.2}$$

where $\dot{+}$ is the addition modulo 2π. Let τ be the boundary operator defined by

$$H^{1,2}(0, 2\pi) \ni u \mapsto \tau u = u(2\pi-) - u(0+) \in \mathbb{R}. \tag{0.10.3}$$

Note that by the Sobolev embedding theorem $H^{1,2}(0, 2\pi) \hookrightarrow C([0, 2\pi])$ and hence $u(2\pi-)$ and $u(0+)$ make sense for each $u \in H^{1,2}(0, 2\pi)$. Finally, we assume that $\xi =$

$\big(\xi(t)\big)_{t\geq 0}$ is an $\mathbb{R}$-valued càdlàg process defined on some complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$.

With all these notation we can now present an abstract form of problem (0.1.3), i.e.

$$\begin{cases} \dfrac{\partial u(t)}{\partial t} = \mathcal{A}u(t) & \text{for } t \in (0,\infty), \\ u(0,\cdot) = u_0 & \text{on } (0,2\pi), \\ \tau\big[u(t)\big] = \dfrac{\mathrm{d}\xi(t)}{\mathrm{d}t} & \text{for } t \in (0,\infty). \end{cases} \tag{0.10.4}$$

The problem needs to be reformulated as, for example, on the one hand there is an expression $\mathcal{A}u(t)$ and on the other hand $\tau\big[u(t)\big]$ may be different from 0, and hence $u(t)$ may not belong to $\mathrm{D}(\mathcal{A})$. In order to give a proper definition of a mild solution to the above problem we will argue heuristically as in Section 0.5. For this we need to introduce a counterpart of the Dirichlet map D_τ from Section 0.4. To this aim let $z_0 \in H^{1,2}(0,2\pi)$ be the unique solution of the following problem

$$\frac{\mathrm{d}}{\mathrm{d}x} z_0(x) = z_0(x),\; x \in (0,2\pi), \qquad z_0(2\pi) - z_0(0) = 1.$$

Thus $z_0(x) = (\mathrm{e}^{2\pi}-1)^{-1}\,\mathrm{e}^x$, $x \in (0,2\pi)$.

Then we define a map $D_\tau \colon \mathbb{R} \ni \alpha \mapsto \alpha z_0 \in H^{1,2}(0,2\pi)$. Note that the map D_τ satisfies the following

$$u \in H^{1,2}(0,2\pi),\; \frac{\mathrm{d}u}{\mathrm{d}x} = u \quad \text{on } \; (0,2\pi) \; \text{ and } \; \tau(u) = \alpha \iff D_\tau(\alpha) = u.$$

In other words, $\tau \circ D_\tau$ is the identity operator on $\mathbb{R}$.

Let u be a solution to (0.10.1). As in Section 0.5 we consider a new process y defined by the following formula

$$y(t) := u(t) - D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(t), \qquad t > 0. \tag{0.10.5}$$

Clearly $\tau y(t) = 0$ for $t > 0$ and $y(0) = u_0 - D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(0)$. Next we have

$$\begin{aligned} \frac{\partial y}{\partial t} &= \frac{\partial u}{\partial t} - \frac{\partial}{\partial t}\left(D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(t)\right) \\ &= \mathcal{A}\left(y(t) + D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(t)\right) - \frac{\partial}{\partial t}\left(D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(t)\right) \\ &= \mathcal{A}y(t) + \left[D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(t) - \frac{\partial}{\partial t}\left(D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(t)\right)\right]. \end{aligned}$$

Continuing as in Section 0.5 we obtain

$$y(t) = U(t)y(0) + \int_0^t U(t-s)\left[D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}r}(r) - \frac{\partial}{\partial r}\Big(D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}r}(r)\Big)\right]\mathrm{d}r.$$

On the other hand by integration by parts formula

$$-\int_0^t U(t-r)\frac{\partial}{\partial r}\left(D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}r}(r)\right)\mathrm{d}r = -D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(t) \\ +U(t)D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(0) - \int_0^t \mathcal{A}U(t-r)D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}r}(r)\mathrm{d}r.$$

Hence, we infer that

$$y(t) = U(t)\left(y(0) + D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(0)\right) - D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}t}(t) \\ + \int_0^t (I-\mathcal{A})\, U(t-r)\left[D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}r}(r)\right]\mathrm{d}r, \quad t \geq 0.$$

Therefore, in view of (0.10.5) we arrive at the following heuristic formula

$$u(t) = U(t)u_0 + \int_0^t (I-\mathcal{A})\, U(t-r)\left[D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}r}(r)\right]\mathrm{d}r, \quad t \geq 0.$$

Since $(\mathcal{A}^*)'$ is an extension of $\mathcal{A}$, to obtain a meaningful version of the above equation one should write

$$u(t) = U(t)u_0 + \int_0^t (I-(\mathcal{A}^*)')\, U(t-r)\left[D_\tau \frac{\mathrm{d}\xi}{\mathrm{d}r}(r)\right]\mathrm{d}r, \quad t \geq 0, \tag{0.10.6}$$

where U is the extension of the original group to $(\mathrm{D}(\mathcal{A}^*))'$.

Summing up, we have shown (for more details see[4]) that the transport problem can be written in the abstract form (0.1.1) with $E = D_\tau$ and $\lambda = 1$. It is known (see[4]) that the condition (0.2.1) is satisfied.

It follows from[4] Proposition 1.1, p. 459, that the above integral defines a function belonging to $C([0,\infty);\mathcal{H})$ provided $\frac{\mathrm{d}\xi}{\mathrm{d}r} \in L^2_{\mathrm{loc}}(0,\infty)$. However, we are interested in cases when this condition is no longer satisfied. Hence, if we perform integration by parts in the integral in (0.10.6) we obtain the following Anzatz for the solution to problem (0.10.1):

$$u(t) = U(t)u_0 + (I-(\mathcal{A}^*)')\left[D_\tau\xi(t) - U(t)D_\tau\xi(0)\right] \\ + \int_0^t (I-\mathcal{A}^*)')\,\mathcal{A}U(t-r)\left[D_\tau\xi(r)\right]\mathrm{d}r, \qquad t \geq 0. \tag{0.10.7}$$

Let us observe that formula (0.10.7) is a counterpart of formula (0.3.5) from Section 0.3. In some sense, this could be seen as a stochastic counterpart of a deterministic result from,[4] see Proposition 1.1 on p. 459.

As far as the problem (0.10.1) is concerned, it remains to identify the space $(\mathrm{D}(\mathcal{A}^*))'$ with an appropriate space of distributions. We have the following.

Proposition 0.3. *The space* $(\mathrm{D}(\mathcal{A}^*))'$ *is equal to* $H^{-1,2}(0,2\pi)$ *and the operator* $(\mathcal{A}^*)'$ *is equal to the weak derivative. In particular,*

$$((\mathcal{A}^*)'u,\varphi) = -\left(u, \frac{\mathrm{d}\varphi}{\mathrm{d}x}\right), \qquad u \in \mathcal{H},\ \varphi \in H^{1,2}(0,2\pi).$$

Recall that $D_\tau(\alpha) = \alpha z_0$, where $z_0(x) = \left(e^{2\pi} - 1\right)^{-1} e^x$, $x \in (0, 2\pi)$. Thus, by the proposition above, for any $\varphi \in H^{1,2}(0, 2\pi)$,

$$
\begin{aligned}
((\mathcal{A}^*)' z_0, \varphi) &= -\int_0^{2\pi} z_0(x) \frac{d\varphi}{dx}(x) dx \\
&= -z_0(2\pi)\varphi(2\pi) + z_0(0)\varphi(0) + \int_0^{2\pi} z_0(x)\varphi(x) dx,
\end{aligned}
$$

and hence

$$
(I - (\mathcal{A}^*)')\, D_\tau[\alpha] = \left(\frac{e^{2\pi}}{e^{2\pi} - 1} \delta_{2\pi} - \frac{1}{e^{2\pi} - 1} \delta_0 \right) \alpha. \qquad (0.10.8)
$$

Next not that, for any test function $\varphi \in H^{1,2}(0, 2\pi)$,

$$
\begin{aligned}
\left(\int_0^t U(t-s)\delta_{2\pi} d\xi(s), \varphi \right) &= \int_0^t \left(\delta_{2\pi}, \varphi(t - s \dot{+} \cdot) \right) d\xi(s) \\
&= \int_0^t \varphi(t - s \dot{+} 2\pi) d\xi = \int_0^t \varphi(t - s \dot{+} 0) d\xi \\
&= \left(\int_0^t U(t-s)\delta_0 d\xi(s), \varphi \right).
\end{aligned}
$$

Therefore, by (0.10.8),

$$
\left(\int_0^t U(t-s) \left(I - (\mathcal{A}^*)'\right) D_\tau d\xi(s), \varphi \right) = \int_0^t \varphi(t - s \dot{+} 0) d\xi(s),
$$

and in other words, we have the following result.

Proposition 0.4. *The solution u to problem* (0.1.3) *is an $H^{-1,2}(0, 2\pi)$-valued process such that for any test function $\varphi \in H^{1,2}(0, 2\pi)$,*

$$
(u(t), \varphi) = (u_0, \varphi) + \int_0^t \varphi(t - s \dot{+} 0) d\xi(s), \qquad t \geq 0.
$$

Example 0.3. Assume that ξ is a compound Poisson process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the jump measure ν; that is

$$
\xi(t) = \sum_{k=1}^{\Pi(t)} X_k, \qquad t \geq 0,
$$

where Π is Poisson process with intensity $\nu(\mathbb{R})$ and X_k are independent random variable with the distribution $\nu/\nu(\mathbb{R})$. Let τ_k be the moments of jumps of Π. Then

$$
\int_0^t \varphi(t - s \dot{+} 0) d\xi(s) = \sum_{\tau_k \leq t} \varphi(t - \tau_k \dot{+} 0) X_k.
$$

Since each τ_k has a absolutely continuous distribution (exponential), the formula can be extended to any $\varphi \in L^2(0, 2\pi)$, and hence the solution is a cylindrical process in $L^2(0, 2\pi)$; that is for each $t > 0$, $u(t)$ is a bounded linear operator from $L^2(0, 2\pi)$ to $L^2(\Omega, \mathcal{F}, \mathbb{P})$.

Example 0.4. Let ξ be a Wiener process. Then again u is a distribution valued process and a cylindrical Gaussian random process in $L^2(0, 2\pi)$.

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