ON SOME SMOOTHENING EFFECTS OF THE TRANSITION SEMIGROUP OF A LÉVY PROCESS

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Abstract. Let (P_t) be the transition semigroup of a Lévy process (L_t) taking values in a Hilbert space H. Let ν and $\tilde{N}$ respectively be the Lévy measure and compensated Poisson random measure of (L_t). It is shown that for any bounded and measurable function f,

$$A_q P_t f(x) = \frac{1}{t}\mathbb{E}\left[f(L_t^x)\int_0^t\int_H q(y)\tilde{N}(\mathrm{d}s,\mathrm{d}y)\right] \qquad \text{for all } t>0,\, x\in H.$$

where A_q is some non-local operator. As a corollary,

$$\int_H |P_t f(x+y) - P_t f(x)|^2\,\nu(\mathrm{d}y) \leq \frac{1}{t}P_t f^2(x) \qquad \text{for all } t>0,\, x\in H.$$

As ν can be infinite this formula establishes some smoothening effect of the semigroup (P_t). In the paper some applications of the formula will be presented as well.
Key words: Bismut–Elworthy–Li formula, Lévy processes, Smoothening effect.

1. Introduction

Let (X_t) be the unique solution to an SDE on a Hilbert space H driven by a non-degenerate Wiener process W. Let

$$P_t f(x) = \mathbb{E}\left(f(X_t^x)\right), \qquad f\in B_b(H),\ t\geq 0,$$

be the corresponding transition semigroup defined on the space of bounded measurable functions $B_b(H)$. Then the following Bismut–Elworthy–Li formula holds (see [7] or [12])

$$\nabla_v P_t f(x) = \frac{1}{t}\mathbb{E}\left(f(X_t^x)\int_0^t K(s;v)\mathrm{d}W_s\right), \tag{1.1}$$

where $K(s,v)$ is an adapted stochastic processes independent of f. This formula implies the strong Feller property of (P_t), and therefore is very useful for studying its ergodic properties. For other applications we refer to e.g. [18, 2, 8].

In this paper, we derive the following, similar to (1.1), derivative formula for a family of Lévy processes $L_t^x = x + L_t$, $t\geq 0$, $x\in H$, taking values in a Hilbert space H;

$$A_q P_t f(x) = \frac{1}{t}\mathbb{E}\left[f(L_t^x)\int_0^t\int_H q(y)\tilde{N}(\mathrm{d}s,\mathrm{d}y)\right]. \tag{1.2}$$

Above A_q is a certain *non-local* operator A_q corresponding to a function q playing a similar role as v in (1.1). Thus the associated transition semigroup (P_t) transforms

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$B_b(H)$ into the intersection of domains of some non-local operators (see Theorem 3.1 for more details). Next, see Corollary 3.2 below, we show that

$$\int_H |P_tf(x+y) - P_tf(x)|^2 \nu(\mathrm{d}y) \le \frac{1}{t}P_tf^2(x) \quad \text{for } f \in B_b(H),\, t>0,\, x\in H. \tag{1.3}$$

Note that if the Lévy measure ν of (L_t) is infinite, then for any open ball $B_\varepsilon(0)$ with the center at 0 and radius $\varepsilon > 0$ one has $\nu(B_\varepsilon(0)) = +\infty$. Therefore (1.3) means that for any $x \in H$, $P_tf(x+y)$, $y \in B_\varepsilon(0)$, is in a certain sense close to $P_tf(x)$.

By choosing a special q in (1.2) we obtain a 'fractional gradient' estimate for some symmetric Lévy processes. From this estimate, we further get the 'fractional gradient' estimate for the perturbed stable type stochastic systems. By Corollary 3.2 and a generalized Campanato's theorem, we calculate modulus of continuity of the transition semigroups of 'log-stable' processes.

The paper is organized as follows: the next section includes some preliminary facts on Lévy processes. The main general results are formulated in Section 3. The last three sections are devoted to applications. In the appendix we prove the generalized Campanato theorem of harmonic analysis.

2. Preliminary facts on Lévy processes

Recall some preliminary facts on Lévy processes (for details see e.g. [1, 13, 14]). Let $(L_t)_{t\ge0}$ be an H-valued Lévy process. It is well known that there is a vector $m \in H$, a symmetric positive definite trace class operator $Q\colon H \mapsto H$, and a Borel measure ν on H satisfying

$$\nu(\{0\}) = 0, \qquad \int_H 1 \wedge |y|_H^2 \nu(\mathrm{d}y) < +\infty, \tag{2.1}$$

such that

$$\mathbb{E}\, \mathrm{e}^{\mathrm{i}\langle x, L_t\rangle_H} = \mathrm{e}^{-t\psi(x)}, \qquad x \in H,$$

where the so-called *Lévy exponent* ψ of (L_t) is given by the following Lévy–Khinchin representation:

$$\psi(x) = i\langle x, m\rangle_H + \frac{1}{2}\langle Qx, x\rangle_H + \int_H \left[\mathrm{e}^{\mathrm{i}\langle x,y\rangle_H} - 1 - \mathrm{i}\langle x, y\rangle_H \mathbf{1}_{\{|y|_H\le 1\}}\right] \nu(\mathrm{d}y).$$

We call ν the *Lévy measure* of L and (m, Q, ν) the *generating triplet* of L.

The *Poisson random measure* associated with (L_t) is defined by

$$N(t,\Gamma) := \sum_{s\in(0,t]} \mathbf{1}_\Gamma(L_s - L_{s-}), \qquad \Gamma \in \mathcal{B}(H),\ t>0,$$

and the *compensated Poisson random measure* is given by

$$\tilde{N}(t,\Gamma) = N(t,\Gamma) - t\nu(\Gamma).$$

By the *Lévy–Itô decomposition* (cf. [1, p.108, Theorem 2.4.16] or [13, p. 53, Theorem 4.23]), one has

$$L_t = mt + W_Q(t) + \int_{\{0<|x|_H\le 1\}} x\tilde{N}(\mathrm{d}t, \mathrm{d}x) + \int_{\{|x|_H>1\}} xN(\mathrm{d}t,\mathrm{d}x), \qquad t \ge 0,$$

where W_Q is a Wiener process in H with covariance operator Q.

Let $(\mathfrak{F}_t)$ be the filtration generated by (L_t), and let us denote by $\mathcal{L}^2_{\mathrm{loc}}$ the space of all predictable stochastic process ψ satisfying

$$\mathbb{E}\int_0^t\int_H |\psi(s,y)|^2_H \nu(\mathrm{d}y)\mathrm{d}s < \infty \qquad \text{for } t>0.$$

Then for any $\psi\in\mathcal{L}^2_{\mathrm{loc}}$ the stochastic integral $\int_0^t\int_H \psi(s,y)\tilde{N}(\mathrm{d}s,\mathrm{d}y)$ is a well-defined square integrable and mean zero martingale. Moreover, the following Itô isometry holds (see e.g. [1, p. 200] or [13, Section 8.7])

$$\begin{aligned}\mathbb{E}&\left[\int_0^t\int_H \psi(s,y)\tilde{N}(\mathrm{d}s,\mathrm{d}y)\int_0^t\int_H \varphi(s,y)\tilde{N}(\mathrm{d}s,\mathrm{d}y)\right]\\ &=\mathbb{E}\int_0^t\int_H \psi(s,y)\varphi(s,y)\nu(\mathrm{d}y)\mathrm{d}s, \qquad \psi,\varphi\in\mathcal{L}^2_{\mathrm{loc}}.\end{aligned} \tag{2.2}$$

Let $L=(L_t)$ be a Lévy process with a generating triplet (m,Q,ν). Consider the Markov family

$$L^x_t = x+L_t, \qquad t\geq 0,\ x\in H. \tag{2.3}$$

Its transition semigroup (P_t) is given as follows

$$P_tf(x)=\mathbb{E}\,f(L^x_t), \qquad t\geq 0,\ x\in H,\ f\in B_b(H). \tag{2.4}$$

Observe that (P_t) is strongly continuous on the space $UC_b(H)$ of uniformly continuous bounded functions on H equipped with the supremum norm $\|\cdot\|_\infty$, i.e., P_0 is the identity operator, $P_{t+s}=P_tP_s$ for all $t,s\geq 0$, and $\lim_{t\downarrow 0}\|P_tf-f\|_\infty=0$ for all $f\in UC_b(H)$ (see [13, p. 80] for more details). Moreover, the domain of its generator $\mathcal{L}$ contains the space $UC^2_b(H)$, and

$$\begin{aligned}\mathcal{L}f(x)&=\langle Df(x),m\rangle_H+\frac{1}{2}\mathrm{Trace}\,QD^2f(x)\\ &+\int_H\big(f(x+y)-f(x)-\mathbf{1}_{\{|x|_H<1\}}\langle Df(x),y\rangle_H\big)\,\nu(\mathrm{d}y), \qquad f\in UC^2_b(H),\ t\geq 0,\ x\in H.\end{aligned}$$

3. Main results

Let L be a Lévy process on H with the generating triplet (m,Q,ν). Let (L^x_t), $t\geq 0$, $x\in H$, be the Markov family given by (2.3), and let (P_t) given by (2.4) be the transition semigroup of (L^x_t).

Given $q\in L^2(H,\mathcal{B}(H),\nu)$, define

$$\mathcal{D}_q:=\left\{f\in B_b(H)\colon \sup_{x\in H}\int_H |f(x+y)-f(x)|\,|q(y)|\nu(\mathrm{d}y)<\infty\right\}.$$

Next, let

$$A_qf(x):=\int_H [f(x+y)-f(x)]\,q(y)\nu(\mathrm{d}y) \qquad \text{for } f\in\mathcal{D}_q,\ x\in H. \tag{3.1}$$

Taking into account (2.1), we see that $C_b^1(H) \subset \mathcal{D}_q$, A_q is a bounded linear operator form $C_b^1(H)$ into $B_b(H)$ and

$$
\begin{aligned}
\|A_q f\|_\infty &:= \sup_{x\in H} |A_q f(x)| \\
&\le 2\|f\|_\infty \left(\int_{|y|_H\ge 1} \nu(\mathrm{d}y)\right)^{\frac{1}{2}} \left(\int_{|y|_H\ge 1} q^2(y)\nu(\mathrm{d}y)\right)^{\frac{1}{2}} \\
&\quad + \|Df\|_\infty \left(\int_{\{|y|_H<1\}} |y|_H^2 \nu(\mathrm{d}y)\right)^{\frac{1}{2}} \left(\int_{\{|y|_H<1\}} q^2(y)\nu(\mathrm{d}y)\right)^{\frac{1}{2}} < \infty.
\end{aligned}
$$

Theorem 3.1. *Let $q \in L^2(H, \mathcal{B}(H), \nu)$. Then for all $t > 0$ and $f \in B_b(H)$, $P_t f \in \mathcal{D}_q$ and*

$$
A_q P_t f(x) = \frac{1}{t}\mathbb{E}\left[f(L_t^x) \int_0^t \int_H q(y)\tilde{N}(\mathrm{d}s, \mathrm{d}y)\right].
$$

Moreover,

$$
|A_q P_t f(x)|^2 \le \frac{1}{t} P_t f^2(x) \int_H q^2(y)\nu(\mathrm{d}y) \qquad \textit{for all } x \in H.
$$

Proof. We follow the spirit of the proof of Theorem 2.1 in [7]. Assume $f \in UC_b^2(H)$ and consider $P_{t-s}f(L_s^x)$ with $0 \le s \le t$. Applying Kolmogorov's backward equation to $P_{t-s}f$ and Itô formula to L_s^x (see e.g. [1]), we obtain

$$
\begin{aligned}
f(L_t^x) - P_t f(x) &= -\int_0^t \mathcal{L}P_{t-s}f(L_s^x)\mathrm{d}s + \int_0^t \mathcal{L}P_{t-s}f(L_s^x)\mathrm{d}s \\
&\quad + \int_0^t \int_H \left[P_{t-s}f(L_s^x + y) - P_{t-s}f(L_s^x))\right] \tilde{N}(\mathrm{d}y, \mathrm{d}s) \\
&\quad + \int_0^t \langle DP_{t-s}f(L_s^x), \mathrm{d}W_Q(s)\rangle_H \\
&= \int_0^t \int_H \left[P_{t-s}f(L_s^x + y) - P_{t-s}f(L_s^x)\right] \tilde{N}(\mathrm{d}y, \mathrm{d}s) \\
&\quad + \int_0^t \langle DP_{t-s}f(L_s^x), \mathrm{d}W_Q(s)\rangle_H.
\end{aligned} \tag{3.2}
$$

Multiplying the both sides of (3.2) by

$$
\int_0^t \int_H q(y)\tilde{N}(\mathrm{d}y, \mathrm{d}s)
$$

and taking into account (2.2), we further get

$$
\begin{aligned}
\mathbb{E}\left[f(L_t^x) \int_0^t \int_H q(y)\tilde{N}(\mathrm{d}y, \mathrm{d}s)\right] &= \mathbb{E}\int_0^t \int_H \left[P_{t-s}f(L_s^x + y) - P_{t-s}f(L_s^x)\right] q(y)\nu(\mathrm{d}y)\mathrm{d}s \\
&= \int_0^t \int_H \left[P_s P_{t-s}f(x + y) - P_s P_{t-s}f(x)\right] q(y)\nu(\mathrm{d}y)\mathrm{d}s \\
&= t\int_H \left[P_t f(x + y) - P_t f(x)\right] q(y)\nu(\mathrm{d}y) = tA_q P_t f(x).
\end{aligned}
$$

This clearly implies the first formula in the theorem.

From the previous formula, by the Hölder inequality and Itô isometry we obtain

$$
\begin{aligned}
t\,|A_qP_tf(x)| &\le \left(\mathbb{E}\, f^2(L_t^x)\right)^{1/2}\left(\int_0^t\int_H q^2(y)\nu(\mathrm{d}y)\mathrm{d}s\right)^{1/2}\\
&\le \left(P_tf^2(x)\right)^{1/2} t^{1/2}\left(\int_H q^2(y)\nu(\mathrm{d}y)\right)^{1/2}.
\end{aligned}
$$

Thus the desired estimate holds for any $f\in UC_b^2(H)$. Assume that $f\in B_b(H)$. Let $x\in H$. Then there is a sequence $(f_n)\subset UC_b^2(H)$ such that

$$
\lim_{n\to\infty} P_tf_n^2(x) = P_tf^2(x),
$$

and

$$
\lim_{n\to\infty} P_tf_n(x+y) = P_tf(x+y) \quad \text{for } \nu \text{ almost all } y.
$$

Consequently, the desired estimate for f follows from the Fatou lemma. □

Corollary 3.2. *For arbitrary $f\in B_b(H)$ we have*

$$
\int_H |P_tf(x+y)-P_tf(x)|^2\,\nu(\mathrm{d}y) \le \frac{1}{t}P_tf^2(x), \qquad x\in H,\ t>0.
$$

Proof. By (3.1), we have

$$
\begin{aligned}
&\int_H |P_tf(x+y)-P_tf(x)|^2\,\nu(\mathrm{d}y)\\
&= \sup\left\{\left|\int_H (P_tf(x+y)-P_tf(x))\,q(y)\nu(\mathrm{d}y)\right|^2,\quad q\colon \int_H q^2(y)\nu(\mathrm{d}u)\le 1\right\} \qquad (3.3)\\
&= \sup\left\{|A_qP_tf(x)|^2,\quad q\colon \int_H q^2(y)\nu(\mathrm{d}u)\le 1\right\},
\end{aligned}
$$

and the estimate follows from Theorem 3.1. □

Given $f\in B_b(H)$ we define the difference operator $\nabla^n_{y_1,\ldots,y_n}f(x)$, $x,y_1,\ldots,y_n\in H$ putting

$$
\begin{aligned}
\nabla_yf(x) &= f(x+y)-f(x),\\
\nabla^{n+1}_{y_1,\ldots,y_{n+1}}f(x) &= \nabla_{y_{n+1}}\left(\nabla^n_{y_1,\ldots,y_n}f\right)(x).
\end{aligned}
$$

Corollary 3.3. *For any $f\in B_b(H)$ and $n\in\mathbb{N}$,*

$$
\sup_{x\in H}\int_H\ldots\int_H \left|\nabla_{y_1,\ldots,y_n}\left(P_tf\right)(x)\right|^2\nu(\mathrm{d}y_1)\ldots\nu(\mathrm{d}y_n) \le \left(\frac{n}{t}\right)^n\|f\|_\infty^2.
$$

Proof. It is enough to show the estimate for $f\in UC_b^2(H)$. Let $q_1,\ldots,q_n\in L^2(H,\mathcal{B}(H),\nu)$. We claim that the operators A_q and P_s commute. Indeed, by Fubini theorem and the fact $L_t^x+y=L_t^{x+y}$, for all $f\in UC_b^2(\mathbb{R}^d)$ we have

$$
\begin{aligned}
P_tA_qf(x) = \mathbb{E}\,A_qf(L_t^x) &= \mathbb{E}\int_H \left(f(L_t^x+y)-f(L_t^x)\right)q(y)\nu(\mathrm{d}y)\\
&= \int_H \left[\mathbb{E}f(L_t^{x+y})-\mathbb{E}f(L_t^x)\right]q(y)\nu(\mathrm{d}y)\\
&= \int_H \left[P_tf(x+y)-P_tf(x)\right]q(y)\nu(\mathrm{d}y)\\
&= A_qP_tf(x).
\end{aligned}
$$

Thus, by Theorem 3.1 and commutative property between A_q and P_t,

$$\begin{aligned}\|A_{q_1}\dots A_{q_n}P_tf\|_\infty^2 &= \sup_{x\in H}|A_{q_1}\dots A_{q_n}P_tf(x)|^2\\ &= \sup_{x\in H}\left|(A_{q_1}P_{t/n})\dots(A_{q_n}P_{t/n})f(x)\right|^2\\ &\le \frac{n}{t}\|(A_{q_2}P_{t/n})\dots(A_{q_n}P_{t/n})f\|_\infty^2\|q_1\|^2_{L^2(H,\mathcal{B}(H),\nu)}\\ &\le \left(\frac{n}{t}\right)^n\|f\|_\infty^2\prod_{k=1}^n\|q_k\|^2_{L^2(H,\mathcal{B}(H),\nu)}.\end{aligned}$$

Since

$$\begin{aligned}&\int_H\dots\int_H|\nabla_{y_1,\dots,y_n}(P_tf)(x)|^2\nu(\mathrm{d}y_1)\dots\nu(\mathrm{d}y_n)\\ &= \sup\left\{|A_{q_n}\dots A_{q_1}P_tf(x)|^2 : q_i : \int_H q_i^2(y)\nu(dy)\le 1\right\}\end{aligned}\tag{3.4}$$

the desired inequality holds. □

4. Application 1: Short time behaviour of the semigroup

We investigate the fractional gradient estimate of α-stable and truncated α-stable processes. In this and next sections $H=\mathbb{R}^d$. The norm on $\mathbb{R}^d$ will be denoted by $|\cdot|$.

Theorem 4.1. *Let the Lévy measure ν be of the form*

$$\nu(\mathrm{d}x)=\frac{1}{|x|^{d+\alpha}}\mathbf{1}_{\{|x|<K\}}\mathrm{d}x,$$

where $\alpha\in(0,2]$ and $K\in(0,\infty]$. Then for any $\beta\in(\alpha/2,\alpha)$ we have

$$\|(-\Delta)^{\frac{\alpha-\beta}{2}}P_tf\|_\infty\le C(1+t^{-1/2})\|f\|_\infty,\qquad \forall\, t>0,\ f\in B_b(\mathbb{R}^d),\tag{4.1}$$

where $C=C_{\alpha,\beta}$ only depends on α and β.

Proof. Without any loss of generality, we may assume that $K=1$. Choose q such that

$$q(y)=|y|^\beta\qquad \forall\,|y|\le 1,\qquad \int_{\{|y|>1\}}q^2(y)\nu(\mathrm{d}y)<\infty.$$

It is easy to see that

$$\int_{\{|y|\le1\}}q^2(y)\nu(\mathrm{d}y)=\int_{\{|y|\le1\}}|y|^{-d-\alpha+2\beta}\mathrm{d}y\le C_{\alpha,\beta}.\tag{4.2}$$

Therefore, $q\in L^2(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),\nu)$.

Observe that

$$A_qf(x)=\int_{\{|y|\le1\}}\frac{f(x+y)-f(x)}{|y|^{\alpha+d-\beta}}\mathrm{d}y+\int_{\{|y|>1\}}[f(x+y)-f(x)]\,q(y)\nu(\mathrm{d}y)$$

and

$$\begin{aligned}\left|(-\Delta)^{\frac{\alpha-\beta}{2}}f(x)\right| &= \left|\int_{\mathbb{R}^d}\frac{f(x+y)-f(x)}{|y|^{\alpha+d-\beta}}\mathrm{d}y\right|\\ &\le \left|\int_{\{|y|>1\}}\frac{f(x+y)-f(x)}{|y|^{\alpha+d-\beta}}\mathrm{d}y\right|+\left|\int_{\{|y|\le1\}}\frac{f(x+y)-f(x)}{|y|^{\alpha+d-\beta}}\mathrm{d}y\right|.\end{aligned}$$

It is easy to see that

$$\left|\int_{\{|y|>1\}} \frac{f(x+y)-f(x)}{|y|^{\alpha+d-\beta}}\mathrm{d}y\right| \leq c_{\alpha,\beta}\|f\|_\infty,$$

and that

$$\begin{aligned}\left|\int_{\{|y|\leq 1\}} \frac{P_tf(x+y)-P_tf(x)}{|y|^{\alpha+d-\beta}}\mathrm{d}y\right| &\leq |A_qP_tf(x)| + \left|\int_{\{|y|>1\}} [P_tf(x+y)-P_tf(x)]\,q(y)\nu(\mathrm{d}y)\right| \\ &\leq t^{-1/2}\|f\|_\infty\|q\|_{L^2(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),\nu)} + C_{\alpha,d}\|f\|_\infty\|q\|_{L^2(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d),\nu)},\end{aligned}$$

where the last inequality follows from Theorem 3.1 and Hölder's inequality. Collecting the previous inequalities, we get the desired one. □

5. Application 2: Estimate for a perturbed dynamics

Let X_t^x be the value at t of the solution to the following stochastic differential equation

$$\mathrm{d}X_t = b(X_{t-})\mathrm{d}Y_t + \mathrm{d}Z_t, \qquad X_0 = x, \tag{5.1}$$

where $b \in C_b^2(\mathbb{R}^d)$ and Z_t, Y_t are both symmetric stable processes with the parameters $\alpha, \beta \in (0,2)$. Our result below is also true if b is a bounded measurable function, to avoid the complicated differentiability issue and stress the idea, we assume $b \in UC_b^2(\mathbb{R}^d)$.

Eq. (5.1) in more general setting has been intensively studied (see e.g. [6, 11, 4, 5] and the references therein) and has a unique weak solution. Let

$$P_tf(x) = \mathbb{E}f(X_t^x), \qquad \forall\, f \in B_b(\mathbb{R}^d),$$

be the corresponding transition semigroup. The semigroup is C_0 on the space $UC_b(\mathbb{R}^d)$. Let $\mathcal{L}$ be the generator of (P_t) considered on $UC_b(\mathbb{R}^d)$. It is well known that $UC_b^2(\mathbb{R}^d) \subset \mathrm{Dom}\,(\mathcal{L})$, and that for all $f \in UC_b^2(\mathbb{R}^d)$,

$$\mathcal{L}f = -(-\Delta)^{\alpha/2}f - |b(x)|^\beta(-\Delta)^{\beta/2}f,$$

and the following backward Kolmogorov equation holds

$$\partial_tP_tf = \mathcal{L}P_tf. \tag{5.2}$$

We use (4.1) to show some properties of the associated backward Kolmogorov equation.

Theorem 5.1. *If $\beta \in (0, \alpha/2)$, then there exists $t_0 \in (0,1)$ depending on α, β and $\|b\|_\infty$, such that for any $f \in UC_b(\mathbb{R}^d)$,*

$$\begin{aligned} &\|(-\Delta)^{\beta/2}P_tf\|_\infty \leq C_1t^{-1/2}\|f\|_\infty, \qquad t \leq t_0, \\ &\|(-\Delta)^{\beta/2}P_tf\|_\infty \leq C_2\|f\|_\infty, \qquad t > t_0,\end{aligned} \tag{5.3}$$

where C_1, C_2 depend on α, β and t_0.

Proof. Since $UC_b^2(\mathbb{R}^d)$ is dense in $UC_b(\mathbb{R}^d)$ and $(-\Delta)^{\beta/2}$ is closable, it suffices to show (5.3) for $f \in UC_b^2(\mathbb{R}^d)$.

For any $f \in UC_b^2(\mathbb{R}^d)$, define $P_t^0f(x) = \mathbb{E}[f(Z_t+x)]$. It satisfies

$$\partial_tP_t^0f = -(-\Delta)^{\alpha/2}P_t^0f. \tag{5.4}$$

$(P_t^0)_{t\geq 0}$ can be extended to a Markov strongly continuous semigroup on $UC_b(\mathbb{R}^d)$. Due to (5.2) and (5.4), using the classical Duhamel principle we obtain

$$P_tf(x) = P_t^0f(x) - \int_0^t P_{t-s}^0[|b|^\beta(-\Delta)^{\beta/2}P_sf](x)\mathrm{d}s. \tag{5.5}$$

Since $b \in UC_b^2(\mathbb{R}^d)$ and $f \in UC_b^2(\mathbb{R}^d)$, $P_t f \in UC_b^2(\mathbb{R}^d)$ and $P_t^0 f \in UC_b^2(\mathbb{R}^d)$ both hold. By (4.1), we have

$$\|(-\Delta)^{\beta/2} P_t^0 f\|_\infty \le C(1+t^{-1/2})\|f\|_\infty \le 2Ct^{-1/2}\|f\|_\infty, \qquad \forall\, t<1, \tag{5.6}$$

which, together with (5.5), yields

$$\begin{aligned}\|(-\Delta)^{\beta/2} P_t f\|_\infty &\le Ct^{-1/2}\|f\|_\infty + C\int_0^t (t-s)^{-1/2}\|b\|_\infty^\beta \|(-\Delta)^{\beta/2}P_s f\|_\infty \mathrm{d}s\\ &\le Ct^{-1/2}\|f\|_\infty + C\int_0^t (t-s)^{-1/2}\|b\|_\infty^\beta \|(-\Delta)^{\beta/2}P_s f\|_\infty \mathrm{d}s\\ &= Ct^{-1/2}\|f\|_\infty + C\int_0^t (t-s)^{-1/2}s^{-1/2}\|b\|_\infty^\beta s^{1/2}\|(-\Delta)^{\beta/2}P_s f\|_\infty \mathrm{d}s.\end{aligned}$$

Define

$$L_T := \sup_{0\le t\le T} t^{1/2}\|(-\Delta)^{\beta/2}P_t f\|_\infty$$

with $T>0$ to be chosen later. From the previous inequality we have

$$\begin{aligned}L_T &\le C\|f\|_\infty + CT^{\frac12}\sup_{0\le t\le T}\int_0^t s^{-\frac12}(t-s)^{-\frac12}\mathrm{d}s\|b\|_\infty^\beta L_T\\ &\le C\|f\|_\infty + CB(3/2,3/2)T^{\frac12}\|b\|_\infty^\beta\, L_T,\end{aligned} \tag{5.7}$$

where B is the beta function. Choosing $t_0\in(0,1)$ (depending on α, β, and $\|b\|_\infty^\beta$) such that $CB(3/2,3/2)t_0^{\frac12}\|b\|_\infty^\beta < \frac12$, we obtain

$$L_{t_0} \le 2C\|f\|_\infty.$$

This immediately implies the first estimate in the theorem.

For the second estimate, taking $P_{t_0/2}f$ rather than f as the initial data, by the same procedure as above we have

$$\begin{aligned}\|(-\Delta)^{\beta/2}P_t f\|_\infty &\le C_1\left(t-\frac{t_0}{2}\right)^{-\frac12}\|P_{t_0/2}f\|_\infty\\ &\le C_1\left(t-\frac{t_0}{2}\right)^{-\frac12}\|f\|_\infty, \qquad \forall\, t\in(t_0/2, 3t_0/2).\end{aligned}$$

Therefore

$$\|(-\Delta)^{\beta/2}P_t f\|_\infty \le C_1\left(\frac{t_0}{2}\right)^{-\frac12}\|f\|_\infty, \qquad \forall\, t\in(t_0, 3t_0/2).$$

Now taking $P_{t_0}f$ as the initial data, we obtain

$$\|(-\Delta)^{\beta/2}P_t f\|_\infty \le C_1\left(\frac{t_0}{2}\right)^{-\frac12}\|f\|_\infty, \qquad \forall\, t\in(3t_0/2, 2t_0).$$

Iterating the above argument, we finally get

$$\|(-\Delta)^{\beta/2}P_t f\|_\infty \le C_1\left(\frac{t_0}{2}\right)^{-\frac12}\|f\|_\infty, \qquad \forall\, t\ge t_0,$$

which is the desired second estimate. □

6. Application 3: modulus of continuity of transition semigroup of a family of Lévy processes

When the Lévy measure ν on $\mathbb{R}^d$ satisfies

$$\int_{\{|x|<r\}} \nu(\mathrm{d}x) = \infty, \qquad \forall\, r > 0, \tag{6.1}$$

the process $(L_t)_{t\geq 0}$ has infinitely many small jumps in any time interval $[t, t+\delta)$. Then (see e.g. [14, Theorem 27.4]), the law $\mathfrak{L}(L_t)$ of L_t is continuous but not necessarily absolutely continuous with respect to Lebesgue measure. However, if additionally ν is radially absolutely continuous with some divergence condition, then $\mathfrak{L}(L_t)$ is absolutely continuous (see e.g. [14, Theorem 27.10]). Let us also recall, see e.g. [10], that the law $\mathfrak{L}(L_t)$ is absolutely continuous if and only if the corresponding semigroup satisfies $P_t\colon B_b(\mathbb{R}^d) \mapsto C_b(\mathbb{R}^d)$. Thus the absolute continuity is equivalent to the strong Feller property.

Below we provide some estimates for the moduli of continuity of the transition semigroup which is beyond the scope of the α-stable type process. Namely, assume that the Lévy measure ν is absolutely continuous with respect to Lebesgue measure on the ball $B_1(0) = \{x\colon |x| < 1\}$ and

$$\frac{\nu(\mathrm{d}x)}{\mathrm{d}x} \geq \frac{|\log_2 |x||^{2\gamma}}{|x|^d}, \qquad x \in B_1(0), \tag{6.2}$$

where $\gamma \in (1, \infty)$ is a constant.

Theorem 6.1. *Let $(L_t)_{t\geq 0}$ be a Lévy process with Lévy measure ν satisfying (6.2). Then $(L_t)_{t\geq 0}$ is strong Feller. Moreover, there exists an $r_0 > 0$ such that*

$$|P_t f(x) - P_t f(y)| \leq \frac{C}{|\log_2 |x-y||^{\gamma-1}}, \qquad |x-y| \leq r_0, \tag{6.3}$$

where C depends on γ, d, t and r_0.

Let $\Omega \subset \mathbb{R}^d$ be an open set. Given a function $f\colon \Omega \to \mathbb{R}$, $x \in \Omega$ and $r > 0$ such that the ball $B_r(x) \subset \Omega$, define

$$\bar{f}_{x,r} := \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\mathrm{d}y.$$

To prove the theorem, we need to use the following lemma, which is a generalized Campanato theorem. The proof of the lemma is deferred to the appendix.

Lemma 6.2. *Let $\Omega \subset \mathbb{R}^d$ be open and bounded, and let $f\colon \Omega \to \mathbb{R}$ be a bounded function. Assume that there are constants $C > 0$ and $\gamma > 1$ such that*

$$\int_{B_r(x)} |f(y) - \bar{f}_{x,r}|^2 \mathrm{d}y \leq C \frac{r^d}{|\log_2 r|^{2\gamma}}, \tag{6.4}$$

for any ball $B_r(x) \subset \Omega$. Then f is uniformly continuous and there exists $r_0 > 0$ such that for any open $\tilde{\Omega} \subset \Omega$ with $\operatorname{diam}(\tilde{\Omega}) < r_0$ and $\operatorname{dist}(\tilde{\Omega}, \partial\Omega) > r_0$, we have

$$|\bar{f}_{x,r} - f(x)| \leq \frac{\hat{C}}{|\log_2 r|^{\gamma-1}}, \qquad \forall\, x \in \tilde{\Omega}, \tag{6.5}$$

where $\hat{C}$ depends on d, γ and C. Moreover,

$$\sup_{x,y\in\tilde{\Omega}, x\neq y} |f(x) - f(y)||\log_2 |x-y||^{\gamma-1} \leq \tilde{C}, \tag{6.6}$$

where $\tilde{C}$ depends on C, $\operatorname{dist}(\tilde{\Omega}, \partial\Omega), d, \gamma$ *and* r_0*.*

Remark 6.3. It is interesting to point out that (6.5) is a Poincaré type inequality, which might be of independent interest.

Proof of Theorem 6.1. Let $f \in B_b(\mathbb{R}^d)$ and set $g(x) = P_t f(x)$. For any $t > 0$, we have

$$\begin{aligned}\int_{B_r(x)} |g(y) - \bar{g}_{x,r}|^2 \mathrm{d}y &= \int_{\{|y-x|\le r\}} \frac{1}{|B_r(x)|^2} \left| \int_{\{|z-x|\le r\}} (g(y) - g(z))\, \mathrm{d}z \right|^2 \mathrm{d}y \\ &\le \int_{\{|y-x|\le r\}} \frac{1}{|B_r(x)|} \int_{\{|z-x|\le r\}} |g(y) - g(z)|^2 \, \mathrm{d}z\mathrm{d}y \\ &\le \int_{\{|y-x|\le r\}} \frac{1}{|B_r(x)|} \int_{\{|y-z|\le 2r\}} |g(y) - g(z)|^2 \, \mathrm{d}z\mathrm{d}y,\end{aligned} \tag{6.7}$$

where the last inequality follows from the inclusion

$$\{|y-x| \le r\} \cap \{|z-x| \le r\} \subset \{|y-x| \le r\} \cap \{|y-z| \le 2r\}.$$

From (6.7) we further get

$$\begin{aligned}&\int_{B_r(x)} |g(y) - \bar{g}_{x,r}|^2 \mathrm{d}y \\ &\le \int_{\{|y-x|\le r\}} \frac{1}{|B_r(x)|} \int_{\{|y-z|\le 2r\}} |g(y) - g(z)|^2 \frac{|z-y|^d}{|\log_2 |y-z||^{2\gamma}} \frac{|\log_2 |z-y||^{2\gamma}}{|z-y|^d} \mathrm{d}z\mathrm{d}y \\ &= \int_{\{|y-x|\le r\}} \frac{1}{|B_r(x)|} \int_{\{|z|\le 2r\}} |g(y+z) - g(y)|^2 \frac{|z|^d}{|\log_2 |z||^{2\gamma}} \frac{|\log_2 |z||^{2\gamma}}{|z|^d} \mathrm{d}z\mathrm{d}y \\ &\le \int_{\{|y-x|\le r\}} \frac{1}{|B_r(x)|} \int_{\{|z|\le 2r\}} |g(y+z) - g(y)|^2 \frac{|z|^d}{|\log_2 |z||^{2\gamma}} \nu(\mathrm{d}z)\mathrm{d}y\end{aligned}$$

Since $\frac{r^d}{|\log_2 r|^{2\gamma}}$ is decreasing as $r < r_0/2$ for small $r_0 > 0$, the above inequality further gives

$$\begin{aligned}\int_{B_r(x)} |g(y) - \bar{g}_{x,r}|^2 \mathrm{d}y &\le C \frac{r^d}{|\log_2 r|^{2\gamma}} \frac{1}{|B_r(x)|} \int_{\{|y-x|\le r\}} \int_{\{|z|\le 2r\}} |g(y+z) - g(y)|^2 \, \nu(\mathrm{d}z)\mathrm{d}y \\ &\le C \frac{r^d}{|\log_2 r|^{2\gamma}} \sup_{y\in B_r(x)} \int_{\mathbb{R}^d} |g(y+z) - g(y)|^2 \, \nu(\mathrm{d}z) \\ &\le C \frac{r^d}{|\log_2 r|^{2\gamma}} t^{-1} \|f\|_\infty^2,\end{aligned}$$

where the last inequality follows from Corollary 3.2. Combining the estimate above and Lemma 6.2 we obtain the desired conclusion. □

7. Proof of Lemma 6.2

We follow [9]. For $0 < r_2 < r_1 < \min\{\text{dist}\,(\tilde{\Omega}, \partial\Omega), 1\}$ and any $x \in \tilde{\Omega}$, we have

$$
\begin{aligned}
|\bar{f}_{x,r_1} - \bar{f}_{x,r_2}| &\leq \frac{1}{|B_{r_2}|}\int_{B_{r_2}(x)} |f(y) - \bar{f}_{x,r_2}|\mathrm{d}y + \frac{1}{|B_{r_2}|}\int_{B_{r_2}(x)} |f(y) - \bar{f}_{x,r_1}|\mathrm{d}y \\
&\leq \sqrt{\frac{1}{|B_{r_2}|}\int_{B_{r_2}(x)} |f(y) - \bar{f}_{x,r_2}|^2\mathrm{d}y} + \sqrt{\frac{1}{|B_{r_2}|}\int_{B_{r_2}(x)} |f(y) - \bar{f}_{x,r_1}|^2\mathrm{d}y} \\
&\leq \sqrt{\frac{1}{|B_{r_2}|}\int_{B_{r_2}(x)} |f(y) - \bar{f}_{x,r_2}|^2\mathrm{d}y} + \sqrt{\frac{|B_{r_1}|}{|B_{r_2}|}\frac{1}{|B_{r_1}|}\int_{B_{r_1}(x)} |f(y) - \bar{f}_{x,r_1}|^2\mathrm{d}y} \\
&\leq C\left[\frac{1}{|\log_2 r_2|^\gamma} + \left(\frac{r_1}{r_2}\right)^{d/2}\frac{1}{|\log_2 r_1|^\gamma}\right] \\
&\leq C\left[1 + \left(\frac{r_1}{r_2}\right)^{d/2}\right]\frac{1}{|\log_2 r_1|^\gamma},
\end{aligned}
\tag{7.1}
$$

where the last inequality is by the assumption of the lemma. For all $0 < r_n < r_m < \text{dist}\,(\tilde{\Omega}, \partial\Omega)$, define

$$
N := \left[\log_2\left(\frac{r_m}{r_n}\right)\right],
$$

without loss of generality we assume $r_m < 1/2$. By (7.1) we have

$$
\begin{aligned}
|\bar{f}_{x,r_n} - \bar{f}_{x,r_m}| &\leq \sum_{k=1}^{N} |\bar{f}_{x,2^{-k}r_m} - \bar{f}_{x,2^{-k+1}r_m}| + |\bar{f}_{x,2^{-N}r_m} - \bar{f}_{x,r_n}| \\
&\leq C_d \sum_{k=1}^{N} \frac{1}{|k - 1 - \log_2 r_m|^\gamma} + C_d\frac{1}{|N - \log_2 r_m|^\gamma} \\
&\leq C_{d,\gamma}\frac{1}{|\log_2 r_m|^{\gamma-1}}.
\end{aligned}
\tag{7.2}
$$

Hence, there exists an $\tilde{f}$ such that

$$
\lim_{r\to 0} \bar{f}_{x,r} = \tilde{f}(x), \qquad \forall\, x \in \tilde{\Omega},
$$

and there exists an $r_0 > 0$ such that as $r < r_0$,

$$
|\bar{f}_{x,r} - \tilde{f}(x)| \leq C_{d,\gamma}\frac{1}{|\log_2 r|^{\gamma-1}}, \qquad \forall\, x \in \tilde{\Omega}. \tag{7.3}
$$

On the other hand, by the Lebesgue theorem,

$$
\lim_{r\to 0} \bar{f}_{x,r} = f(x) \qquad \text{for almost all}\, x \in \tilde{\Omega}.
$$

By (7.3), all the points in $\tilde{\Omega}$ are Lebesgue points. Hence, $\bar{f}_{x,r} \to f(x)$ uniformly for $x \in \tilde{\Omega}$ as $r \to 0$ with

$$
|\bar{f}_{x,r} - f(x)| \leq C_{d,\gamma}\frac{1}{|\log_2 r|^{\gamma-1}}, \qquad \forall\, x \in \tilde{\Omega}. \tag{7.4}
$$

Now for $x, y \in \tilde{\Omega}$, denote $r = |x - y|$, we have

$$
|\bar{f}_{y,2r} - \bar{f}_{x,2r}| \leq |\bar{f}_{y,2r} - f(z)| + |f(z) - \bar{f}_{x,2r}|, \qquad z \in B_{2r}(x) \cap B_{2r}(y). \tag{7.5}
$$

Since $B_{2r}(x) \cap B_{2r}(y)$ contains a ball with radius r, as $r < r_0/2$,

$$
\begin{aligned}
|\bar f_{y,2r} - \bar f_{x,2r}| &\le \frac{1}{|B_r|} \int_{B_{2r}(x)\cap B_{2r}(y)} |\bar f_{y,2r} - f(z)| + |f(z) - \bar f_{x,2r}| \mathrm{d}z \\
&\le \frac{1}{|B_r|} \int_{B_{2r}(y)} |\bar f_{y,2r} - f(z)| \mathrm{d}z + \frac{1}{|B_r|} \int_{B_{2r}(x)} |f(z) - \bar f_{x,2r}| \mathrm{d}z \\
&\le C \frac{1}{|\log_2 r|^\gamma},
\end{aligned} \tag{7.6}
$$

where the last inequality is by the assumption of the lemma. Observe that

$$
|f(x) - f(y)| \le |f(x) - \bar f_{x,2r}| + |f(y) - \bar f_{y,2r}| + |\bar f_{y,2r} - \bar f_{x,2r}|. \tag{7.7}
$$

This, together with (7.6) and (7.4), immediately implies the desired inequality.

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