Stochastic Partial Differential Equations with Lévy Noise (a few aspects)

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Abstract. These are the notes for my two 90 minutes talks on some aspects of SPDEs with Lévy noise presented during the semester on SPDEs in EPF Lausanne and then in the Institute of Applied Mathematics, Chinese Academy of Sciences. The first talk was devoted to analytical aspects of the theory: the form of the generator of a Markov semigroup in finite and infinite dimensional spaces, properties of the transition semigroup of a Lévy process and the Lévy–Khinchin formula. The second talk was concerned with stochastic integration with respect to a Poisson random measure on L^p-spaces, and with time regularity of solutions to SPDEs driven by Lévy processes.

1. Introduction

In this paper we can only focus on a few aspects of the theory of SPDEs driven by Léve noise. The choice of topics is to some extend arbitrary and reflexes the current area of interest of the author.

Therefore we start with the classical Courrège theorem giving the form of the generator of a Markov transition semigroup on a state space $\mathbb{R}^d$. Then, following Itô, we show that the corresponding Markov family is defined by a stochastic ordinary differential equation. Let A be the generator of a Markov transition semigroup on $\mathbb{R}^d$. Define $\mathcal{A}\psi = A\psi(0)$. Obviously, $\mathcal{A}$ is a linear functional satisfying the minimum principle: $\mathcal{A}\psi \geq 0$ if ψ has a global minimum at 0. We present a theorem from the recent book of Stroock [29] which characterizes linear functionals satisfying the minimum principle. We conclude the topic of the representation of Markov semigroups and families with results (Theorems 3 and 4) dealing with a semigroup on a possibly infinite-dimensional state space.

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The problem of the representation of Markov families is one of the motivation to study Léve processes. Therefore in Section 3 we recall the definition of a Lévy process, and then we investigate analytical properties of its transition semigroup. In particular we are interested on which function space the transition semigroup is strongly continuous, and what is its generator. Section 4 is devoted to the Lévy–Khinchin decomposition and Lévy–Khinchin formula. We recall the classical results of Kruglov and De Acosta on the existence of exponential moments of a Lévy process. We recall also the Kinney theorem on the existence of a càdlàg modification of a Markov process taking values in a metric space.

In Section 5 we introduce the theory of integration with respect to a square integrable Lévy process taking values in a Hilbert space. We study also the problem of the existence of a solution to SPDE driven by a square integrable Lévy process. Our general existence result (Theorem 13) will be applied to the stochastic heat equation.

In Section 6 we discuss the problem whether in stochastic integration integrands must be predictable. It is an important issue in the case of SDEs in infinite dimensional spaces since in infinite dimensional case very often the solution does not have càdlàg modification or left limits. Therefore one cannot write the diffusion term in the typical for finite dimensional case form $b(u(t-))\mathrm{d}L(t)$.

Section 7 deals with stochastic integration with respect to Poisson random measure. We give examples of equations. We introduce the concept of an impulsive white noise, which to some extend is a jump analog of the Brownian sheet.

The last section is concerned with the existence of càdlàg solutions to SPDEs driven by a Lévy process or Poisson random measure. We show that in general the solution does not need to be càdlàg. However, we present also some criteria for the existence of a càdlàg solution.

2. Representation of Markov processes

We recall the Courrège result (see [5]) on the form of the generator A of a Markov semigroup on $\mathbb{R}^d$. Then we will try to find a Markov family defined by a stochastic differential equation whose generator is A. We will finish this section with some partial results valid in infinite dimensional spaces.

2.1. Finite dimensional case

Let $P_t(x,\cdot)$, $t \geq 0$, $x \in E$, be a transition probability. Then the corresponding transition semigroup (P_t) is given by

$$P_t\psi(x) = \int_E P_t(x,\mathrm{d}y)\psi(y), \qquad \psi \in B_b(E),$$

where $B_b(E)$ is the space of all bounded measurable real-valued functions on E.

Let us denote by $C_0(\mathbb{R}^d)$ the space of continuous functions having 0 limit at infinity and by $C_0^\infty(\mathbb{R}^d)$ the space of infinitely differentiable functions with all derivatives of orders ≥ 0, continuous and having limit 0 at infinity. We denote by $\|\cdot\|_\infty$ the supremum norm, and by $M_s^+(d\times d)$ the space of all symmetric and non-negative definite matrices of dimension $d\times d$. Finally, D denotes the derivative (or gradient) operator.

Theorem 1. (Courrège's 1965/66) *Let (P_t) be a transition semigroup on $B_b(\mathbb{R}^d)$. Assume that:*

(i) *(P_t) satisfies the Feller property, that is $P_t\colon C_0(\mathbb{R}^d)\mapsto C_0(\mathbb{R}^d)$.*

(ii) *(P_t) is strongly continuous(C_0 for short) on $\big(C_0(\mathbb{R}^d),\|\cdot\|_\infty\big)$, that is for any $\psi\in C_0(\mathbb{R}^d)$,*

$$\lim_{t\downarrow 0}\|P_t\psi-\psi\|_\infty = 0.$$ [1]

(iii) *For any $\psi\in C_0^\infty(\mathbb{R}^d)$ and for any $x\in\mathbb{R}^d$, the function*

$$[0,+\infty)\ni t\mapsto P_t\psi(x)\in\mathbb{R},$$

is differentiable.

Then there are measurable mappings $a\colon\mathbb{R}^d\mapsto\mathbb{R}^d$, $Q\colon\mathbb{R}^d\mapsto M_s^+(d\times d)$, and a family $\nu(x,\cdot)$, $x\in\mathbb{R}^d$, of non-negative but not necessery finite measures on $\big(\mathbb{R}^d\setminus\{0\},\mathcal{B}(\mathbb{R}^d\setminus\{0\})\big)$ satisfying

$$\int_{\mathbb{R}^d}|y|^2\wedge 1\nu(x,\mathrm{d}y)<\infty,\qquad\forall\, x\in\mathbb{R}^d,$$

such that for any $\psi\in C_0^\infty(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$,

$$\begin{aligned}A\psi(x) := \lim_{t\downarrow 0}\frac{P_t\psi(x)-\psi(x)}{t} &= \langle a(x), D\psi(x)\rangle + \frac{1}{2}\,Tr\,Q(x)D^2\psi(x)\\ &+\int_{\mathbb{R}^d}\big(\psi(x+y)-\psi(x)-\chi_{[0,1]}(|y|)\langle y, D\psi(x)\rangle\big)\,\nu(x,\mathrm{d}y).\end{aligned}$$

The Courrège theorem gives the form of the generator of a transition semigroup satisfying mild and natural conditions. A natural question is whether for given a, Q and $\nu(x,\cdot)$, $x\in\mathbb{R}^d$, there is a transition semigroup with prescribed generator? To answer this question assume that the family of measures $\nu(x,\cdot)$, $x\in\mathbb{R}^d$, is a transport of a single measure μ by a family of mappings $F(x,y)$, i.e. there is a one measure μ on $\mathbb{R}^m\setminus\{0\}$ such that $\nu(x,\cdot)=F(x,\cdot)\circ\mu$, $x\in\mathbb{R}^d$, where $F\colon\mathbb{R}^d\times\mathbb{R}^m\mapsto\mathbb{R}^d$. Assume also that $\int_{\mathbb{R}^m}|y|^2\wedge 1\mu(\mathrm{d}y)<\infty$. Then

$$\begin{aligned}&A\psi(x) = \langle a(x), D\psi(x)\rangle + \frac{1}{2}\mathrm{Tr}\,Q(x)D^2\psi(x)\\ &+\int_{\mathbb{R}^m}\big(\psi(x+F(x,y))-\psi(x)-\chi_{[0,1]}(|F(x,y)|)\langle F(x,y), D\psi(x)\rangle\big)\,\mu(\mathrm{d}y).\end{aligned}$$

[1] It turns out that uniform convergence follows from the pointwise convergence, see the book of Rogers and Williams [25], p. 241, Lemma 6.7.

Hence

$$
\begin{aligned}
A\psi(x) &= \langle \tilde a(x), D\psi(x)\rangle + \frac{1}{2}\mathrm{Tr}\, Q(x)D^2\psi(x)\\
&\quad + \int_{\mathbb{R}^m} \left(\psi(x+F(x,y)) - \psi(x) - \chi_{[0,1]}(|y|)\langle F(x,y), D\psi(x)\rangle\right)\mu(\mathrm{d}y),
\end{aligned}
$$

where

$$
\tilde a(x) = a(x) - \int_{\mathbb{R}^m} \left(\chi_{[0,1]}(|F(x,y)|) - \chi_{[0,1]}(|y|)\right) F(x,y)\mu(\mathrm{d}y),
$$

Let $\pi(\mathrm{d}t,\mathrm{d}y)$ be the Poisson random measure with intensity measure $\mathrm{d}t\mu(\mathrm{d}y)$, and let

$$
\tilde\pi(\mathrm{d}t,\mathrm{d}y) = \chi_{\{|y|>1\}}\pi(\mathrm{d}t,\mathrm{d}y) + \chi_{\{|y|\le 1\}}\left(\pi(\mathrm{d}t,\mathrm{d}y) - \mu(\mathrm{d}y)\mathrm{d}t\right),
$$

be the compensated measure. Under suitable assumptions on $\tilde a$, Q and F, for any $x\in\mathbb{R}^d$, the following stochastic ordinary equation

$$
\begin{aligned}
&\mathrm{d}X(t) = \tilde a(X(t)\mathrm{d}t + \sqrt{Q(X(t))}\mathrm{d}W(t) + \int_{\mathbb{R}^m} F(X(t-),y)\tilde\pi(\mathrm{d}t,\mathrm{d}y),\\
&X(0) = x,
\end{aligned}
$$

has a unique solution. It turns out that (P_t) is the corresponding Markov semigroup. This construction should be attributed to K. Itô.

2.2. Representation theorem from the book of Stroock

It is an open problem how to extend either Courrège's theorem or Itô's construction to infinite dimensional spaces. It seems that for this purpose, the following result valid still in finite dimensional spaces could be a starting point. The result comes from the book by Stroock [29]. It gives the form of an arbitrary linear operator satisfying the minimum principle. Here we denote by $C_c^\infty(\mathbb{R}^d)$ the space of all infinitely differentiable functions on $\mathbb{R}^d$ having compact support.

Theorem 2. *A linear operator* $\mathcal{A}\colon C_c^\infty(\mathbb{R}^d)\oplus\mathbb{R}\mapsto\mathbb{R}$ *satisfies the hypothesis:*

(i) *minimum principle;* $\mathcal{A}\psi\ge 0$ *if* ψ *has a global minimum at* 0,
(ii) *tightness; for any* $\psi\in C_c^\infty(\mathbb{R}^d)\oplus\mathbb{R}$, $\mathcal{A}\psi_\varepsilon\to 0$ *as* $\varepsilon\downarrow 0$, *where* $\psi_\varepsilon(x) = \psi(x\varepsilon)$, $x\in\mathbb{R}^d$,

if and only if there are $a\in\mathbb{R}^d$, $Q\in M_s^+(d\times d)$ *and a measure* ν *on* $\mathbb{R}^d\setminus\{0\}$ *satisfying* $\int_{\mathbb{R}^d}|y|^2\wedge 1\nu(\mathrm{d}y)<\infty$ *such that*

$$
\begin{aligned}
\mathcal{A}\psi &= \langle a, D\psi(0)\rangle + \frac{1}{2}\mathit{Tr}\,QD^2\psi(0)\\
&\qquad + \int_{\mathbb{R}^d}\left(\psi(y) - \psi(0) - \chi_{\{[0,1]\}}(|y|)\langle y, D\psi(0)\rangle\right)\nu(\mathrm{d}y).
\end{aligned}
$$

Obviously, in the infinite dimensional case a substitute of the tightness property (ii) should be found.

2.3. Representation theorem in the infinite dimensional case

Let E and U be linear topological spaces. Later we will need to assume that E is additionally Polish. The following result, borrowed from the book by Peszat and Zabczyk [22], deals with the simplest stochastic evolution equation driven by a compound Poisson process. It provides the existence of the unique solution and the form of the generator.

Theorem 3. *Assume that:*

(i) *L is a compound Poisson process on U with intensity of jump measure ν,*

(ii) *$G(x)$, $x \in E$, is a family of continuous linear mappings from U to E. We assume that G is strongly measurable in the sense that for any $v \in U$, the mapping $E \ni x \mapsto G(x)v \in E$ is measurable.*

Then for any $x \in E$ the following stochastic equation

$$\mathrm{d}X(t) = G(X(t-))\mathrm{d}L(t), \qquad X(0) = x, \tag{1}$$

has a unique solution X^x. Moreover, $(X^x,\ x \in E)$ is a Markov family on E, and for any $\psi \in B_b(E)$, uniformly in $x \in E$,

$$\lim_{t \downarrow 0} \frac{P_t\psi(x) - \psi(x)}{t} = \int_E [\psi(x+y) - \psi(x)]\, \nu(x, \mathrm{d}y), \tag{2}$$

where $\nu(x,\cdot)$ is the transport of ν by the mapping $G(x)$; $\nu(x,\mathrm{d}y) = G(x)(\cdot) \circ \nu(\mathrm{d}y)$.

Skech of the proof. The proof of the existence and uniqueness is simple. Namely, let $\tau_1, \tau_2, \ldots$ be the consecutive jump times of L. Then $X(t) = x$ for $t \in [0, \tau_1)$, $X(t) = x + G(x)L(\tau_1)$ for $t \in [\tau_1, \tau_2)$, and generally $X(t) = X(\tau_{n-1}) + G(X(\tau_{n-1}))(L(\tau_n) - L(\tau_{n-1})$ for $t \in [\tau_n, \tau_{n+1})$. To see the form of the generator take $\psi \in B_b(E)$ and $x \in E$. Take $\tau_0 = 0$. Then

$$\begin{aligned}
\mathbb{E}\psi(X^x(t)) &= \sum_{n=0}^{\infty} \mathbb{P}\left(t \in [\tau_n, \tau_{n+1})\right) \mathbb{E}\psi(X(\tau_n)) \\
&= \sum_{n=0}^{\infty} \mathrm{e}^{-\alpha t} \frac{(t\alpha)^n}{n!} \mathbb{E}\psi(X(\tau_n)),
\end{aligned}$$

where $\alpha = \nu(U) < \infty$. Therefore, since ψ is bounded,

$$\begin{aligned}
\lim_{t\downarrow 0} \frac{P_t\psi(x) - \psi(x)}{t} &= \lim_{t\downarrow 0} \frac{\mathbb{E}\psi(X^x(t)) - \psi(x)}{t} \\
&= \lim_{t\downarrow 0} \frac{1}{t} \left[\sum_{n=0}^{\infty} \mathrm{e}^{-\alpha t} \frac{(t\alpha)^n}{n!} \mathbb{E}\psi(X(\tau_n)) - \psi(x) \right] \\
&= \lim_{t\downarrow 0} \frac{1}{t} \left(\mathrm{e}^{-\alpha t} - 1\right) \psi(x) + \alpha \mathbb{E}\psi\left(x + G(x)L(\tau_1)\right) \\
&= \int_U [\psi(x + G(x)y) - \psi(x)]\, \nu(\mathrm{d}y). \qquad \square
\end{aligned}$$

The question is whether given a family of measures $\nu(x, \mathrm{d}y)$ on $(E, \mathcal{B}(E))$ there is a Markov family (defined by an equation driven by a compound Poisson process) whose generator is as on the right hand side of (2). The answer is affirmative at least if E is a Polish space, and $\nu(x, \mathrm{d}y)$ is a *transition probability kernel*; that is $\nu(x, \mathrm{d}y)$ is a probability measure on $(E, \mathcal{B}(E))$ and for any $\Gamma \in \mathcal{B}(E)$, the function $E \ni x \mapsto \nu(x, \Gamma) \in [0, 1]$ is measurable. To see this, let U be the space of all finite point measures on $[0, 1]$ with topology of bounded variation, and let $\mathfrak{m}$ be the transport of the Lebesgue measure ℓ_1 on $[0, 1]$ by the mapping $f\colon [0, 1] \ni x \mapsto \delta_x \in U$. Clearly $\mathfrak{m}$ is a measure on U. The following result was shown in [22].

Theorem 4. *Assume that $\nu(x, \mathrm{d}y)$ is a probability kernel. Let L be a compound Poisson process on the space U with the intensity measure $\mathfrak{m}$. Then there is a strongly measurable family $G(x)$, $x \in E$, of bounded mappings from U to E, such that the generator of the transition semigroup of the family given by* (1) *is given by* (2).

Proof. It is know that any time homogeneous Markov chain (X_n) on a Polish space E can be represented in the form $X_{n+1} = F(X_n, \xi_{n+1})$, where (ξ_n) are independent identically distributed random variables on $[0, 1]$ and F is a measurable mapping from $E \times [0, 1]$ to E. Let (X_n) be the Markov family with the transition probability $\nu(x, \mathrm{d}y)$. Then $\nu(x, \cdot) = F(x, \cdot) \circ \ell_1$. Obviously F does not need to be linear in y! To overcame this difficulty it is enough to take

$$G(x)v = \int_0^1 F(x, y)v(\mathrm{d}y), \qquad v \in U.$$

□

A much simpler representation can be obtain by taking a Poisson random measure π with intensity measure $\mathrm{d}t\ell_1(\mathrm{d}y)$. Then

$$\mathrm{d}X(t) = \int_0^1 F(X(t-), y)\pi(\mathrm{d}t, \mathrm{d}y), \qquad X(0) = x.$$

2.4. Other results

By the Courrège theorem any time homogeneous Markov family on $\mathbb{R}^d$, which is Gaussian, Feller and such that the functions $t \mapsto P_t\psi(x)$, $x \in \mathbb{R}^d$ and $\psi \in C_0^\infty(\mathbb{R}^d)$, are differentiable on $\mathbb{R}^d$ is given by the equation

$$\mathrm{d}X = a(X)\mathrm{d}t + \sqrt{Q}\mathrm{d}W, \qquad X(0) = x \in \mathbb{R}^d,$$

where W is a Wiener process in $\mathbb{R}^d$, $Q \in M_s^+(d \times d)$ and $a\colon \mathbb{R}^d \mapsto \mathbb{R}^d$. In fact one can deduce from the Gaussianity, that there are: a linear map $A\colon \sqrt{Q}(\mathbb{R}^d) \mapsto \sqrt{Q}(\mathbb{R}^d)$, a vector $b \in \sqrt{Q}(\mathbb{R}^d)$, and a (possibly) nonlinear mapping $F\colon \sqrt{Q}(\mathbb{R}^d)^\perp \mapsto \sqrt{Q}(\mathbb{R}^d)^\perp$ such that

$$a(x) = (A\Pi x + b) + F(\Pi^\perp x),$$

where $\Pi\colon \mathbb{R}^d \mapsto \sqrt{Q}(\mathbb{R}^d)$ is a linear orthogonal projection.

What has beed done in infinite dimensions? We have thew paper of Itô [12], who proved that any stationary time homogeneous Gaussian Markov family is defined by an infinite-dimensional Ornstein–Uhlenbeck equation. Then in [10] is shown that any time homogeneous Gaussian Markov family X^x is a solution to Ornstein–Uhlenbeck equation provided that for any $t > 0$ and $x \in E$, the support of the law $\mathcal{L}(X^x(t))$ is equal to E.

3. Levý process and its transition semigroup

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space.

Definition 1. A stochastic process L with values in H is *Lévy* if:

(i) $L(0) = 0$,
(ii) L has stationary independent increments,
(iii) L is stochastically continuous.

Let L be a Lévy process and let μ_t be the law of $L(t)$. Then:

(i') $\mu_0 = \delta_0$,
(ii') $\mu_{t+s} = \mu_t * \mu_s$, $t, s \geq 0$,
(iii') $\mu_t(\{x\colon |x|_H \geq r\}) = \mathbb{P}(|L(t)|_H \geq r) \to 0$ as $t \downarrow 0$ for any $r > 0$.

Clearly (iii') can be stated equivalently that μ_t converges weakly to δ_0 as $t \downarrow 0$.

Definition 2. The family of probability measures satisfying (i') to (iii') is called *convolution semigroup of measures* or *infinitely divisible family.* Sometimes μ_1 is called *infinitely divisible measure.*

Any Lévy process is Markov with transition probability $P_t(x, \Gamma) = \mu_t(\Gamma - x)$. The corresponding semigroup is given by

$$P_t\psi(x) = \int_H \psi(x+y)\mu_t(\mathrm{d}y).$$

Theorem 5. *Every Lévy process has a càdlàg modification. This modification is a Lévy process.*

The theorem follows from the following general result of Kinney [13]. Here $B(x, r)$ denotes the closed ball of radius r with centre at x and $B^c(x, r)$ denotes its complement.

Theorem 6. (Kinney 1953) *Assume that X is a Markov process with transition probabilities $P_t(x, \mathrm{d}y)$, $x \in H$, $t \geq 0$. If*

$$\lim_{t\downarrow 0} \sup_{x\in H} P_t(x, B^c(x, r)) = 0, \qquad \forall\, r > 0,$$

then X has a càdlàg modification in H.

Let us now apply the Kinney theorem to the Lévy process. Let $r > 0$. By (iii') we have

$$\lim_{t\downarrow 0} \sup_{x\in H} P_t(x, B^c(x, r)) = \lim_{t\downarrow 0} \sup_{x\in H} \mu_t(B^c(x, r) - x) = \lim_{t\downarrow 0} \mu_t(B^c(0, r)) = 0.$$

Note that the Kinney theorem cannot be applied to the family given by the generalised Ornstein–Uhlenbeck equation

$$\mathrm{d}X = AX\mathrm{d}t + \mathrm{d}L, \qquad X(0) = x,$$

even in finite dimensional case. For

$$\limsup_{t\downarrow 0}\sup_{x\in H} P_t(x, B^c(x,r)) = \limsup_{t\downarrow 0}\sup_{x\in H} \mathbb{P}\left(\left|\mathrm{e}^{At}x - x + \int_0^t \mathrm{e}^{A(t-s)}\mathrm{d}L(s)\right| \geq r\right).$$

Clearly the right hand side of the identity above equals $+\infty$ unless $A = 0$.

3.1. Semigroups

Let (μ_t) be a convolution semigroup of measures on H. Let $C_b(H)$ and $UC_b(H)$ be the spaces of bounded continuous and bounded uniformly continuous functions on H equipped with the supremum norm $\|\cdot\|_\infty$. The following result was shown in [30].

Theorem 7. (Tessitore and Zabczyk 2001) *Transition semigroup (P_t) is C_0 on $C_b(H)$ if and only if (μ_t) corresponds to a compound Poisson process or $\mu_t \equiv \delta_0$.*

Proof. If part is simple. Namely if (μ_t) corresponds to a compound Poisson process L, then denoting by τ_n its consecutive jump times we have

$$\begin{aligned} P_t\psi(x) &= \sum_{n=0}^{\infty} \mathbb{P}\left(t \in [\tau_n, \tau_{n+1})\right)\mathbb{E}\psi(L(\tau_n)) \\ &= \mathrm{e}^{-\alpha t}\sum_{n=0}^{\infty}\frac{(\alpha t)^n}{n!}\mathbb{E}\psi(L(\tau_n)). \end{aligned}$$

Consequently

$$\limsup_{t\downarrow 0}\|P_t\psi - \psi\|_\infty \leq \limsup_{t\downarrow 0}\mathrm{e}^{-\alpha t}\sum_{n=1}^{\infty}\frac{(\alpha t)^n}{n!}\|\psi\|_\infty = 0.$$

Assume now that neither $\mu_t \equiv \delta_0$ nor (μ_t) corresponds to a compound Poisson process. Let $\varepsilon \in (0,1)$. Then there is a sequence $t_n \downarrow 0$ such that

$$\mu_{t_n}\left(|x|_H \leq \frac{1}{2}\right) = \mathbb{P}\left(|L(t_n)|_H \leq \frac{1}{2}\right) \geq 1 - \varepsilon$$

and (at this moment we use the assumption that L is not a compound Poisson nor $L \equiv 0$) such that for each n we can find an $0 < r_n < \frac{1}{2}$ such that

$$\mu_{t_n}\left(|x|_H \leq r_n\right) = \mathbb{P}\left(|L(t_n)|_H \leq r_n\right) \leq \varepsilon.$$

Let (x_n) be a sequence of elements of H such that $|x_n - x_m|_H \geq 1$ if $n \neq m$. Let $\psi_n \in C_b(H)$ be such that $0 \leq \psi_n(x) \leq 1$ for all x, $\psi_n(x_n) = 1$ and $\psi_n(x) = 0$ if $|x - x_n|_H \geq r_n$. Let $\psi = \sum \psi_n$. Then $\psi \in C_b(H)$, $0 \leq \psi(x) \leq 1$ for all x, and $\psi(x_n) = 1$ or all n. We have

$$\|P_{t_n}\psi - \psi\|_\infty \geq |P_{t_n}\psi(x_n) - \psi(x_n)| = |1 - P_{t_n}\psi(x_n)|.$$

Since

$$\begin{aligned} P_{t_n}\psi(x_n) =& P_{t_n}\psi_n(x_n) + \sum_{m\neq n} P_{t_n}\psi_m(x_n) \\ & \leq \mu_{t_n}\left(|x|_H \leq r_n\right) + \mu_{t_n}\left(|x|_H > \frac{1}{2}\right) \leq 2\varepsilon, \end{aligned}$$

we have $\|P_{t_n}\psi - \psi\|_\infty \geq 1 - 2\varepsilon$. □

Definition 3. *A semigroup (P_t) of continuous linear operators on $UC_b(H)$ is* translation invariant *if for all $a \in H$, $t \geq 0$, and $\psi \in UC_b(H)$, $P_t\tau_a\psi = \tau_a P_t\psi$, where τ_a is the translation on a vector a; $\tau_a\psi(x) = \psi(x+a)$.*

For a simple proof of the following result we refer the reader to [22].

Theorem 8. (*i*) *The transition semigroup of a Lévy process is C_0 on $UC_b(H)$.*
(*ii*) *A Markov transition semigroup on $UC_b(H)$ is translation invariant if and only it is the transition semigroup of a Lévy process on H.*

4. Lévy–Khinchin decomposition

The so-called Lévy–Khinchin decomposition and Lévy–Khinchin formula play fundamental roles in the theory of Lévy processes. Let L be a Lévy process taking values in a Hilbert space H. Taking if necessary a modification we may assume that L is càdlàg, see Theorem 5. Define $\Delta L(s) := L(s) - L(s-)$,

$$\mu(A) := \mathbb{E}\sum_{s\leq 1} \chi_A(\Delta L(s)), \qquad A \in \mathcal{B}(H \setminus \{0\}).$$

Next, given $A \in \mathcal{B}(H)$ such that $\mathrm{dist}(A, \{0\}) > 0$ write

$$L_A(t) := \sum_{s\leq t} \chi_A(\Delta L(s))\Delta L(s), \qquad t \geq 0.$$

Note that due to the fact that L has càdlàg trajectories in H, $\chi_A(\Delta L(s)) \neq 0$ only for a finite number of $s \leq t$. For a proof of the following result we refer the reader to e.g. [9].

Theorem 9. (Lévy–Khinchin) (*i*) *μ is a measure satisfying*

$$\int_H |y|_H^2 \wedge 1\mu(\mathrm{d}y) < \infty.$$

(*ii*) *For any $A \in \mathcal{B}(H)$ such that* $\mathrm{dist}(A, \{0\}) > 0$, *$L_A$ is a compound Poisson process with jump intensity measure μ_A being equal to μ restricted to A.*
(*iii*) *For an arbitrary sequence (r_n) degreasing to* 0,

$$L(t) = at + W(t) + \sum_{n=1}^{\infty}\left(L_{A_n}(t) - t\int_{A_n} y\mu(\mathrm{d}y)\right) + L_{A_0}(t), \tag{3}$$

where the series converges $\mathbb{P}$-a.s. uniformly in t on any bounded interval $[0,T]$, $a \in H$, W is a Wiener process in H, $A_0 = \{|x|_H \geq r_0\}$, and $A_n = \{r_n \leq |x|_H < r_{n-1}\}$. All components are independent, and W does not depend on the choice of (r_n).

Sketch of the proof. First assume that L is a Lévy process in H with continuous trajectories. Can we show that $L(t) = at + W(t)$, where $a \in H$ and W is a Wiener process? To do this observe that L is square integrable. This follows from the following result, whose relatively easy proof can be found in the book of Protter [24] (see also the book of Peszat and Zabczyk [22]).

Theorem 10. (Kruglov 1972) *Assume that L is a càdlàg Lévy process in a Banach space E with jumps bounded by a fixed constant $C > 0$; that is there is a $C > 0$ such that $|\Delta L(t)|_E \le C$ for all $t > 0$. Then there is a constant $\beta > 0$ such that*

$$\mathbb{E} e^{\beta |L(t)|_E} < \infty, \qquad \forall\, t \ge 0. \tag{4}$$

Remark 1. De Acosta, see [7], showed that under the hypothesis of Kruglov's theorem, (4) holds for any $\beta \ge 0$.

Going back to the proof of the Lévy–Khinchin theorem, we see that any continuous Lévy process L is in particular square integrable. Then, since L has stationary and independent increments, the function $f\colon [0,\infty) \mapsto H$ given by $f(t) = \mathbb{E}\, L(t)$, $t \ge 0$, satisfies $f(t+s) = f(t) + f(s)$. Since, by the Fubini theorem, f is measurable, $f(t) = f(1)t$, $t \ge 0$. Therefore $\widehat{L}(t) := L(t) - t\mathbb{E}\, L(1)$, $t \ge 0$, is a square integrable martingale in H. Let Q be the covariance operator of $\widehat{L}(1)$:

$$\langle Q\psi, \phi\rangle_H = \mathbb{E}\langle \widehat{L}(1), \psi\rangle_H \langle \widehat{L}(1), \phi\rangle_H, \qquad \psi, \phi \in H.$$

Then for all $\psi, \phi \in H$,

$$\langle \widehat{L}(t), \psi\rangle_H \langle \widehat{L}(t), \phi\rangle_H - t\langle Q\psi, \phi\rangle_H, \qquad t \ge 0,$$

is a martingale. Therefore, by the Lévy characterisation $\widehat{L}$ is a Wiener process with covariance Q.

In the second part of the proof we would like to subtract from L its jumps. Note that if $A \in \mathcal{B}(H)$ is such that $\mathrm{dist}(A, \{0\}) > 0$, then

$$\Delta(L - L_A)(t) \notin A, \qquad \forall\, t \ge 0.$$

In particular $L - L_{A_0}$ does not have jumps of the size bigger than r_0. One can show that L_A is a Lévy process. It is piecewise constant as it has isolated jumps. Therefore L_A is a compound Poisson process. The intensity of L_A is

$$\mu_A(\Gamma) = \mathbb{E} \sum_{s \le 1} \chi_\Gamma(\Delta L_A(s)) = \mathbb{E} \sum_{s\le 1} \chi_{\Gamma\cap A}(\Delta L(s)), \qquad \Gamma \in \mathcal{B}(H).$$

Therefore μ_A is the restriction of μ to A. Moreover, it can be shown that $L - L_A$ and L_A are independent. Therefore we may expect that

$$L - L_{A_0} - \sum_{n=1}^{\infty} L_{A_n}$$

is a continuous Lévy process. We need however to prove the convergence of the series. It terns out, that the sum

$$\sum_{n=1}^{\infty}\left(L_{A_n}(t)-t\int_{A_n} y\mu(\mathrm{d}y)\right), \qquad t\ge 0,$$

converges in H, $\mathbb{P}$-a.s. uniformly in t from any bounded interval! Indeed, let

$$M_n(t)=L_{A_n}(t)-t\int_{A_n} y\mu(\mathrm{d}y).$$

Then, M_n is a square integrable martingale (also a Lévy process), and

$$\mathbb{E}\,|M_n(t)|_H^2=\int_A |y|_H^2\mu(\mathrm{d}y).$$

The proof of this is not difficult as each L_A is a compound Poisson process. The convergence follows from the Doob maximal inequality for submartingales

$$r\mathbb{P}\left(\sup_{0\le t\le T}\sum_{n=N}^{K}|M_n(t)|_H^2\ge r\right)\le \mathbb{E}\sum_{n=N}^{K}|M_n(T)|_H^2=\int_{\bigcup_{n=N}^{K}A_n}|y|_H^2\mu(\mathrm{d}y).$$

From this we obtain the convergence in probability uniform in $t\in[0,T]$. The convergence $\mathbb{P}$-a.s. follows from the following result:

Theorem 11. (Itô–Nisio 1968) *If X_n, $n\in\mathbb{N}$, are independent random vectors in a not necessarily separable Banach space E, then the convergence of $\sum_{n=1}^{\infty}X_n$ in probability and $\mathbb{P}$-a.s. are equivalent.*

In fact we apply the Itô–Nisio theorem to $X(n)=(M_n(t);t\in[0,T])$ and $E=D([0,T];H)$, $D([0,T];H)$ is the space of all càdlàg H-valued mappings. The space E is equipped with the supremum norm. E is then complete but not separable!

4.1. Poisson random measure

Define

$$\pi([0,t]\times A):=\sum_{0\le s\le t}\chi_A(\Delta L(s)).$$

Then π is a Poisson random measure with intensity measure $\mathrm{d}t\mu(\mathrm{d}z)$, and

$$L_A(t)=\sum_{0\le s\le t}\chi_A(\Delta L(s))\Delta L(s)=\int_0^t\int_A z\pi(\mathrm{d}s,\mathrm{d}z).$$

Therefore we arrive at the following representation formula:

$$L(t)=at+W(t)+\int_0^t\int_H z\tilde{\pi}(\mathrm{d}s,\mathrm{d}z),$$

where

$$\tilde{\pi}(\mathrm{d}s,\mathrm{d}z):=\pi(\mathrm{d}s,\mathrm{d}z)|_{[0,\infty)\times A_0}+\left(\pi(\mathrm{d}s,\mathrm{d}z)-\mathrm{d}s\mu(\mathrm{d}x)\right)|_{[0,\infty)\times(H\setminus A_0)}.$$

4.2. Generator of a Lévy process

Using either Itô formula ore direct calculation as in [22] on obtains the following result.

Theorem 12. *Assume that A is the generator of the transition semigroup on $UC_b(H)$ of a Lévy process L with the Lévy–Khinchin decomposition* (3)*. Then $UC_b^2(H) \subset Dom\, A$, and*

$$A\psi(x) = \langle a, D\psi(x)\rangle_H + \frac{1}{2} QD^2\psi(x) + \int_H \left(\psi(x+y) - \psi(x) - \chi_{\{|y|_H \le 1\}}(y)\langle D\psi(x), y\rangle_H\right) \mu(\mathrm{d}y).$$

Remark 2. In 1973 Nemirovskii and Semenov showed (see [21]) that $UC_b^2(H)$ is dense in $UC_b(H)$ if and only if H is finite dimensional. Therefore, in infinite dimensional case the theorem above gives the description of the generator on a non dense subset of its domain!

5. Stochastic integration

5.1. With respect to a square integrable Lévy martingale

In this and next sections U, H, and V are real separable Hilbert spaces. We denote by $L(U,H)$ the space of all bounded linear operators form U into H, and by $L_{(HS)}(U,H)$ its subspace of Hilbert–Schmidt operators. Recall that $\alpha \in L(U,H)$ belongs to $L_{(HS)}(U,H)$ if

$$\|\alpha\|^2_{L_{(HS)}(U,H)} := \sum_{k=1}^{\infty} |\alpha e_k|^2_H < \infty$$

for any, or equivalently for some orthonormal basis (e_k) of U.

Assume that L is a square integrable Lévy process (large jumps removed) taking values U. Then

$$M(t) = L(t) - t\mathbb{E}\, L(1), \qquad t \ge 0,$$

is a square integrable martingale. Let Q be the covariance operator of $L(1)$. Let

$$\psi = \sum_k \alpha_k \chi_{(t_k, t_{k+1}]}$$

be a simple function; α_k are $L(U,H)$-valued random variables, $\alpha_k(u)$ is $\mathcal{F}_{t_k}$ measurable for any $u \in U$. We define

$$\int_0^t \psi(s)\mathrm{d}M(s) := \sum_k \alpha_k \left(M(t\wedge t_{k+1}) - M(t \wedge t_k)\right).$$

Then after simple calculation we have

$$\mathbb{E}\left|\int_0^t \psi(s)\mathrm{d}M(s)\right|^2_H = \int_0^t \mathbb{E}\|\psi(s)Q^{1/2}\|^2_{L_{(HS)}(U,H)}\mathrm{d}s.$$

Let $\mathcal{H} = Q^{1/2}(U)$ be the image of $Q^{1/2}$. On $\mathcal{H}$ we consider the scalar product inherited from U by $Q^{1/2}$. We call $\mathcal{H}$ the *Reproducing Kernel Hilbert Space* of L. We extend the integral to the completion of the class of simple function with respect to the family of semi-norms

$$|||\psi|||_T := \sqrt{\int_0^T \mathbb{E}\|\psi(s)Q^{1/2}\|^2_{L_{(HS)}(U,H)}\mathrm{d}s}, \qquad T > 0.$$

Thus the space of integrands is the space of all predictable square integrable random processes

$$\psi\colon \Omega \times [0,\infty) \mapsto L_{(HS)}(\mathcal{H}, H).$$

satisfying $|||\psi|||_T < \infty$ for any $T > 0$.

The isometry formula holds

$$\mathbb{E}\left|\int_0^t \psi(s)\mathrm{d}M(s)\right|^2_H = \int_0^t \mathbb{E}\|\psi\|^2_{L_{(HS)}(\mathcal{H},H)}\mathrm{d}s.$$

5.2. Existence and uniqueness to SPDE

Assume that a Hilbert space H is continuously imbedded into a Hilbert space V. Consider SPDE

$$\mathrm{d}u = (Au + F(u))\,\mathrm{d}t + B(u)\mathrm{d}M, \qquad u(0) = u_0 \in H, \tag{5}$$

where $(A, D(A))$ generates a C_0-semigroup S on H, $F\colon H \mapsto V$, and for any $x \in H$, $B(x)$ is a linear operator (not necessarily bounded) from $\mathcal{H}$ to H. We have the following simple existence result.

Theorem 13. *Assume that for any $t > 0$, the semigroup $S(t)$ has a (unique) extension to a bounded linear map from V into H, and that*

$$|S(t)(F(x) - F(y))|_H \le b(t)|x - y|_H,$$
$$\|S(t)\,(B(x) - B(y))\,\|_{L_{(HS)}(\mathcal{H},H)} \le a(t)|x - y|_H$$

and

$$|S(t)F(x)|_H \le b(t)\,(1 + |x|_H)\,,$$
$$\|S(t)B(x)\|_{L_{(HS)}(\mathcal{H},H)} \le a(t)\,(1 + |x|_H)\,,$$

where

$$\int_0^T \left(b(t) + a^2(t)\right)\mathrm{d}t < \infty, \qquad \forall\, T > 0.$$

Then there is a unique adapted process u such

$$\sup_{0\le t\le T} \mathbb{E}\,|u(t)|^2_H < \infty, \qquad \forall\, T > 0,$$

and for all $t \ge 0$,

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u(s))\mathrm{d}s + \int_0^t S(t-s)F(u(s))\mathrm{d}M(s), \qquad \mathbb{P}\text{-}a.s.$$

Sketch of the proof. Let us fix a finite time horizon $T > 0$. Let $\mathcal{X}_T$ be the space of all square-integrable adapted processes $X\colon \Omega \times [0,T] \mapsto H$ such

$$[0,T] \ni t \to \mathbb{E}\,|X(t)|_H^2 \in \mathbb{R}$$

is continuous. On $\mathcal{X}_T$ consider the family of equivalent norms

$$\|X\|_\beta := \sup_{0\le t\le T} \mathrm{e}^{-\beta t}\sqrt{\mathbb{E}|X(t)|_H^2}, \qquad \beta > 0.$$

Then $\mathcal{X}_T$ equipped with $\|\cdot\|_\beta$ is a Banach space. Consider the mapping

$$\Psi(X)(t) = S(t)u_0 + \int_0^t S(t-s)F(X(s))\mathrm{d}s + \int_0^t S(t-s)B(X(s))\mathrm{d}M(s).$$

Then $\Psi\colon \mathcal{X}_T \mapsto \mathcal{X}_T$. Moreover, for β large enough Ψ is a contraction. Thus the desired conclusion follows from the Banach fixed point theorem. □

5.3. Typical example

As an example consider stochastic heat equation

$$\mathrm{d}u = (\Delta u + f(u))\,\mathrm{d}t + b(u)\mathrm{d}M, \qquad u(0) = u_0,$$

considered on a bounded region $\mathcal{O} \subset \mathbb{R}^d$ with 0-Dirichlet boundary conditions. Assume that the RKHS $\mathcal{H}$ of M is a subset of $H = L^2(\mathcal{O})$, and $f, b\colon \mathbb{R} \mapsto \mathbb{R}$. Then we are in the framework of equation (5), with A being the Laplace operator on $H = L^2(\mathcal{O})$ with the Dirichlet boundary conditions, and F and B of the Nemytskii type operators

$$F(\psi)(x) = f(\psi(x)), \quad B(\psi)[\phi](x) = b(\psi(x))\phi(x),$$

for $\psi \in L^2(\mathcal{O})$, $\phi \in \mathcal{H}$, $x \in \mathcal{O}$.

Note the if $f\colon \mathbb{R} \mapsto \mathbb{R}$ is Lipschitz then the corresponding $F\colon L^2(\mathcal{O}) \mapsto L^2(\mathcal{O})$ is Lipchitz as well. As far as B is concerned, then $B(u)$ is a bounded linear operator from $L^2(\mathcal{O})$ to $L^2(\mathcal{O})$ if and only if $b(u) \in L^\infty(\mathcal{O})$. Therefore B is an $L(L^2(\mathcal{O}), L^2(\mathcal{O}))$-valued if and only if b is bounded. Assume now that b is bounded. Note that

$$B\colon L^2(\mathcal{O}) \mapsto L(L^2(\mathcal{O}), L^2(\mathcal{O}))$$

is continuous if and only if b is constant. For

$$\begin{aligned}\|B(u) - B(v)\|^2_{L(L^2(\mathcal{O}),L^2(\mathcal{O}))} &= \sup_{|\psi|_{L^2(\mathcal{O})}\le 1} \int_{\mathcal{O}} \left(b(u(x)) - b(v(x))\right)^2 \psi^2(x)\mathrm{d}x \\ &= \|b(u) - b(v)\|^2_\infty.\end{aligned}$$

Let $a_1 \ne a_2 \in \mathbb{R}$ and let $\mathcal{O}_\varepsilon$ be a subset of $\mathcal{O}$ of Lebesgue measure ε. Take $u_\varepsilon(x) = a_1\chi_{\mathcal{O}_\varepsilon}(x)$ and $v_\varepsilon(x) = a_2\chi_{\mathcal{O}_\varepsilon}(x)$ for $x \in \mathcal{O}$. Then $\|b(u_\varepsilon) - b(v_\varepsilon)\|_\infty = |b(a_1) - b(a_2)|$. On the other hand

$$|u_\varepsilon - v_\varepsilon|_{L^2(\mathcal{O})} = |a_1 - a_2|\sqrt{\varepsilon}.$$

Note that $B(u)$ is Hilbert–Schmidt if and only if $b \equiv 0$.

Let G be the Green kernel. Then

$$
\begin{aligned}
&\|S(t)\left(B(u)-B(v)\right)\|_{L(L^2(\mathcal{O}),L^2(\mathcal{O}))}\\
&= \sup_{\|\psi\|_{L^2(\mathcal{O})}\le 1}\int_{\mathcal{O}}\psi(x)S(t)\left(B(u)-B(v)\right)(x)\mathrm{d}x\\
&= |S(t)\left(B(u)-B(v)\right)|_{L^\infty(\mathcal{O})}\\
&= \sup_{x\in\mathcal{O}}\int_{\mathcal{O}} G(t,x,y)|b(u(y))-b(v(y))|\mathrm{d}y\\
&\le |b(u)-b(v)|_{L^2(\mathcal{O})}\sup_{x\in\mathcal{O}}\left(\int_{\mathcal{O}} G^2(t,x,y)\mathrm{d}y\right)^{1/2}.
\end{aligned}
$$

Recall that d is the dimension of the domain $\mathcal{O}$. Taking into account the Arronson estimates for the Green kernel, see Arronson [2], Eidelman [8], Solonnikov [27] and [28],

$$
G(t,x,y)\le C_1 t^{-d/2}\exp\left\{-C_2\frac{|x-y|^2}{t}\right\}
$$

we arrive at the estimate

$$
\sup_{x\in\mathcal{O}}\left(\int_{\mathcal{O}} G^2(t,x,y)\mathrm{d}y\right)^{1/2}\le C_3 t^{-d/4}.
$$

On the other hand

$$
\begin{aligned}
&\|S(t)\left(B(u)-B(v)\right)\|^2_{L_{(HS)}(L^2(\mathcal{O}),L^2(\mathcal{O}))}\\
&\quad=\int_{\mathcal{O}}\int_{\mathcal{O}} G^2(t,x,y)|b(u(y))-b(v(y))|^2\mathrm{d}y\mathrm{d}x\\
&\quad\le |b(u)-b(v)|^2_{L^2(\mathcal{O})}\sup_{y\in\mathcal{O}}\int_{\mathcal{O}} G^2(t,x,y)\mathrm{d}x\le C_3t^{-d/2}\,|b(u)-b(v)|^2_{L^2(\mathcal{O})}.
\end{aligned}
$$

Therefore, if $d=1$, then the existence of the solution follows from Theorem 13.

For $d=1$ on can also use the following arguments, let (e_k) be the orthonormal basis of $L^2(\mathcal{O})$ of eigenvectors of Δ and let $(-\lambda_k)$ be the corresponding sequence of eigenvalues. Then

$$
\begin{aligned}
&\|S(t)\left(B(u)-B(v)\right)\|^2_{L_{(HS)}(L^2(\mathcal{O}),L^2(\mathcal{O}))}\\
&=\sum_{k,j}\langle S(t)\left((b(u)-b(v))e_k\right),e_j\rangle^2_{L^2(\mathcal{O})}=\sum_{k,j}\langle (b(u)-b(v))e_k,S(t)e_j\rangle^2_{L^2(\mathcal{O})}\\
&=\sum_{k,j}\mathrm{e}^{-2\lambda_j t}\langle (b(u)-b(v))e_k,e_j\rangle^2_{L^2(\mathcal{O})}=\sum_j \mathrm{e}^{-2\lambda_j t}\,|(b(u)-b(v))e_j|^2_{L^2(\mathcal{O})}\\
&\le\sum_j \mathrm{e}^{-2\lambda_j t}\,|b(u)-b(v)|^2_{L^2(\mathcal{O})}.
\end{aligned}
$$

Since λ_j is of order j^2, there is a constant C such that $\sum_j \mathrm{e}^{-2\lambda_j t}\le Ct^{-1/2}$.

6. Predictability

It is know that if we integrate with respect to a Wiener process, then it is enough to assume that the integrand is measurable, adapted and locally square integrable with respect to time with probability 1. The following examples shows also that in general the integrand should be predictable or the stochastic integration differs from the Lebesgue–Stieltjes integral in the case of the integration with respect to a process with bounded variation.

Example 1. Let Π be a Poisson process with intensity λ. Let τ be the moment of the first jump of Π. Then $\chi_{[0,\tau)}$ is a measurable adapted process. We note that $\chi_{[0,\tau)}$ is not predictable. Clearly a predictable process is $\chi_{[0,\tau]}$. Note that $\chi_{[0,\tau]}$ is a modification of $\chi_{[0,\tau)}$.

Let $\widehat{\Pi}$ be the compensated process. Then, if we treat the integral as the Lebesgue–Stieltjes integral with respect to a process $\widehat{\Pi}$ with bounded variation, then

$$X(t) := \int_0^t \chi_{[0,\tau)}(s)\mathrm{d}\widehat{\Pi}(s) = -\lambda t \wedge \tau + \int_0^t \chi_{[0,\tau)}(s)\mathrm{d}\Pi(s) = -\lambda t \wedge \tau.$$

Note that X is not a martingale, nor a local martingale. It has decreasing trajectories. On the other hand, the process

$$Y(t) := \int_0^t \chi_{[0,\tau]}(s)\mathrm{d}\widehat{\Pi}(s) = -\lambda t \wedge \tau + \int_0^t \chi_{[0,\tau]}(s)\mathrm{d}\Pi(s) = -\lambda t \wedge \tau + \chi_{\{t\geq\tau\}}$$

is a martingale.

Obviously if X is càdlàg and adapted, then $X(t-)$, $t \geq 0$, is predictable. Unfortunately, in important cases X does not have a càdlàg modification. It can be mean square continuous, that is

$$\lim_{s\uparrow t} \mathbb{E}\,|X(t) - X(s)|_H^2 = 0, \qquad \forall\, t \geq 0.$$

Then there is its predictable modification due to the following general result (see Gikhmann and Skorokhod [9] or Peszat and Zabczyk [22], Prop. 3.21).

Theorem 14. *Any measurable stochastically continuous adapted process has a predictable modification.*

The problem of predictability of integrands is treated in more details by Albeverio, Mandrekar, and Rüdiger [1] and by Mandrekar and Rüdiger [15], [16], and [17].

7. Poisson random measures

Let $(E,\mathcal{E})$ be a measurable space. Let π be the Poisson random measure on $[0,\infty)\times E$ with the intensity measure $\mathrm{d}t\mu(\mathrm{d}z)$, and let $\widehat{\pi}(\mathrm{d}t,\mathrm{d}\xi) := \pi(\mathrm{d}t,\mathrm{d}\xi) - \mu(\mathrm{d}\xi)\mathrm{d}t$ be the compensated measure. We would like to integrate with respect to π a random field $X(t,\xi)$, $t\geq 0$, $\xi\in E$. Here $X(t,\xi)$ can be real valued or taking values in a Banach space V. Define the filtration

$$\mathcal{F}_t := \sigma\left(\pi([0,s]\times A)\colon 0\leq s\leq t,\ A\in\mathcal{E}\right), \qquad t\geq 0.$$

In the first step we integrate *simple fields*; that is the fields of the form

$$X=\sum_{j=1}^{K} X_j\chi_{(t_j,t_{j+1}]}\chi_{A_j},$$

where $K\in\mathbb{N}$, $\mu(A_j)<\infty$, X_j are bounded and X_j is $\mathcal{F}_{t_j}$-measurable. Namely we write

$$I^{\pi}_t(X) := \int_0^t\int_E X(s,\xi)\pi(\mathrm{d}s,\mathrm{d}\xi) = \sum_{j=1}^{K} X_j\pi\left((t_j\wedge t, t_{j+1}\wedge t]\times A_j\right).$$

In the same way we define $I^{\widehat{\pi}}_t(X)$. Observe, that in the sum on above, X_j does not depend on the random variable $\pi\left((t_j\wedge t, t_{j+1}\wedge t]\times A_j\right)$ having the Poisson distribution with intensity $\mu(A_j)\left(t\wedge t_{j+1}-t\wedge t_j\right)$. Therefore

$$\begin{aligned}\mathbb{E}I^{\pi}_t(X) &= \sum_{j=1}^{K}\mathbb{E}\,X_j\mu(A_j)\left(t\wedge t_{j+1}-t\wedge t_j\right)\\ &= \mathbb{E}\int_0^t\int_E X(s,\xi)\mathrm{d}s\mu(\mathrm{d}\xi).\end{aligned}$$

Next since each X_j is bounded $I^{\pi}_t(X)$ has all moments finite. Obviously

$$\mathbb{E}\left|I^{\pi}_t(X)\right|_V \leq \mathbb{E}I^{\pi}_t\left(|X|_V\right).$$

Assume now that the integrand is real-valued.

Lemma 1. *For any simple real-valued field X, the process $I^{\widehat{\pi}}_t(X)$, $t\geq 0$, is a square integrable real valued martingale with the quadratic variation*

$$\left[I^{\widehat{\pi}}(X), I^{\widehat{\pi}}(X)\right]_t = I^{\pi}_t(X^2), \qquad t\geq 0.$$

We have now the following result of Saint Loubert Bié. It plays a fundamental role in the L^p-theory of SPDEs with Lévy noise, see e.g. [22] and the original paper by Saint Loubert Bié [26].

Lemma 2. *Let $p\in[1,2]$. Then there is a constant C_p such that for arbitrary simple field X and $T>0$,*

$$\mathbb{E}\sup_{0\leq t\leq T}\left|I^{\widehat{\pi}}_t(X)\right|^p \leq C_p\mathbb{E}\int_0^T\int_E |X(t,\xi)|^p\,\mathrm{d}t\mu(\mathrm{d}\xi).$$

Proof. By the Burkholder–Davis–Gundy inequality

$$\mathbb{E}\sup_{0\leq t\leq T}\left|I^{\widehat{\pi}}_t(X)\right|^p \leq C_p\mathbb{E}\left[I^{\widehat{\pi}}(X), I^{\widehat{\pi}}(X)\right]_T^{p/2} = C_p\mathbb{E}\left(I^{\pi}_T(X^2)\right)^{p/2}.$$

Now

$$I^{\pi}_T(X^2) = \sum_{j=1}^{K} X_j^2\pi\left((t_j\wedge t, t_{j+1}\wedge t]\times A_j\right).$$

But $\pi\left((t_j\wedge t, t_{j+1}\wedge t]\times A_j\right)$ are non-negative integers! Therefore since $p/2\le 1$,

$$\left(\sum_{j=1}^K X_j^2\pi\left((t_j\wedge t, t_{j+1}\wedge t]\times A_j\right)\right)^{p/2} \le \sum_{j=1}^K |X_j|^p\pi\left((t_j\wedge t, t_{j+1}\wedge t]\times A_j\right).$$

Hence $\left(I^\pi_T(X^2)\right)^{p/2}\le I^\pi_T\left(|X|^p\right)$, and consequently

$$\mathbb{E}\left(I^\pi_T(X^2)\right)^{p/2}\le \mathbb{E}I^\pi_T\left(|X|^p\right)=\mathbb{E}\int_0^t\int_E |X(s,\xi)|^p\mathrm{d}s\mu(\mathrm{d}\xi).$$

□

Having defined the stochastic integral of a simple field we would like to extend it to a more general class of random fields. Namely, given $T<\infty$, we denote by $\mathcal{P}_{[0,T]}$ the σ-field of predictable sets in $[0,T]\times\Omega$. Define

$$\mathcal{L}^p_{\mu,T}:=L^p\left([0,T]\times\Omega\times E,\mathcal{P}_{[0,T]}\otimes\mathcal{E},\mathrm{d}t\mathbb{P}\mu\right).$$

The space $\mathcal{L}^p_{\mu,T}$ is equipped with the norm

$$\|X\|_{\mathcal{L}^p_{\mu,T}}=\left(\int_0^T\int_E\mathbb{E}\,|X(s,\xi)|^p\mathrm{d}s\mu(\mathrm{d}\xi)\right)^{1/p}.$$

The simple fields are dense in $\mathcal{L}^p_{\mu,T}$, yielding the following consequence of Lemmas 1 and 2.

Theorem 15. 1. *For $p\in[1,2]$ and $t\in[0,T]$ there is a unique extension of the stochastic integral $I^{\widehat{\pi}}_t$ to a bounded linear operator, denoted also by $I^{\widehat{\pi}}_t$, from $\mathcal{L}^p_{\mu,t}$ into $L^p(\Omega,\mathcal{F}_t,\mathbb{P})$.*

2. *There is a unique extension of the mapping $\mathcal{L}_0\ni X\mapsto I^\pi_t(X)\in L^1(\Omega,\mathcal{F}_t,\mathbb{P})$ to a bounded linear operator from $\mathcal{L}^1_{\mu,t}$ into $L^1(\Omega,\mathcal{F}_t,\mathbb{P})$. The value of this operator at X is given by*

$$\int_0^t\int_E X(s,\xi)\pi(\mathrm{d}s,\mathrm{d}\xi),$$

or by $I^\pi_t(X)$.

3. *For $X\in\mathcal{L}^1_{\mu,T}$ and $0\le s\le t\le T$,*

$$\mathbb{E}\left|I^{\widehat{\pi}}_t(X)-I^{\widehat{\pi}}_s(X)\right|\le c_1\int_s^t\int_E\mathbb{E}\,|X(r,\xi)|\mathrm{d}r\mu(\mathrm{d}\xi)$$

and

$$\mathbb{E}\,|I^\pi_t(X)-I^\pi_s(X)|\le\int_s^t\int_E\mathbb{E}\,|X(r,\xi)|\mathrm{d}r\mu(\mathrm{d}\xi).$$

Hence the processes $I^{\widehat{\pi}}(X)$ and $I^\pi(X)$ admit predictable modifications.

4. *If* $X \in \mathcal{L}^2_{\mu,T}$ *then* $\left(I_t^{\widehat{\pi}}(X),\, t\in[0,T]\right)$ *is a square integrable martingale. Moreover, for* $X, Y \in \mathcal{L}^2_{\mu,T}$ *and* $t\in[0,T]$, $[I^{\widehat{\pi}}(X), I^{\widehat{\pi}}(Y)]_t = I_t^{\pi}(XY)$.

As for the case of simple fields, we write $\int_0^t\int_E X(s,\xi)\widehat{\pi}(\mathrm{d}s,\mathrm{d}\xi)$ instead of $I_t^{\widehat{\pi}}(X)$.

7.1. Example of equations

Consider the following heat equation on a bounded region $\mathcal{O}\subset\mathbb{R}^d$;

$$\mathrm{d}u(t,x) = \Delta u(t,x)\mathrm{d}t + \int_S b(u(t,x),\sigma)\widehat{\pi}(\mathrm{d}t,\mathrm{d}x,\mathrm{d}\sigma), \qquad u(0,x)=u_0(x),$$

with homogeneous Dirichlet or Neumann boundary conditions. In the equation π is a Poisson random measure on $[0,\infty)\times\mathcal{O}\times S$, with intensity measure $\mathrm{d}t\mathrm{d}x\nu(\mathrm{d}\sigma)$, σ is a measure on a measurable space $(S,\mathcal{S})$. Equations of this type were investigated in e.g. [19, 20, 26, 22]. The mild formulation of our problem is

$$u(t,x) = \int_{\mathcal{O}} G(t,x,y)u_0(y)\mathrm{d}y + \int_0^t\int_{\mathcal{O}}\int_S G(t-s,x,y)b(u(s,y),\sigma)\widehat{\pi}(\mathrm{d}s,\mathrm{d}y,\mathrm{d}\sigma).$$

A much simple problem is when the random Poisson measure does not depend on space variable x;

$$\mathrm{d}u(t,x) = \Delta u(t,x)\mathrm{d}t + \int_S b(u(t,x),\sigma)\widehat{\pi}(\mathrm{d}t,\mathrm{d}\sigma), \qquad u(0,x)=u_0(x).$$

Its mild form is

$$u(t,x) = \int_{\mathcal{O}} G(t,x,y)u_0(y)\mathrm{d}y + \int_0^t\int_{\mathcal{O}}\int_S G(t-s,x,y)b(u(s,y),\sigma)\widehat{\pi}(\mathrm{d}s,\mathrm{d}\sigma).$$

Then, roughly speaking

$$\int_0^t\int_{\mathcal{O}}\int_S G(t-s,\cdot,y)b(u(s,y),\sigma)\widehat{\pi}(\mathrm{d}s,\mathrm{d}\sigma) = \int_0^t S(t-s)\mathrm{d}M(s),$$

where

$$M(s) = \int_0^t\int_S b(u(s),\sigma)\widehat{\pi}(\mathrm{d}s,\mathrm{d}\sigma), \qquad t\geq 0,$$

is a martingale. It turns out that in the first case the solution does not have a càdlàg modification in $L^2(\mathcal{O})$ whereas in the second case it does, see Section 9.

8. Impulsive white noise

Let $\mathcal{O}$ be an open not necessarily bounded domain in $\mathbb{R}^d$ (possibly $\mathcal{O} = \mathbb{R}^d$). Let π be a Poisson random measure on $[0,\infty) \times \mathcal{O} \times \mathbb{R}$ with intensity of jump measure $\mathrm{d}t\mathrm{d}x\nu(\mathrm{d}\sigma)$. Assume that $\int_{\mathbb{R}} \sigma^2 \wedge 1\nu(\mathrm{d}\sigma) < \infty$. Consider the distributions-valued process

$$Z(t) = \int_0^t \int_{\{|\sigma|<R\}} \sigma\widehat{\pi}(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma) + \int_0^t \int_{\{|\sigma|\geq R\}} \sigma\pi(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma).$$

Taking into account the representation

$$\pi(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma) = \sum \delta_{\tau_k,x_k,\sigma_k},$$

we obtain the following a bit formal expression for Z;

$$Z(t) = \left[\sum_{|\sigma_k|<R,\tau_k\leq t} \sigma_k\delta_{\tau_k,x_k} - t\int_{|\sigma|<R} \sigma \mathrm{d}x\nu(\mathrm{d}\sigma)\right] + \sum_{|\sigma_k|\geq R,T_k\leq t} \sigma_k\delta_{\tau_k,x_k}.$$

Intuitively, at random points (τ_k, x_k) at time and space Z gives random impulses of random size σ_k.

Remark 3. One can show that

$$M(t) = \int_0^t \int_{\{|\sigma|<R\}} \sigma\widehat{\pi}(\mathrm{d}s\mathrm{d}x\mathrm{d}\sigma)$$

is a square integrable martingale in a sufficiently large space, and that its RKHS equals

$$\mathcal{H} = L^2(\mathcal{O}, \mathcal{B}(\mathcal{O}), a_R\mathrm{d}x), \qquad a_R := \int_{\{|\sigma|<R\}} \sigma^2\nu(\mathrm{d}\sigma).$$

Thus, in particular, M takes values in any Hilbert space V such that the embedding $\mathcal{H} \hookrightarrow V$ is Hilbert–Schmidt.

Remark 4. The jump measure μ of Z is the image of the measure $\mathrm{d}x\nu(\mathrm{d}\sigma)$ under the transformation $\mathcal{O} \times \mathbb{R} \ni (x,\sigma) \mapsto \sigma\delta_x \in \mathcal{D}(\mathcal{O})$, where $\mathcal{D}(\mathcal{O})$ is the space of distribution on $\mathcal{O}$.

Therefore our definition is the following.

Definition 4. *Impulsive cylindrical (or white) noise with intensity of jumps measure* $\mathrm{d}x\nu(\mathrm{d}\sigma)$ is the Lévy process on the space of distributions with the Lévy measure μ being the image of $\mathrm{d}x\nu(\mathrm{d}\sigma)$ under the transformation $(x,\sigma) \mapsto \sigma\delta_x$.

Remark 5. Impulsive cylindrical process L takes values in a Hilbert space U provided that $\int_U |u|_U^2 \wedge 1\mu(\mathrm{d}u) < \infty$. Let $U = H^{-\alpha}$ be the Sobolev space of order $-\alpha$ with an $\alpha > d/2$. Then, by Sobolev embedding, $C :=$

$\sup_{x\in\mathcal{O}} |\delta_x|_{H^{-\alpha}} < \infty$. Therefore

$$\int_{H^{-\alpha}} |u|^2_{H^{-\alpha}} \wedge 1\mu(\mathrm{d}u) = \int_{\mathcal{O}}\int_{\mathbb{R}} \sigma^2 |\delta_x|_{H^{-\alpha}} \mathrm{d}x\nu(\mathrm{d}\sigma) \\ \leq C\int_{\mathbb{R}} \sigma^2 \wedge 1\nu(\mathrm{d}\sigma) < \infty.$$

Consequently, L takes values in $H^{-\alpha}$. For more details on SPDEs driven by impulsive cylindrical process we refer the reader to Mueller [19], Mytnik [20], or [22].

9. Regularity of stochastic convolution

We start this section with results on the lack of a càdlàg modification for SPDEs driven by a process whose jump measure is not supported on the state space. Then we present different tools useful for study regularity of stochastic convolutions.

9.1. Lack of càdlàg modification

As the following example shows in some cases the solution to linear stochastic evolution equation does not have a càdlàg modification.

Example 2. Let U and H be Hilbert spaces such that

(i) H is densely embedded into U.

(ii) One has

$$\int_0^T \|S(s)\|^2_{L_{(HS)}(H,H)}\, ds < \infty, \qquad \forall T > 0.$$

(iii) For any $t > 0$, $S(t)$ has a continuous extension to an operator $S(t) \in L(U, H)$.

(iv) For any $u \in U \setminus H$, $\lim_{t\downarrow 0} |S(t)u|_H = \infty$.

Let Z be a square integrable mean zero random variable in U with RKHS H, and let L be a compound Poisson process with Lévy measure ν which is the distribution of Z. Then

$$X(t) = \int_0^t S(t-s)\mathrm{d}L(s) = \sum_{\tau_n<t} S(t-\tau_n)Z_n,$$

where τ_n are the jump times of L and Z_j are independent copies of Z. Then, by (ii), $\sup_{t\leq T} \mathbb{E}\,|X(t)|^2_H < \infty$ but

$$\lim_{t\downarrow\tau_n} |X(t)|_H = \lim_{t\downarrow\tau_n} |S(t-\tau_n)Z_n|_H = \infty,$$

since Z_n take values in $U \setminus H$.

Explicitly, take $H = L^2(0,1)$, $U = W_0^{-1,2}(0,1)$, S the heat semigroup generated by the Laplace operator with Dirichlet boundary conditions, and $Z = \eta\delta_\xi$, where $\xi \in (0,1)$, and η is a mean zero random variable.

The following have ben proven by Brzezniak and Zabczyk [3], and Peszat and Zabczyk [22].

Theorem 16. *If the jump measure of the noise is not supported on E then the stochastic convolution does not have càdlàg trajectories in E.*

9.2. Factorisation

Stochastic integral with respect to the square integrable martingale as a square integrable martingale has a càdlàg modification. This is not always true for stochastic convolution processes

$$X(t) := \int_0^t S(t-s)\Psi(s)\mathrm{d}M(s), \qquad t \geq 0,$$

where the integrand depends on t.

One way to show the continuity of its trajectories is to use the so-called *Da Prato–Kwapień–Zabczyk factorisation*, see the original paper by Da Prato, Kwapień, and Zabczyk [6], or [22],

$$X(t) = \Gamma(1) I_\alpha(X_\alpha)(t),$$

where

$$X_\alpha(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} S(t-s)\psi(s)\mathrm{d}M(s), \qquad t \geq 0,$$

and I_α is the *fractional derivative* operator given by

$$I_\alpha \psi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} S(t-s)\psi(s)\mathrm{d}s,$$

and Γ is the Euler Γ-function. It is easy to show that

$$I_\alpha \in L(L^p(0,T;H), C([0,T];H))$$

provided that $1/q < \alpha < 1$.

For the Wiener integral it is usually not hard to show that X_α has trajectories in $L^q(0,T;H)$ with some $1/q < \alpha < 1$. Therefore the continuity of trajectories of X follows. However, for discontinuous Lévy process, X_α does not have trajectories in $L^q(0,T;H)$ with any $1/q < \alpha < 1$. This can been seen as a consequence of the Bichteler–Jacod estimate (see e.g. [18]).

9.3. Kotelenez regularity result

Kotelenez [14] proved the regularity of stochastic convolution

$$\int_0^t S(t-s)\mathrm{d}M(s), \qquad t \geq 0,$$

driven by an arbitrary square integrable martingale in H for a generalized contraction semigroup S. Recall that for any C_0-semigroup S there are constants $\beta > 0$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\|_{L(H,H)} \leq \beta \mathrm{e}^{\omega t}, \qquad t \geq 0. \tag{6}$$

If (6) holds with $\beta = 1$, then S is a *generalized contraction semigroup*. If moreover, $\omega \leq 0$, then S is a *contraction semigroup*.

We outline here the proof of Kotelenez result due to Hausenblas and Seidler [11]. Their method is based on the Nagy dilation theorem.

Theorem 17. (Nagy) *If S is a C_0-semigroup of contractions on H, then there is a Hilbert space $\tilde{H}$ containing H and a unitary group R on $\tilde{H}$ such that $S = PR$, where $P \in L(\tilde{H}, H)$ is a projection.*

Proof of the Kotelenez regularity result. If S is a generalized contraction semigroup then for ω large enough $\mathrm{e}^{-\omega t}S(t)$, $t \geq 0$, is a semigroup of contractions. Then by the Nagy theorem

$$\mathrm{e}^{-\omega t}S(t) = PR(t), \qquad t \geq 0,$$

where $P \in L(\tilde{H}, H)$ and R is a unitary C_0-group. Hence

$$\begin{aligned}\int_0^t S(t-s)\mathrm{d}M(s) &= \int_0^t \mathrm{e}^{\omega(t-s)}\mathrm{e}^{-\omega(t-s)}S(t-s)\mathrm{d}M(s)\\ &= \int_0^t \mathrm{e}^{\omega(t-s)}PR(t-s)\mathrm{d}M(s)\\ &= \mathrm{e}^{\omega t}PR(t)\int_0^t R(-s)\mathrm{d}M(s).\end{aligned}$$

Since

$$Y(t) := \int_0^t R(-s)\mathrm{d}M(s), \qquad t \geq 0,$$

is a square integrable martingale, Y has càdlàg trajectories in $\tilde{H}$ and consequently X has càdlàg trajectories in H as R is strongly continuous.

9.4. Criterion for the absence of discontinuities of the second kind

The following criterion of the Chentsov type (see [4]), follows from a certain more general result (see Gikhman and Skorokhod [9], Chapter 3). For its proof we refer the reader to Gikhman and Skorokhod [9], or [23].

Let $\xi = (\xi(t),\, t \in [0,T])$ be a separable process taking values in a metric space (U, ρ). We extent ξ on $\mathbb{R}$ putting $\xi(t) = \xi(0)$ for $t < 0$ and $\xi(t) = \xi(T)$ for $t \geq T$.

Theorem 18. *Assume that there are $p, r, K > 0$ such that for all $t \in [0,T]$ and $h > 0$,*

$$\mathbb{E}\left[\rho\left(\xi(t), \xi(t-h)\right)\rho\left(\xi(t), \xi(t+h)\right)\right]^p \leq Kh^{1+r}. \tag{7}$$

Then with probability 1, ξ has no discontinuities of the second kind. Moreover, for any $1 \leq q < 2p$,

$$\mathbb{E}\sup_{t,s\in[0,T]}\left(\rho(\xi(t), \xi(s))\right)^q \leq (2G)^q\mathbb{E}\left(\rho(\xi(T), \xi(0))\right)^q + R, \tag{8}$$

where $0 < r' < r$,

$$G = \sum_{n=1}^{\infty}(T2^{-n})^{r'/(2p)} < \infty, \tag{9}$$

and $R := 1 + \frac{q}{2p-q}\frac{K(2G)^{2p}T^{1+r-r'}}{1-2^{r'-r}}$.

The criterion yields the following result (see [23]) on the existence of a càdlàg in H solution to linear equation with the noise taking values in a bigger space $U \hookleftarrow H$. For more specific examples we refer the reader to [23].

Theorem 19. *Let X be the solution to the following linear equation*

$$\mathrm{d}X = AX\mathrm{d}t + \mathrm{d}Z,$$

where A is the generator of an exponentially stable analytic semigroup S on a Hilbert space H and Z is a pure jump Lévy process taking values in a Hilbert space $U = H_{-\rho}$ for a certain $\rho < 1/2$. Assume that the Lévy measure ν of Z satisfies $\nu(H_{-\rho} \setminus H) = 0$ and that

$$\int_H \left(|z|^2_{-\rho} + |z|^4_\varepsilon\right) \nu(\mathrm{d}z) < \infty$$

for a certain $\varepsilon > 0$. Then X has a càdlàg modification in H and

$$\mathbb{E} \sup_{0\le t\le T} |X(t)|^q_H < \infty, \qquad \forall\, T < \infty,\ \forall\, q \in [1, 4).$$

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