TIME REGULARITY FOR STOCHASTIC VOLTERRA EQUATIONS BY THE DILATION THEOREM

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Abstract. The dilation theorem of Nagy is applied to establish time regularity of the solutions to a class of stochastic evolutionary Volterra equations.

1. Introduction

This note is concerned with the path regularity of solutions to the following stochastic Volterra equation

$$X(t) = X_0 + \int_0^t v(t-s)AX(s)\mathrm{d}s + L(t), \qquad t \in [0,T], \tag{1}$$

where $(A, D(A))$ is a closed, densely defined operator on a real separable Hilbert space $(H, \langle \cdot, \cdot \rangle_H)$, L is a semimartingale with càdlàg (or continuous) trajectories in H and v is a locally integrable real-valued function. If the operator A is bounded then the existence of a regular in time solution to (1) can be obtained in a very simple way. Namely we have the following result. Actually this result is valid for any measurable H-valued function L. The semimartingale property of L is not used.

Proposition 1. *Assume that v is locally integrable, and that A is a bounded linear operator on H. Then for any $X_0 \in H$ there is a unique strong solution X to* (1). *Moreover, the solution is càdlàg (res. continuous) in t.*

Proof. Define $I_0(t) = X_0 + L(t)$, $t \geq 0$, and

$$I_{n+1}(t) = \int_0^t v(t-s)AI_n(s)\mathrm{d}s, \qquad t \geq 0, \ n = 0, 1, \ldots.$$

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Then each I_n is càdlàg (resp. continuous) in H. Given $\omega > 0$ and $f\colon [0,T] \mapsto H$, let $\|f\|_\omega := \sup_{t\in[0,T]} \mathrm{e}^{-\omega t}|f(t)|_H$. Then for $n = 0, 1, \ldots$,

$$\mathrm{e}^{-\omega t}\,|I_{n+1}(t)|_H \le \|A\|_{L(H)} \int_0^t \mathrm{e}^{-\omega(t-s)}|v(t-s)|\mathrm{e}^{-\omega s}\,|I_n(s)|_H\,\mathrm{d}s,$$

and consequently, we have

$$\begin{aligned}\|I_n\|_\omega &\le \|A\|_{L(H)} \sup_{0\le t\le T} \int_0^t \mathrm{e}^{-\omega(t-s)}|v(t-s)|\mathrm{e}^{-\omega s}|I_n(s)|_H\mathrm{d}s\\ &\le \|A\|_{L(H)} \int_0^T \mathrm{e}^{-\omega t}|v(t)|\mathrm{d}t\,\|I_n\|_\omega\\ &\le \left(\|A\|_{L(H)} \int_0^T \mathrm{e}^{-\omega t}|v(t)|\mathrm{d}t\right)^n \|x+L\|_\omega, \qquad n = 0, 1, \ldots.\end{aligned}$$

Taking ω big enough we obtain

$$\|A\|_{L(H)} \int_0^T \mathrm{e}^{-\omega t}|v(t)|\mathrm{d}t < 1.$$

Thus $\sum_{n=0}^\infty \|I_n\|_\omega < \infty$, and consequently the series $\sum_{n=0}^\infty I_n(t)$ converges in H uniformly in $t \in [0,T]$, to the unique solution having càdlàg (resp. continuous) trajectories. □

If the operator A is unbounded we should work rather with weak solutions.

Definition 1. Process X is a *weak solution* to (1) if for any $a^* \in D(A^*)$,

$$\langle X(t), a^*\rangle = \langle X_0 + L(t), a^*\rangle + \int_0^t v(t-s)\langle X(s), A^*a^*\rangle_H \mathrm{d}s, \qquad t \in [0,T].$$

Note that in the special case when A is the infinitesimal generator of a strongly continuous semigroup S, and $v \equiv 1$, then (1) becomes

$$\mathrm{d}X(t) = AX(t)\mathrm{d}t + \mathrm{d}L(t), \qquad t \in [0,T],$$

and the weak solution X is given by the stochastic convolution:

$$X(t) = S(t)X_0 + \int_0^t S(t-s)\mathrm{d}L(s), \qquad \in [0,T]. \tag{2}$$

If S is a semigroup of generalized contractions (see Definition 3), then one can prove the existence of a càdlàg (resp. continuous) modification of X, given by (2), using either the Kotelenez submartingale inequality [12] or, following Hausenblas and Seidler, the so-called dilation theorem of Nagy. The case of general strongly continuous semigroup is open with the exception of the case when L is a Wiener process when the so-called factorization method works (see the original paper [3] or [4, 5, 15]). Path regularity of solutions to linear evolution equations driven by Lévy process is studied in [1, 13, 15, 16].

The study of the regularity of solutions to the stochastic Volterra equation was initiated by Clément and Da Prato [2] and continued by Karczewska and Zabczyk [11] and Karczewska [8, 9, 10], mostly for Wiener perturbation.

In the note we prove the existence of a càdlàg version of X for several classes of functions v and self-adjoint operators A using the dilation theorem. Our aim is to prove the following result valid for an arbitrary semimartingale L.

Theorem 1. *Let $X_0 \in H$. Assume that A is a self-adjoint negative semi-definite operator on a Hilbert space H and that a locally integrable function v of at most exponential growth at infinity satisfies one of the following conditions*

(a) $v\colon (0,+\infty) \to \mathbb{R}$ *is nonincreasing and positive,*

(b) $v\colon [0,+\infty) \to \mathbb{R}$ *is a function of locally bounded variation $v(0) \geq 0$ and the generated measure* $\mathrm{d}v$ *is positive definite.*

If L is càdlàg (or continuous) semimartingale in H, then the weak solution to (1) *exists, is unique, and has a càdlàg (resp. continuous) modification.*

Example 1. Take $v \equiv 1$, then X is given by (2) and we recover a well known result for stochastic convolutions.

Example 2. Let $v(t) = t^{-\alpha}$ for some $\alpha \in (0,1)$. Then v is locally integrable, strictly decreasing, and positive. Therefore it is of the type (a) from Theorem 1.

Example 3. Let $v(t) = t$, $t \geq 0$. Then $v' \equiv 1$ is positive definite, and thus v is of type (b). In this case the solution is given in the following explicit form

$$X(t) = \cos(\sqrt{-A}t)X_0 + \int_0^t \cos(\sqrt{-A}(t-s))\mathrm{d}L(s), \qquad t \geq 0,$$

see Example 6 and Proposition 5 for more details.

Example 4. Assume that $v \in C^1([0,+\infty))$, v' is non-negative, non-increasing and concave. Then, by the Bochner theorem, $t \mapsto v'(|t|)$ is positive definite as required. In particular, if $v(0) \geq 0$, $v'(t) \geq 0$, $v''(t) \leq 0$ and $v'''(t) \geq 0$, then v is of type (b).

Example 5. Let $a, b > 0$, and let

$$v'(t) = \begin{cases} a - \frac{a}{b}t, & t \in [0,b], \\ 0, & t \geq b. \end{cases}$$

Then v' is non-negative, non-increasing and concave. Note that

$$v(t) = \begin{cases} v(0) + at - \frac{a}{2b}t^2, & t \in [0,b], \\ v(0) + \frac{ab}{2}, & t \geq b, \end{cases}$$

is of type (b) provided that $v(0) \geq 0$.

The proof of the theorem is given in Section 3. It is based on the following three ingredients: a representation formula similar to (2) in which the semigroup S is replaced by the so-called resolvent family R of (1); see Definition 2, the dilation theorem (see Theorem 2) for positive definite families of operators, both presented in Preliminaries. The third ingredient and the most essential part of the proof, is a result stating sufficient conditions under which the resolvent is positive definite.

2. Preliminaries

2.1. **Resolvents.** In this subsection A is a possibly unbounded densely defined closed operator on H.

Definition 2. A family $R(t)$, $t \ge 0$, of bounded linear operators on H is called *resolvent* to the equation

$$y(t) = f(t) + \int_0^t v(t-s)Ay(s)\mathrm{d}s, \qquad t \ge 0, \tag{3}$$

if the following conditions are satisfied

(i) R is strongly continuous on $\mathbb{R}_+ = [0, +\infty)$ and $R(0)$ equals the identity operator I,
(ii) $R(t)D(A) \subset D(A)$ and $AR(t)x = R(t)Ax$ for all $t \ge 0$ and $x \in D(A)$,
(iii) for all $x \in D(A)$,

$$R(t)x = x + \int_0^t v(t-s)AR(s)x\mathrm{d}s, \qquad t \ge 0. \tag{4}$$

The following elementary result is crucial for our proofs.

Proposition 2. *Assume that the resolvent to* (3) *exists and is denoted by R. If L is a semimartingale with càdlàg trajectories in H, then the weak solution to* (1) *exists and is given by the formula:*

$$X(t) = R(t)X_0 + \int_0^t R(t-s)\mathrm{d}L(s), \qquad t \in [0,T]. \tag{5}$$

A proof of Proposition 2 is similar to that of Theorem 9.15 of [15].

2.2. **Dilation theorem and regularity.**

Definition 3. We say that a family $R(t)$, $t \in \mathbb{R}$, of bounded linear operators on a Hilbert space $(H, \langle\cdot,\cdot\rangle_H)$ is *positive definite* if for any finite sequences (t_j) in $\mathbb{R}$, and (ψ_j) in H,

$$\sum_{j,k} \langle R(t_j - t_k)\psi_j, \psi_k\rangle_H \ge 0.$$

We say that the family R is *strongly continuous* if for any $\psi \in H$ and $t \in \mathbb{R}$,

$$\lim_{s\to t} |R(s)\psi - R(t)\psi|_H = 0.$$

Theorem 2. ***(Nagy dilation theorem)*** *Assume that R is a strongly continuous positive definite family of bounded operators on a Hilbert space H, such that $R(0)$ equals the identity operator I. Then there exist: a Hilbert space $\mathbb{H}$ containing isometrically H and a strongly continuous unitary group $U(t)$, $t \in R$, on $\mathbb{H}$, such that*

$$R(t)\psi = \Pi U(t)\psi, \qquad t \geq 0, \ \psi \in H,$$

where Π is the orthogonal projection of $\mathbb{H}$ onto H.

For the proof we refer to [14]. The dilation theorem is related to the regularity problem through the following proposition.

Proposition 3. *Assume that for some ω the family $e^{-\omega|t|}R(|t|)$, $t \in \mathbb{R}$ is positive definite, $t \to R(|t|)$ is strongly continuous, and $R(0) = I$. If L is càdlàg (or continuous) semimartingale than the process X,*

$$X(t) = R(t)X_0 + \int_0^t R(t-s)\mathrm{d}L(s), \qquad t \geq 0,$$

is càdlàg (or continuous) as well.

Proof. We use the Hausenblas and Seidler arguments from [7]. Namely, by the Nagy dilation theorem (Theorem 2),

$$\begin{aligned} X(t) &= R(t)X_0 + e^{\omega t}\int_0^t e^{-\omega s}e^{-\omega(t-s)}R(t-s)\mathrm{d}L(s) \\ &= R(t)X_0 + e^{\omega t}\int_0^t e^{-\omega s}\Pi U(t-s)\mathrm{d}L(s) \\ &= R(t)X_0 + e^{\omega t}\Pi U(t)\int_0^t e^{-\omega s}U(-s)\mathrm{d}L(s), \end{aligned}$$

where U is a C_0-unitary group in a Hilbert space $\mathcal{H} \hookleftarrow H$ and Π is a continuous projection of $\mathcal{H}$ into H. Therefore the result follows from the regularity of stochastic integrals. $\square$

3. Proof of Theorem 1

It is enough to show that the resolvent R exists and the operator valued mapping $\mathbb{R} \ni t \mapsto e^{-\omega|t|}R(|t|) \in L(H)$ is positive definite. The properties that $R(0) = I$ and R is strongly continuous are included in the definition of the resolvent, see Definition 2. For doing this we will analyze first a close relation between the properties of Eq. (1) and the following family of scalar Volterra equations

$$s(t) + \mu\int_0^t v(t-\tau)s(\tau)\mathrm{d}\tau = 1, \qquad t \geq 0, \tag{6}$$

parametrized by elements μ of the spectrum $\sigma(-A)$ of the operator $-A$.

3.1. **Scalar Volterra equation.** Assume that $s(t;\mu)$, $t\geq 0$, solves (6) and note that if μ is an eigenvalue of $-A$ corresponding to the eigenvector x, then $R(t)x = s(t;\mu)x$, $t\geq 0$, solves the resolvent equation (4). Therefore heuristically $R(t)=s(t;-A)$, $t\geq 0$.

Let us observe that taking in Proposition 1, $H=\mathbb{R}$, $B=-\mu$, $l\equiv 0$ and $X_0=1$ we obtain the following.

Proposition 4. *Assume that v is locally integrable. Then for any $\mu\in\mathbb{R}$ there is a unique solution $s(t;\mu)$, $t\geq 0$, to* (6)*. Moreover, $s(t;\mu)$ is a continuous function of t.*

A proof of the following result can be found in Corollary 1.2 and Theorem 1.1 of [17], see also Theorem 1 from [6].

Proposition 5. *Assume that A is self-adjoint negative semi-definite with the associated spectral measure $E(\tau)$, $\tau\in\sigma(A)$. Then the following assertions are valid.*
(*a*) *Let $v\colon[0,+\infty)\to\mathbb{R}$ be a function of bounded variation such that $\mathrm{d}v$ is a positive definite measure. Then for any $\mu\geq 0$ and $t\geq 0$, $|s(t;\mu)|\leq 1$. Moreover, the resolvent $R(t)$, $t\geq 0$, to* (3) *exists, and is given by the formula*

$$R(t)=s(t;-A)=\int_{\sigma(A)} s(t;-\mu)E(\mathrm{d}\mu). \tag{7}$$

Finally, R is differentiable and

$$\|R(t)\|_{L(H)}\leq 1,\quad \left|\frac{d}{\mathrm{d}t}R(t)x\right|\leq \mathrm{Var}\, a\,\big|_0^t\,|Ax|_H\,,\qquad t\geq 0,\ a\in D(A).$$

(*b*) *Let v be locally integrable, non-negative and non-increasing. Then for any $\mu\geq 0$ and $t\geq 0$, $0\leq s(t;\mu)\leq 1$. Moreover, the resolvent $R(t)$, $t\geq 0$, to* (3) *exists, it is given by* (7)*, it is self-adjoint and satisfies $0\leq\langle R(t)x,x\rangle_H\leq |x|^2_H$, $x\in H$.*

In some cases the solution $s(|\cdot|;\mu)$ can be found explicitly.

Example 6. If $v\equiv 1$ then $s(t;\mu)=e^{-\mu t}$, $t\geq 0$. If $v(t)=e^{-t}$, then

$$s(t;\mu)=(1+\mu)^{-1}[1+\mu e^{-(1+\mu)t}],\ t\geq 0.$$

Let $v(t)=t$, $t\geq 0$, be as in Example 3. Then for $\mu\geq 0$, $s(t;\mu)=\cos(\sqrt{\mu}t)$.

Proposition 6. *Assume that the resolvent is given by* (7) *and that there is an $\omega\geq 0$ such that for each $\mu\in\sigma(-A)$, $\mathbb{R}\ni t\mapsto \mathrm{e}^{-\omega|t|}s(|t|;\mu)$ is positive definite. Then*

$$\mathrm{e}^{-\omega|t|}R(|t|)=\mathrm{e}^{-\omega|t|}s(|t|;-A),\qquad t\in\mathbb{R},$$

is positive definite.

Proof. For a fixed $h \in H$ consider the projection $U\psi = \langle \psi, h\rangle_H h$. Let $\tilde{s}(t) = \mathrm{e}^{-\lambda|t|}s(|t|;\mu)$. Then the operator valued mapping $V(t) = \tilde{s}(t)U$, $t \in \mathbb{R}$, is positive definite. In fact

$$\begin{aligned}\sum_{i,j}\langle V(t_i - t_j)\psi_i, \psi_j\rangle_H &= \sum_{i,j}\langle U\psi_i, \psi_j\rangle_H \tilde{s}(t_i - t_j)\\ &= \sum_{i,j}\langle \psi_i, h\rangle_H \langle \psi_j, h\rangle_H \tilde{s}(t_i - t_j) \geq 0.\end{aligned}$$

This can be easily generalized to finite dimensional orthogonal projections and then to arbitrary projections. The spectral decomposition formula (7) valid also for $\tilde{R}$ completes the proof. □

3.2. **Positive definiteness of** $\mathrm{e}^{-\omega|\cdot|}s(|\cdot|;\mu)$**.** Assume that $f\colon [0,+\infty) \mapsto \mathbb{R}$ is measurable and such that $[0,+\infty) \ni t \mapsto \mathrm{e}^{-\omega t}\ |f(t)| \in \mathbb{R}$ is integrable. Write

$$I(f;\omega + \mathrm{i}\beta) := \int_0^{+\infty} \mathrm{e}^{-\omega t}\left(\omega \cos\beta t + \beta \sin \beta t\right) f(t)\mathrm{d}t.$$

Given $\mu \in \mathbb{R}$, let $s(t;\mu)$, $t \geq 0$, be the solution to the scalar Volterra equation (6). The following lemmas constitute the most essential part of the proof of the main theorem.

Lemma 1. *Let $\mu \in \mathbb{R}$. Assume that the functions $[0,+\infty) \ni t \mapsto s(t;\mu) \in \mathbb{R}$ and $[0,+\infty) \ni t \mapsto v(t) \in \mathbb{R}$ have at most exponential growth. Let $\omega > 0$ be such that $\mathrm{e}^{-t\omega}s(t;\mu)$ and $\mathrm{e}^{-t\omega}v(t)$ decay exponentially at $+\infty$. If*

(8) $$\omega + \mu I(v;\omega + \mathrm{i}\beta) > 0, \qquad \forall\, \beta \geq 0,$$

then the function $\mathbb{R} \ni t \mapsto \mathrm{e}^{-\omega|t|}s(|t|;\mu)$ is positive definite.

Proof. Define $\tilde{s}(t) = \mathrm{e}^{-\omega|t|}s(|t|;\mu)$, $t \in \mathbb{R}$. By the Bochner theorem $\tilde{s}$ is positive definite if and only if its Fourier transform

$$\widehat{\tilde{s}}(\beta) := \int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\beta t}\tilde{s}(t)\mathrm{d}t, \qquad \beta \in \mathbb{R},$$

is a non-negative function. Let

$$\widehat{s}_+(\beta) := \int_0^{+\infty} \mathrm{e}^{-i\beta t - \omega t}s(t;\mu)\mathrm{d}t, \qquad \beta \in \mathbb{R}.$$

Clearly

(9) $$\widehat{\tilde{s}}(\beta) = \widehat{s}_+(\beta) + \widehat{s}_+(-\beta) = \widehat{s}_+(\beta) + \overline{\widehat{s}_+(\beta)} = 2\mathrm{Re}\,\widehat{s}_+(\beta).$$

Let $\lambda = \omega + \mathrm{i}\beta$, and let $v_+(t)$ equals $v(t)$ if $t \geq 0$ and 0 if $t < 0$. From the convolution equation

$$\widehat{s}_+(\beta) + \mu\widehat{v}_+(\lambda)\widehat{s}_+(\beta) = \frac{1}{\lambda},$$

where $\widehat{v}_+(\lambda)$ is the Fourier transform of v_+ calculated at λ.

Assume that

$$1+\mu\widehat{v}_+(\lambda)\neq 0. \tag{10}$$

Note that as $\omega>0$, then $\lambda=\omega+\mathrm{i}\beta\neq 0$. Therefore

$$\widehat{s}_+(\beta)=\frac{1}{\lambda\left(1+\mu\widehat{v}_+(\lambda)\right)},$$

and hence, by (9) we have

$$\begin{aligned}\widehat{\widetilde{s}}(\beta)&=\frac{1}{\lambda\left(1+\mu\widehat{v}_+(\lambda)\right)}+\frac{1}{\overline{\lambda}\left(1+\mu\overline{\widehat{v}_+(\lambda)}\right)}=2\mathrm{Re}\,\frac{1}{\lambda\left(1+\mu\widehat{v}_+(\lambda)\right)}\\ &=\frac{2}{\left|\lambda\left(1+\mu\widehat{v}_+(\lambda)\right)\right|^2}\,\mathrm{Re}\,\overline{\lambda}\left(1+\mu\widehat{v}_+(\overline{\lambda})\right).\end{aligned}$$

Clearly

$$\begin{aligned}\mathrm{Re}\,\left(\overline{\lambda}\left(1+\mu\widehat{v}_+(\lambda)\right)\right)&=\mathrm{Re}\,\left(\left(\omega-\mathrm{i}\beta\right)\left(1+\mu\widehat{v}_+(\omega-\mathrm{i}\beta)\right)\right)\\ &=\omega+\mu\mathrm{Re}\,(\omega-\mathrm{i}\beta)\widehat{v}(\omega-\mathrm{i}\beta).\end{aligned}$$

Next, as $v_+(t)=0$ for $t<0$, we have

$$\begin{aligned}&(\omega-\mathrm{i}\beta)\,\widehat{v}\,(\omega-\mathrm{i}\beta)=\int_0^{+\infty}(\omega-\mathrm{i}\beta)\,\mathrm{e}^{-(\omega-\mathrm{i}\beta)t}v(t)\mathrm{d}t\\ &=\int_0^{+\infty}\mathrm{e}^{-\omega t}\left[\left(\omega\cos\beta t+\beta\sin\beta t\right)+\mathrm{i}\left(\omega\sin\beta t-\beta\cos\beta t\right)\right]v(t)\mathrm{d}t.\end{aligned}$$

Thus

$$\mathrm{Re}\,\left(\overline{\lambda}\left(1+\mu\widehat{v}_+(\lambda)\right)\right)=\omega+\mu I(v;\omega+\mathrm{i}\beta).$$

To finish the prove note that (8) implies that

$$\omega+\mu I(v;\omega+\mathrm{i}\beta)>0,\qquad\forall\,\beta\in\mathbb{R},$$

and hence in particular it guarantees (10). □

Lemma 2. *If f is differentiable and $[0,+\infty)\ni t\mapsto\mathrm{e}^{-\omega t}\,|f'(t)|\in\mathbb{R}$ is integrable, then*

$$I(f;\omega+\mathrm{i}\beta)=f(0)+\int_0^{+\infty}f'(t)\mathrm{e}^{-\omega t}\cos\beta t\mathrm{d}t.$$

Proof. Then since

$$\begin{aligned}\int_0^t\mathrm{e}^{-\omega s}\cos\beta s\mathrm{d}s&=\frac{\beta\sin\beta t-\omega\cos\beta t}{\omega^2+\beta^2}\mathrm{e}^{-\omega t}=:g_1(t),\\ \int_0^t\mathrm{e}^{-\omega s}\sin\beta s\mathrm{d}s&=\frac{-\omega\sin\beta t-\beta\cos\beta t}{\omega^2+\beta^2}\mathrm{e}^{-\omega t}=:g_2(t),\end{aligned}$$

we have

$$
\begin{aligned}
I(f;\omega+\mathrm{i}\beta) &= \int_0^{+\infty} \left(\alpha g_1'(t)+\beta g_2'(t)\right) f(t)\mathrm{d}t \\
&= -\left(\alpha g_1(0)-\beta g_2(0)\right) f(0) - \int_0^{+\infty} f'(t)\left[\alpha g_1(t)+\beta g_2(t)\right]\mathrm{d}t \\
&= f(0) + \int_0^{+\infty} f'(t)\mathrm{e}^{-\alpha t}\cos\beta t\mathrm{d}t.
\end{aligned}
$$

□

Lemma 3. *If $f\colon [0,+\infty)\mapsto[0,+\infty)$ is non-increasing then $I(f;\omega+\mathrm{i}\beta)\geq 0$ for all $\omega>0$ and $\beta\geq 0$.*

Proof. Clearly $[0,+\infty)\ni t\mapsto \mathrm{e}^{-\omega t}\,|f(t)|\in\mathbb{R}$ is integrable. Since f can be represented as a limit of increasing sequence of differentiable functions with compact supports we can assume that f itself is differentiable and satisfies the assumptions of Lemma 2. Then $f'(t)\leq 0$ and $\omega\geq 0$, by Lemma 2, we have

$$
\begin{aligned}
I(f;\omega+\mathrm{i}\beta) &= f(0)+\int_0^{+\infty} f'(t)\mathrm{e}^{-\omega t}\cos\beta t\mathrm{d}t \\
&\geq f(0)+\int_0^{+\infty} f'(t)\mathrm{d}t = f(0)-f(0)+\lim_{T\to+\infty} f(T)\geq 0.
\end{aligned}
$$

□

Lemma 4. *Assume that f is differentiable, $f(0)\geq 0$, and the function $t\mapsto f'(|t|)$ is positive definite and bounded from below. Then $I(f;\omega+\mathrm{i}\beta)\geq 0$ for all β.*

Proof. Since $t\mapsto f'(|t|)$ is positive definite it has maximum at 0. Thus $t\mapsto f'(|t|)$ is bounded. Thus in particular $[0,+\infty)\ni t\mapsto \mathrm{e}^{-\omega t}\left(|f(t)|+|f'(t)|\right)\in\mathbb{R}$ is integrable. By Lemma 2,

$$
\begin{aligned}
I(f;\omega+\mathrm{i}\beta) &= f(0)+\int_0^{+\infty}\cos\beta t\mathrm{e}^{-\omega t} f'(t)\mathrm{d}t \\
&= f(0)+\frac{1}{2}\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\beta t}\mathrm{e}^{-\omega|t|}f'(|t|)\mathrm{d}t.
\end{aligned}
$$

Since $\int_{-\infty}^{+\infty}\mathrm{e}^{\mathrm{i}\beta t}\mathrm{e}^{-\omega|t|}f'(|t|)\mathrm{d}t\geq 0$ as the Fourier transform of the convolution of two positive measures: up to a constant α stable and whose Fourier transform is $t\mapsto v'(|t|)$, the desired conclusion follows. □

3.3. **End of the proof.** Taking into account Propositions 2, 5 and 6, it is enough to show that there is an $\omega>0$ such that for any $\mu\geq 0$, the function $\mathbb{R}\ni t\mapsto \tilde{s}(t)=s(|t|;\mu)\in\mathbb{R}$ is positive definite. Assume that v is non-negative and decreasing. Then the desired property follows from Lemmas 1 and 3. If v is of type (b), then $\tilde{s}$ is positive definite, by Lemmas 1 and 4.

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