PASSIVE TRACER IN A FLOW CORRESPONDING TO TWO DIMENSIONAL STOCHASTIC NAVIER STOKES EQUATIONS

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Abstract. In this paper we prove the law of large numbers and central limit theorem for trajectories of a particle carried by a two dimensional Eulerian velocity field. The field is given by a solution of a stochastic Navier–Stokes system with a non-degenerate noise. The spectral gap property, with respect to Wasserstein metric, for such a system has been shown in [9]. In the present paper we show that a similar property holds for the environment process corresponding to the Lagrangian observations of the velocity. In consequence we conclude the law of large numbers and the central limit theorem for the tracer. The proof of the central limit theorem relies on the martingale approximation of the trajectory process.

1. Introduction

Consider the Navier–Stokes equations (N.S.E.) on a two dimensional torus $\mathbb{T}^2$,

$$
\begin{aligned}
&\partial_t\vec{u}(t,x)+\vec{u}(t,x)\cdot\nabla_x\vec{u}(t,x)=\Delta_x\vec{u}(t,x)-\nabla_x p(t,x)+\vec{F}(t,x),\\
&\nabla\cdot\vec{u}(t,x)=0,\\
&\vec{u}(0,x)=\vec{u}_0(x).
\end{aligned}
\tag{1.1}
$$

The two dimensional vector field $\vec{u}(t,x)$ and scalar field $p(t,x)$ over $[0,+\infty)\times\mathbb{T}^2$, are called an Eulerian velocity and pressure, respectively. The forcing $\vec{F}(t,x)$ is assumed to be a Gaussian white noise in t, homogeneous and sufficiently regular in x defined over a certain probability space $(\Omega,\mathcal{F},\mathbb{P})$. The trajectory of a tracer particle is defined as the solution of the ordinary differential equation (o.d.e.)

$$
\frac{dx(t)}{dt}=\vec{u}(t,x(t)),\quad x(0)=x_0,
\tag{1.2}
$$

where $x_0\in\mathbb{R}^2$. Thanks to well known regularity properties of solutions of N.S.E, see e.g. [23], $\vec{u}(t,x)$ possesses continuous modification in x for any $t>0$. However, since $\vec{u}(t,x)$ needs not be Lipschitz in x, the equation might not define $x(t)$, $t\geq 0$, as a stochastic process over $(\Omega,\mathcal{F},\mathbb{P})$, due to possible non-uniqueness of solutions. In our first result we

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construct a solution process (see Proposition 4.6) and show (see Corollary 4.4) that the law of any process satisfying (1.2) and adapted to the natural filtration of $\vec{u}$ is uniquely determined.

The main objective of this paper is to study ergodic properties of the trajectory process. We prove, see part 1) of Theorem 3.5, that the Stokes drift vanishes, i.e.

$$\lim_{t\to+\infty}\frac{x(t)}{t}=0, \tag{1.3}$$

where the limit above is understood in probability. A similar result for a Markovian and Gaussian velocity field $\vec{u}$ (that need not be a solution of a N.S.E.) that decorrelates sufficiently fast in time has been considered in [15]. Next, we investigate the size of "typical fluctuations" of the trajectory around its mean. We prove, see part 3) of the theorem, that

$$Z(t):=\frac{x(t)}{\sqrt{t}}\Rightarrow Z,\qquad \text{as } t\to+\infty \tag{1.4}$$

where Z is a random vector with normal distribution $\mathcal{N}(0,D)$ and the convergence is understood in law. Moreover, we show that the asymptotic variance of $Z(t)$, as $t\to+\infty$, exists and coincides with the covariance matrix D. The question of the law of the iterated logarithm is addressed in our companion paper, see [16].

In our approach a crucial role is played by the *Lagrangian process*

$$\vec{\eta}(t,x):=\vec{u}(t,x(t)+x),\quad t\ge 0,\ x\in\mathbb{T}^2$$

that describes the environment from the vantage point of the moving particle. It turns out that its rotation in x,

$$\omega(t,x)=\operatorname{rot}\vec{\eta}(t,x):=\partial_2\eta_1(t,x)-\partial_1\eta_2(t,x),\quad t\ge 0,\ x\in\mathbb{T}^2,$$

satisfies a stochastic partial differential equation (s.p.d.e.) (4.1) that is similar to the stochastic N.S.E. in the vorticity formulation, see (3.1). The position $x(t)$ of the particle at time t, can be represented as an additive functional of the Lagrangian process, i.e.

$$x(t)=\int_0^t\psi_*(\omega(s))ds,$$

see the begining of Section 6 for the definition of ψ_*. Then, (1.3) and (1.4) become the statements about the law of large numbers and central limit theorem for an additive functional of the process $\eta(\cdot)$.

Following the ideas of Hairer and Mattingly, see [8, 9], we are able to prove, see Theorem 5.1 below, that the transition semigroup of $\omega(\cdot)$ satisfies the spectral gap property in a Wasserstein metric defined over the Hilbert space H of square integrable mean zero functions. If $\psi_*(\cdot)$ were Lipschitz this fact would make the proof of the law of large numbers and central limit theorem standard, in view of [27] (see also [17, 20]). However, in our case the observable ψ_* is not Lipschitz. In fact, it is not even defined on the state space H of the process. Nevertheless, it is a bounded linear functional over another Hilbert space V that is compactly embedded in H. Adopting the approach of Mattingly and Pardoux from [23], see Theorem 5.2 below, we are able to prove that the equation for

ω has regularization properties similar to the N.S.E. and that $\omega(t)$ belongs to V for any $t > 0$. In consequence, one can show that the transition semigroup can be defined on ψ_* and has the same contractive properties as the semigroup defined on Lipschitz functions on H. The law of large numbers can be then shown, Section 6.4, by a modification of the argument of Shirikyan from [27] (see also [17]). To prove the central limit theorem we construct a corrector field χ, see Section 6.1, over the "larger" space H. Then, we proceed with the classical martingale proof of the central limit theorem, see Section 6.4. Such an argument has been used to show this type of a theorem for a Lipschitz observable of the solution of a N.S.E. in [27]. The proof of the existence of the asymptotic variance is done in Section 6.3.

Equation (1.2), that describes one of the most fundamental model of transport of particles in a fluid flow, is sometimes referred to as the equation of a *passive tracer*, see e.g. Chapter V of [31]. The d-dimensional vector field $\vec{u}$ appearing on the right hand side of (1.2) is usually assumed to be random, stationary, and in principle it may have nothing in common with the solution of the N.S.E. Since the fluid flow is incompressible, equation (1.2) is complemented by the condition $\nabla_x \cdot \vec{u}(t,x) \equiv 0$. This model has been introduced by G. Taylor in the 1920-s (see [29] and also [19]) and plays an important role in describing transport phenomena in fluids, e.g. in investigation of ocean currents (see [28]). There exists an extensive literature concerning the passive tracer both from the mathematical and physical points of view, see e.g. [21] and the references therein. In particular, it can be shown (see [26]) that the incompressibility assumption implies that the Lagrangian process $\vec{u}(t,x(t))$, $t \geq 0$, is stationary and if one can prove its ergodicity, the Stokes drift coincides with the mean of the field $v_* = \mathbb{E}\vec{u}(0,0)$. The weak convergence of $(x(t) - v_* t)/\sqrt{t}$ towards a normal law has been shown for flows possessing good relaxation properties either in time, or both in time and space, see [1, 5, 12, 18] for the Markovian case, or [13] for the case of non-Markovian, Gaussian fields with finite decorrelation time. According to our knowledge this is the first result when the central limit theorem has been shown for the tracer in a flow that is given by an actual solution of the two dimensional N.S.E.

2. Preliminaries

2.1. **Some function spaces and operators.** Denote by $\mathbb{T}^2$ the two dimensional torus understood as the product of two segments $[-1/2, 1/2]$ with identified endpoints. Trigonometric monomials $e_k(x) = \mathrm{e}^{2i\pi k \cdot x}$, $k = (k_1, k_2) \in \mathbb{Z}^2$, form the orthonormal base in the space of all complex-valued, square integrable functions $L^2(\mathbb{T}^2)$ with the standard scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$. For a given $w \in L^2(\mathbb{T}^2)$ let $\hat{w}_k = \langle w, e_k \rangle$. Let H be the linear subspace of $L^2(\mathbb{T}^2)$ over the field of reals consisting of those real-valued functions w, for which $\hat{w}_0 = 0$. For any $r \in \mathbb{R}$ let

$$(-\Delta)^{r/2} w := \sum_{k \in \mathbb{Z}^2_*} |k|^r \hat{w}_k e_k, \quad w \in H^r,$$

where H^r consists of such w, for which $\sum_{k \in \mathbb{Z}^2_*} |k|^{2r} |\hat{w}_k|^2 < +\infty$ and $\mathbb{Z}^2_* := \mathbb{Z}^2 \setminus \{(0,0)\}$. We equip H^r with the graph Hilbert norm $|\cdot|_r := |(-\Delta)^{r/2} \cdot|$. Let $V := H^1$ and let V' be the dual to V. Then H can be identified with a subspace of V' and $V \hookrightarrow H \hookrightarrow V'$. We shall also denote by $\|\cdot\|$ the respective norm $|\cdot|_1$. It is well known (see e.g. Corollary 7.11 of [7]) that H^{1+s} is continuously embedded in $C(\mathbb{T}^2)$ for any $s > 0$. Moreover, there exists a constant $C > 0$ such that

$$\|w\|_\infty \le C|w|_{1+s}, \quad \forall\, w \in C^\infty(\mathbb{T}^2). \tag{2.1}$$

Here $\|w\|_\infty := \sup_{x\in\mathbb{T}^2} |w(x)|$. In addition, the following estimate, sometimes referred to as the Gagliardo–Nirenberg inequality, holds, see e.g. p. 27 of [10]. For any $s > 0$, $\beta \in [0,1]$ there exists $C > 0$ such that

$$|w|_{\beta s} \le C|w|^{1-\beta}|w|_s^\beta, \quad \forall\, w \in C^\infty(\mathbb{T}^2). \tag{2.2}$$

Define $\mathcal{K}\colon H^r \to H^{r+1} \times H^{r+1}$ by

$$\mathcal{K}(w) = (\mathcal{K}_1(w), \mathcal{K}_2(w)) := i \sum_{k\in\mathbb{Z}^2_*} |k|^{-2} k^\perp \hat{w}_k e_k. \tag{2.3}$$

We have

$$|\mathcal{K}_i(w)|_{r+1} \le |w|_r, \quad w \in H_r. \tag{2.4}$$

For a given $x \in \mathbb{R}^2$ and $w \in H^r$ we let $\tau_x w \in H^r$ be defined by

$$\tau_x w := w(\cdot + x) = \sum_{k\in\mathbb{Z}^2_*} \mathrm{e}^{-2\pi i k\cdot x} \hat{w}_k e_k.$$

Define also the reflection of w by letting $\mathfrak{s}w(x) := w(-x)$.

2.2. **Homogeneous Wiener process.** Write

$$\mathbb{Z}^2_+ := [(k_1,k_2) \in \mathbb{Z}^2_*\colon k_2 > 0] \cup [(k_1,k_2) \in \mathbb{Z}^2_*\colon k_1 > 0, k_2 = 0]$$

and let $\mathbb{Z}^2_- := -\mathbb{Z}^2_+$. Let $(B_k(t))_{t\ge 0}$, $k \in \mathbb{Z}^2_+$, be independent, standard one dimensional, complex Brownian motions defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define $B_{-k}(t) := B_k^*(t)$ for $k \in \mathbb{Z}^2_+$. Assume that the function $k \mapsto q_k$ is complex even, i.e. $q_{-k} = q_k^*$, $k \in \mathbb{Z}^2_*$. A cylindrical Wiener process in H, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, can be written as

$$W(t) := \sum_{k\in\mathbb{Z}^2_*} B_k(t) e_k, \quad t \ge 0.$$

Let $Q\colon H \to H^r$ be a bounded linear operator given by

$$\widehat{Qw}_k := q_k \widehat{w}_k, \qquad k \in \mathbb{Z}^2_*. \tag{2.5}$$

The Hilbert–Schmidt norm of the operator, see Appendix C of [3], can be computed from formula

$$\|Q\|^2_{L_{(HS)}(H,H^r)} := \sum_{k\in\mathbb{Z}^2_*} \|Qe_k\|^2_{H^r} = \sum_{k\in\mathbb{Z}^d} |k|^{2r}|q_k|^2, \tag{2.6}$$

Proposition 2.1. *If $\|Q\|^2_{L_{(HS)}(H,H^r)} < +\infty$ then the process $(QW(t))_{t\geq 0}$ has realizations in H^r, $\mathbb{P}$-a.s. Its law is invariant under the reflection and translations. The above means that the law of $(QW(t))_{t\geq 0}$ and that of $(\mathfrak{s}QW(t))_{t\geq 0}$ are the same, and the laws of $(\tau_x QW(t))_{t\geq 0}$ are independent of $x \in \mathbb{R}^2$.*

Proof. The first part of the proposition follows directly from Proposition 4.2, p. 88 of [3]. The second part is a simple consequence of the fact that the processes in question have the same covariance operator as $(QW(t))_{t\geq 0}$. □

3. Formulation of the main results

In this section we define rigorously the notion of a solution of (1.2) with vector field $\vec{u}$ given by the solution of the Navier–Stokes equations (1.1) and formulate the main results of the paper dealing with the long time, large scale behavior of the trajectory.

Since, as it turns out, the components of the solution of the N.S.E. belong to V, see [24], if the initial condition $\vec{u}_0 \in V$, we cannot use equation (1.2) for a direct definition of the solution because the point evaluation for the field is not well defined (not to mention the question of the existence and uniqueness of solutions to the o.d.e. in question).

3.1. **Vorticity formulation of the N.S.E.**

Note that the rotation

$$\xi(t) := \mathrm{rot}\,\vec{u}(t) = \partial_2 u_1(t) - \partial_1 u_2(t)$$

of $\vec{u}(t,x) = (\vec{u}_1(t,x), \vec{u}_2(t,x))$, given by (1.1), satisfies

$$\partial_t \xi(t) = \Delta\xi(t) - B_0(\xi(t)) + \mathrm{rot}\vec{F}(t), \qquad \xi(0) = w \in H,$$

where $B_0(\xi) := B_0(\xi,\xi)$, $\xi \in V$, with $B_0(h,\xi) := \vec{v}\cdot\nabla\xi$, and $\vec{v} := \mathcal{K}(h)$. Since $\vec{F}(t,x)$ is homogeneous in space we may assume that $\vec{F}(t,x) = Q\mathrm{d}W(t,x)$, where Q is a Hilbert-Schmidt diagonal of the form (2.5) and W is a cylindrical Wiener process on H. Thus, we suppose that $\xi(t)$ satisfies

$$\mathrm{d}\xi(t) = [\Delta\xi(t) - B_0(\xi(t))]\mathrm{d}t + Q\mathrm{d}W(t), \qquad \xi(0) = w \in H. \tag{3.1}$$

Let $\mathcal{E}_T := C([0,T];H)\cap L^2([0,T];V)$ and let $W(t)$, $t \geq 0$ be non-anticipative with respect to a filtration $\{\mathcal{F}_t,\ t\geq 0\}$.

Definition 3.1. A measurable and $(\mathcal{F}_t)$-adapted, H-valued process $\xi = \{\xi(t),\ t\geq 0\}$ is a solution to (3.1) if for any $T \in (0,+\infty)$, $\xi \in L^2(\Omega, \mathcal{E}_T, \mathbb{P})$ and

$$\xi(t) = \mathrm{e}^{\Delta t} w - \int_0^t \mathrm{e}^{\Delta(t-s)} B_0(\xi(s))ds + \int_0^t \mathrm{e}^{\Delta(t-s)} Q dW(s) \tag{3.2}$$

for all $t \geq 0$.

The following estimate comes from [23], see Lemma A. 3, p. 39.

Proposition 3.2. *For any $T, N > 0$ there exists $C > 0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left(|\xi(t)|^2 + t\|\xi(t)\|^2\right)^N\right] \leq C(1 + |w|^{4N}), \quad \forall\, w \in H. \tag{3.3}$$

Let $\vec{u}(t) := \mathcal{K}(\xi(t))$. Using the above proposition and (2.1) we conclude that

Corollary 3.3. *For any $t > 0$, $\vec{u}(t) \in C(\mathbb{T}^2)$ and*

$$\int_0^t \|\vec{u}(s)\|_\infty ds < +\infty, \quad \mathbb{P} - a.s. \tag{3.4}$$

Proof. The continuity of $\vec{u}(t, x)$ with respect to x, follows from the Sobolev embedding. From (2.4) we conclude that there exists $C > 0$ such that

$$\|\vec{u}(s)\|_\infty \le C\|\xi(s)\|, \quad \forall\, s \ge 0. \tag{3.5}$$

On the other hand from (3.3) we conclude that for any $t > 0$ there exists a random variable $\tilde{C}$ that is almost surely finite and such that $\|\xi(s)\| \le \tilde{C}s^{-1/2}$ for all $s \in (0, t]$. Combining this with (3.5) we conclude (3.4). □

3.2. **Definition of trajectory process and its ergodic properties.**

Definition 3.4. Let $x_0 \in \mathbb{R}^2$. By a *solution to* (1.2) we mean any $(\mathcal{F}_t)$-adapted process $x(t)$, $t \ge 0$, with continuous trajectories, such that

$$x(t) = x_0 + \int_0^t \vec{u}(s, x(s))ds, \quad \forall\, t \ge 0, \qquad \mathbb{P}\text{-a.s.} \tag{3.6}$$

For a given $\nu > 0$ denote $e_\nu(w) := \exp\{\nu|w|^2\}$, $w \in H$.

Theorem 3.5. *Assume that Q in (4.1) belongs to $L_{(HS)}(H, V)$ and has a trivial null space, i.e. $Qw = 0$ implies $w = 0$. Suppose that the initial vorticity is random, distributed on H according to the law μ_0 for which*

$$\int_H e_{\nu_0}(w)\mu_0(dw) < +\infty \tag{3.7}$$

with a certain $\nu_0 > 0$. Finally, assume that $\{x(t; x_0),\, t \ge 0\}$ is a solution of (1.2) corresponding to the initial data $x_0 \in \mathbb{R}^2$. Then, the following are true:

1) *(Weak law of large numbers) for any $x_0 \in \mathbb{R}^2$ we have*

$$\lim_{T\to+\infty} \frac{x(T; x_0)}{T} = 0 \tag{3.8}$$

in probability.

2) *(Existence of the asymptotic variance) there exists $D_{ij} \in [0, +\infty)$ such that*

$$\lim_{T\to+\infty} \frac{1}{T}\mathbb{E}\left[x_i(T; x_0)x_j(T; x_0)\right] = D_{ij}, \quad i, j = 1, 2. \tag{3.9}$$

3) *(Central limit theorem) Random vectors $x(T; x_0)/\sqrt{T}$ converge in law, as $T \to +\infty$, to a zero mean normal law whose co-variance matrix equals $\mathbf{D} = [D_{ij}]$.*

Remark. In our companion paper [16] it is shown that under our assumption about non-degeneracy of the noise, i.e. that $\ker(Q) = \{0\}$, we have $\det\mathbf{D} \ne 0$.

4. Lagrangian and tracer trajectory processes

4.1. **Uniqueness in law of the trajectory process.** Define the *Lagrangian velocity process* as

$$\vec{\eta}(t,x) = (\eta_1(t,x), \eta_2(t,x)) := \vec{u}(t, x(t)+x), \qquad t \geq 0,\ x \in \mathbb{R}^2.$$

Using Itô's formula we obtain that its vorticity, given by,

$$\omega(t,x) := \operatorname{rot} \vec{\eta}(t,x) = \xi(t, x(t)+x)$$

satisfies $\omega(0) = \tau_{x_0} w \in H$ and

$$d\omega(t) = [\Delta\omega(t) - B_0(\omega(t)) + B_1(\omega(t))]dt + Q\mathrm{d}\widetilde{W}(t), \tag{4.1}$$

where $\widetilde{W}$ is some $(\mathcal{F}_t)$-adapted cylindrical Wiener process on H (different from the original W in (3.1)) and

$$B_1(\omega) := B_1(\omega,\omega) \quad \text{and} \quad B_1(h,\omega) := \mathcal{K}(h)(0)\cdot\nabla\omega, \quad \omega \in V,$$

for more details see [6, 14]. Since we have assumed that $\omega \in V$ and, by the Sobolev embedding, $\mathcal{K}(V)$ is embedded into the space $C(\mathbb{T}^2;\mathbb{R}^2)$ of two dimensional, continuous trajectory vector fields on $\mathbb{T}^2$, we see that the evaluation of $\vec{\eta}$ is well defined, and therefore there is no ambiguity in the definition of $B_1(\omega)$ for $\omega \in V$. In what follows we shall omit writing tilde over the cylindrical Wiener process.

Definition 4.1. A measurable, $(\mathcal{F}_t)$-adapted, H-valued process $\omega = \{\omega(t),\, t \geq 0\}$ is a solution to (4.1), with the initial condition $\omega(0) = w$, if for any $T > 0$, $\omega \in L^2(\Omega, \mathcal{E}_T, \mathbb{P})$ and

$$\omega(t) = \mathrm{e}^{\Delta t}w - \int_0^t \mathrm{e}^{\Delta(t-s)}B_0(\omega(s))ds + \int_0^t \mathrm{e}^{\Delta(t-s)}B_1(\omega(s))ds + \int_0^t \mathrm{e}^{\Delta(t-s)}QdW(s), \tag{4.2}$$

$\mathbb{P}$-a.s. for all $t \geq 0$.

Sometimes, when we wish to highlight the dependence on the initial condition and the Wiener process, we shall write $\omega(t; w, W)$. We shall omit writing one, or both of these parameters when they are obvious from the context.

Using a Galerkin approximation argument, as in Section 3 of [24], see also Appendix A below for the outline of the argument, we conclude the following.

Theorem 4.2. *Given an initial condition $w \in H$ and an $(\mathcal{F}_t)$-adapted cylindrical Wiener process $(W(t))_{t\geq 0}$, there exists a unique solution to* (4.1) *in the sense of Definition 4.1. Moreover, processes $\{\omega(t;w),\, t \geq 0\}$ form a Markov family with the corresponding transition probability semigroup $\{P_t,\, t \geq 0\}$ defined on the space $C_b(H)$ of continuous and bounded functions on H.*

Using the Yamada–Watanabe result, see e.g. [32] (Corollary after Theorem 4.1.1), or [11], from the above theorem we can conclude the following result, see [14].

Corollary 4.3. *Solutions of* (4.1) *have the uniqueness in law property, i.e. the laws over $C([0,+\infty);H)$ of any two solutions of* (4.1) *starting with the same initial data (but possibly based on different cylindrical Wiener processes) coincide.*

This immediately implies the uniqueness in law property for solutions of (1.2).

Corollary 4.4. *Suppose that ξ and ξ' are two solutions of* (3.1) *with the identical initial data but possibly based on two cylindrical Wiener processes with the respective filtrations* $(\mathcal{F}_t)$ *and* $(\mathcal{F}'_t)$*. Assume also that* $x(\cdot)$ *and* $x'(\cdot)$ *are the solutions of* (1.2) *corresponding to* $\vec{u}(t) = \mathcal{K}(\xi(t))$ *and* $\vec{u}'(t) = \mathcal{K}(\xi'(t))$*, respectively. Then, the laws of the pairs* $(x(\cdot), \xi(\cdot))$ *and* $(x'(\cdot), \xi'(\cdot))$ *over* $C([0,+\infty),\mathbb{R}^2)\times C([0,+\infty),H)$ *coincide.*

Proof. Both $\omega(t,\cdot) = \xi(t, x(t)+\cdot)$ and $\omega'(t,\cdot) = \xi'(t, x'(t)+\cdot)$ satisfy (4.1). According to Corollary 4.3 they have identical laws on $C([0,+\infty),H)$ with the initial condition $\tau_{x_0}w$. In fact, due to an analogue of Proposition 3.2 that holds for the process $\omega(\cdot)$, see part 1) of Theorem 5.2 this law is actually supported in $L^1_{\rm loc}([0,+\infty),V)$. We can write therefore that $(x(\cdot),\xi(\cdot)) = \Psi(\omega(\cdot))$ and $(x'(\cdot),\xi'(\cdot)) = \Psi(\omega'(\cdot))$, where the mapping

$$\Psi = (\Psi_1,\Psi_2) : L^1_{\rm loc}([0,+\infty),V) \to C([0,+\infty),\mathbb{R}^2)\times C([0,+\infty),H)$$

is defined as

$$\begin{aligned}
&\Psi_1(X)(t) := x_0 + \int_0^t \mathcal{K}(X(s))(0)ds,\\
&\Psi_2(X)(t,x) := X(t, x-\Psi_1(X)(t)), \quad \forall X \in L^1_{\rm loc}([0,+\infty),V),
\end{aligned}$$

and the uniqueness claim made in the corollary follows. □

4.2. **Existence of solution of** (1.2)**.**

Definition 4.5. Suppose that $(\Omega,\mathcal{F},(\mathcal{F}_t),\mathbb{P})$ is a filtered probability space. Let $x_0 \in \mathbb{R}^2$. By *a weak solution to* (1.2) we mean a pair consisting of a continuous trajectory $(\mathcal{F}_t)$-adapted process $x(t)$, $t \ge 0$, and an $(\mathcal{F}_t)$-adapted solution $\xi(t)$, $t \ge 0$, to (3.1) such that (3.6) holds.

Suppose now that we are given a filtration $(\mathcal{F}_t)$ and an $\mathcal{F}_t$-adapted solution ω of (4.1) with the initial condition $\omega(0) = \tau_{x_0}w$. Define $(x(\cdot),\xi(\cdot)) := \Psi(\omega(\cdot))$. One can easily check, using Itô's formula, that $(x(\cdot),\xi(\cdot))$ is a weak solution in the sense of Definition 4.5. Therefore we conclude the following.

Proposition 4.6. *Given a filtered probability space there exists a weak solution of* (1.2)*.*

Since the reflected vorticity $\mathfrak{s}\omega(t)$ satisfies (4.1) with the reflected noise $\mathfrak{s}QW(t)$ we conclude the following.

Proposition 4.7. *If the law of* $\omega(0)$ *is invariant under* $\mathfrak{s}$*, then the laws of* $(\omega(t))$ *and that of* $(\mathfrak{s}\omega(t))$ *over* $C([0,+\infty);H)$ *are identical.*

5. Spectral gap and regularity properties of the transition semigroup

Here we present the basic results that shall be instrumental in the proof of Theorem 3.5 formulated in the previous section. In case of the Navier–Stokes dynamics on a two-dimensional torus, corresponding results have been shown in [9], see Theorem 5.10, Proposition 5.12 and parts 2, 3 of Lemma A.1 from [9]. The proofs of analogous results for the

Lagrangian dynamics are not much different, some additional care is needed due to the presence of function $B_1(\cdot)$, but it usually does not create any difficulty.

Let us introduce the space $C_0^\infty(H)$ consisting of all functionals ϕ, for which there exist $n \geq 1$, a function F from $C_0^\infty(\mathbb{R}^n)$ and vectors $v_1, \ldots, v_n \in H$ such that

$$\phi(v) = F\left(\langle v, v_1\rangle, \ldots, \langle v, v_n\rangle\right), \quad \forall\, v \in H.$$

Given $\nu > 0$ define $\mathcal{B}_\nu$ as the completion of $C_0^\infty(H)$ under the norm

$$\|\phi\|_\nu := \sup_{w \in H} e_{-\nu}(w)\left(|\phi(w)| + \|D\phi(w)\|\right),$$

where, as we recall, $e_\nu(v) = \exp\{\nu|w|^2\}$. Here $\|D\phi(w)\| = \sup_{|\xi|\leq 1}|D\phi(w)[\xi]|$, where $D\phi(w)[\xi]$ denotes the Fréchet derivative of a function $\phi\colon H \to \mathbb{R}$ at w in the direction $\xi \in H$. By $\tilde{\mathcal{B}}_\nu$ we understand the Banach space of all Fréchet differentiable functions ϕ such that $\|\phi\|_\nu < +\infty$. Let $\mathcal{P}(H)$ be the space of all Borel, probability measures on H. Recall also that $\mu_* \in \mathcal{P}(H)$ is called an *invariant measure* for $(P_t)_{t\geq 0}$ if

$$\langle \mu_*, P_t\phi\rangle = \langle \mu_*, \phi\rangle, \quad \forall\, \phi \in C_b(H),\ t \geq 0.$$

Here $\langle \mu, \phi\rangle := \int_H \phi d\mu$ for any $\mu \in \mathcal{P}(H)$ and ϕ that is integrable. Our first result can be stated as follows.

Theorem 5.1. *Under the assumptions of Theorem 3.5 the following are true:*

1) *there exist $\nu_0, C > 0$ such that for any $\nu \in (0, \nu_0]$ we have*

$$\mathbb{E}e_\nu(\omega(t;w)) \leq Ce_\nu(w), \quad \forall\, t \geq 0,\ w \in H, \tag{5.1}$$

2) *there exist a unique Borel probability measure μ_* that is invariant for (P_t), and such that*

$$\int_H e_\nu(w)\mu_*(dw) < +\infty, \quad \forall\, \nu \in (0, \nu_0]. \tag{5.2}$$

This measure is invariant under $\mathfrak{s}$, i.e. $\mu_*\mathfrak{s} = \mu_*$,

3) *the constant ν_0 can be further adjusted in such a way that for any $\nu \in (0, \nu_0]$ the semigroup (P_t) extends to $\tilde{\mathcal{B}}_\nu$ and*

$$P_t(\mathcal{B}_\nu) \subset \mathcal{B}_\nu, \quad \forall\, t \geq 0.$$

In addition, for any ν as above there exist $C, \gamma > 0$ such that

$$\|P_t\phi - \langle \mu_*, \phi\rangle\|_\nu \leq C\mathrm{e}^{-\gamma t}\|\phi\|_\nu, \quad \forall\, t \geq 0,\ \phi \in \tilde{\mathcal{B}}_\nu. \tag{5.3}$$

The property described in (5.3) is referred to as *the spectral gap* of the transition semigroup. Since we shall use an extension of this property to functions defined on a smaller space than H we introduce the following definition. For $N > 0$ and $\phi \in C^1(V)$ define

$$|||\phi|||_N := \sup_{w \in V}\frac{|\phi(w)| + \|D\phi(w)\|}{(1 + \|w\|)^N}$$

and denote by $C_N^1(V)$ the space made of functions, for which $|||\phi|||_N < +\infty$.

Theorem 5.2. *Under the assumptions of Theorem 3.5 the following are true:*

1) *for any $t, N > 0$ there exists $C_{t,N}$ such that*

$$\mathbb{E}\|\omega(t;w)\|^N \le C_{t,N}\left(|w|^{2N}+1\right), \quad \forall\, w \in H, \tag{5.4}$$

2) *the definition of the transition semigroup can be extended to an arbitrary $\phi \in C^1_N(V)$ by letting $P_t\phi(w) := \mathbb{E}\tilde{\phi}(\omega(t;w))$, where $\tilde{\phi}$ is an arbitrary, measurable extension of ϕ from V to H. Moreover, for any $t, N > 0$ there exists $C_{t,N}$ such that for any $\nu > 0$,*

$$\|P_t\phi\|_\nu \le C_{t,N}|\!|\!|\phi|\!|\!|_N, \quad \forall\, \phi \in C^1_N(V). \tag{5.5}$$

Define

$$\mathfrak{p}(w) := \begin{cases} \|w\|^2 & \text{for } w \in V, \\ +\infty & \text{for } w \in H \setminus V. \end{cases}$$

Corollary 5.3. *For any $N > 0$ we have $\langle \mu_*, \mathfrak{p}^N\rangle < +\infty$. Thus, in particular $\mu_*(V) = 1$.*

Proof. Suppose that $\varphi_R\colon [0,+\infty) \to [0, R+1]$ is a continuous function such that $\varphi_R(u) = u$ if $u \in [0,R]$ and it vanishes on $u \ge R+1$. For a fixed $K > 0$ we denote

$$\mathfrak{p}_K(w) := \sum_{0<|k|\le K} |k|^2|\hat{w}(k)|^2.$$

Thanks to part 2) of Theorem 5.1 we have $P_t\mathfrak{p}^N \in \mathcal{B}_\nu$ for any $t > 0$ and therefore from (5.4) and (5.2) we get

$$\langle \mu_*, P_t\mathfrak{p}_K^N\rangle \le \langle \mu_*, P_t\mathfrak{p}^N\rangle < +\infty. \tag{5.6}$$

We have therefore

$$\langle \mu_*, P_t\varphi_R \circ \mathfrak{p}_K^N\rangle = \langle \mu_*, \varphi_R \circ \mathfrak{p}_K^N\rangle \le \langle \mu_*, P_t\mathfrak{p}^N\rangle. \tag{5.7}$$

The first equality follows from the fact that μ_* is invariant. Letting first $K \to +\infty$ and then subsequently $R \to +\infty$ we conclude the corollary. □

Combining the results of Theorem 5.2 with part 2) of Theorem 5.1 we conclude the following.

Corollary 5.4. *For any $N > 0$ there exist $C, \nu_0, \gamma > 0$ such that for any $\nu \in (0, \nu_0]$ we have*

$$\|P_t\phi - \langle \mu_*, \phi\rangle\|_\nu \le C\mathrm{e}^{-\gamma t}|\!|\!|\phi|\!|\!|_N, \quad \forall\, t \ge 0,\, \phi \in C^1_N(V). \tag{5.8}$$

6. Proof of Theorem 3.5

To abbreviate we assume that $x_0 = 0$ and we drop it from our notation. Let $\psi_* = (\psi_*^{(1)}, \psi_*^{(2)})\colon V \to \mathbb{R}^2$ be defined as $\psi_*(\omega) := \mathcal{K}(\omega)(0)$. Since, for any $s > 0$, H_{1+s} is embedded into $C(\mathbb{T}^2)$, for any $s > 0$ there exists $C > 0$ such that

$$|\psi_*^{(i)}(w)| \le C|\mathcal{K}_i(w)|_{1+s} \le C|w|_s, \quad \forall\, w \in H_s,\ i = 1, 2. \tag{6.1}$$

It is clear therefore that the components of ψ_* are bounded linear functional on V and $\psi_* \in C^1_1(V)$. It follows from Corollary 5.3 that the components of ψ_* are integrable with respect to μ_*. In addition, since $\mathcal{K}(\mathfrak{s}\omega) = -\mathfrak{s}\mathcal{K}(\omega)$ and measure μ_* is invariant under $\mathfrak{s}$ we obtain

$$\int \psi_* d\mu_* = \int \psi_* \mathfrak{s} d\mu_* = -\int \mathfrak{s}\psi_* d\mu_* = -\int \psi_* d\mu_*.$$

Thus,

$$\int \psi_* d\mu_* = 0. \tag{6.2}$$

Suppose also that $\omega(t)$ is the solution of (7.14) with the initial data distributed according to μ_0.

6.1. **Proof of part 1).** To prove the weak law of large numbers it suffices only to show that for $i = 1, 2$,

$$\lim_{T\to+\infty} \frac{1}{T}\mathbb{E}x_i(T) = 0 \quad \text{and} \quad \lim_{T\to+\infty} \frac{1}{T^2}\mathbb{E}x_i^2(T) = 0. \tag{6.3}$$

Using the Markov property we can write that

$$\frac{1}{T}\mathbb{E}x_i(T) = \frac{1}{T}\int_0^T \langle \mu_0, P_s\psi_*^{(i)}\rangle ds, \quad i = 1, 2. \tag{6.4}$$

Suppose that ν_0 is chosen in such a way that the conclusions of Theorem 5.1 and Corollary 5.4 hold. Assume also that $\nu \in (0, \nu_0]$. We shall adjust its value later on. By virtue of (5.8) we conclude that there exists a constant $C > 0$ such that

$$|P_t\psi_*(w)| \le C\mathrm{e}^{-\gamma t} e_\nu(w) |\!|\!| \psi_* |\!|\!|_1. \tag{6.5}$$

Hence, the right hand side of (6.4) converges to 0, by estimate (3.7) and the Lebesgue dominated convergence theorem. On the other hand

$$\begin{aligned}\frac{1}{T^2}\mathbb{E}x_i^2(T) &= \frac{1}{T^2}\mathbb{E}\Big(\int_0^T \psi_{*,i}(\omega(t))dt \int_0^T \psi_{*,i}(\omega(s))ds\Big) \\ &= \frac{2}{T^2}\int_0^T\int_0^t \mathbb{E}[\psi_{*,i}(\omega(t))\psi_{*,i}(\omega(s))]dtds.\end{aligned} \tag{6.6}$$

The utmost right hand side of (6.6) equals

$$\frac{2}{T^2}\int_0^T\int_0^t \mathbb{E}\big[\psi_{*,i}(\omega(s))P_{t-s}\psi_{*,i}(\omega(s))\big]dtds = \frac{2}{T^2}\int_0^T\int_0^t \langle \mu_0 P_s, \psi_{*,i}P_{t-s}\psi_{*,i}\rangle dtds. \tag{6.7}$$

Using (6.5) we can estimate the right hand side of (6.7) by

$$\frac{C}{T^2}\int_0^T\int_0^t \mathrm{e}^{-\gamma(t-s)}\langle \mu_0 P_s, |\psi_{*,i}|e_\nu\rangle dtds = \frac{C(1-\mathrm{e}^{-\gamma T})}{\gamma T^2}\int_0^T \langle \mu_0 P_s, |\psi_{*,i}|e_\nu\rangle ds. \tag{6.8}$$

Applying Hölder's inequality with $q \in (1, \nu_0/\nu)$ and an even integer p such that $p^{-1} := 1 - q^{-1}$, we conclude that the right hand side is smaller than

$$\frac{C}{\gamma T^2}\int_0^T \langle \mu_0, P_s|\psi_*|^p\rangle^{1/p}\langle \mu_0 P_s, e_{q\nu}\rangle^{1/q} ds \le \frac{C_1}{\gamma T^2}\int_0^T \langle \mu_0, P_s|\psi_*|^p\rangle^{1/p} ds \tag{6.9}$$

for some constants C, C_1 independent of T. The last inequality follows from (5.1) and (5.2). Since $|\psi_*|^p$ belongs to $C_p^1(V)$ we conclude from Corollaries 5.4, 5.3 and condition (3.7) that the right hand side of the above expression can be estimated by $C_2T/(\gamma T^2)$, with C_2 a constant independent of T, which tends to 0, as $T \to +\infty$. Thus, part 1) follows. $\square$

6.2. **Definition and basic properties of the corrector.** We start with the following.

Proposition 6.1. *Functions*

$$\chi_t(w) = (\chi_t^{(1)}(w), \chi_t^{(2)}(w)) := \int_0^t P_s\psi_*(w)ds, \quad w \in H, \tag{6.10}$$

converge uniformly on bounded sets, as $t \to \infty$. For any $\nu \in (0, \nu_0]$ there is $C > 0$ such that

$$|\chi_t^{(i)}| \leq Ce_\nu, \quad \forall\, t \geq 1,\, i = 1, 2. \tag{6.11}$$

The limit

$$\chi = (\chi^{(1)}, \chi^{(2)}) := \lim_{t \to +\infty} \chi_t = \int_0^{+\infty} P_s\psi_*\, ds, \tag{6.12}$$

called a corrector*, satisfies*

$$|\chi^{(i)}| \leq Ce_\nu, \quad i = 1, 2, \tag{6.13}$$

with the same constant as in (6.11)*.*

Proof. As a consequence of Corollary 5.4 we conclude that the functions

$$\int_1^t P_s\psi_*^{(i)}(w)ds, \quad t \geq 1,\, i = 1, 2,$$

are well defined on H and converge uniformly on bounded sets. The convergence part of the proposition follows from the fact that there exists a constnt $C > 0$ such that for $\nu \in (0, \nu_0]$,

$$\int_0^1 \mathbb{E}\|\omega(s, w)\|^2 ds \leq Ce_\nu(w), \quad \forall\, w \in H, \tag{6.14}$$

see (7.10) below. This estimate together with (6.5) imply both (6.11) and (6.13). $\square$

Proposition 6.2. *One can choose $\nu_0 > 0$ in such a way that $\chi^{(i)} \in \mathcal{B}_\nu$ for any $\nu \in (0, \nu_0]$, $i = 1, 2$.*

Proof. Since $\psi_*^{(i)} \in C_1^1(V)$, $i = 1, 2$, from Corollary 5.4 we conclude that $P_t\psi_*^{(i)} \in \mathcal{B}_\nu$ for $t \geq 1$ and there exists $\nu_0 > 0$ such that for any $\nu \in (0, \nu_0]$ one can find $C, \gamma > 0$, for which

$$\|P_t\psi_*^{(i)}\|_\nu \leq C\mathrm{e}^{-\gamma t}|\!|\!|\psi_*^{(i)}|\!|\!|_1, \quad \forall\, t \geq 1,\, i = 1, 2.$$

This guarantees that $\int_1^{+\infty} P_t\psi_*^{(i)}dt$ belongs to $\mathcal{B}_\nu$. Thanks to estimate (6.13) it suffices only to show that

$$\left|\int_0^1 DP_t\psi_*^{(i)}(w)[\xi]dt\right| \leq Ce_\nu(w), \quad \forall\, w, \xi \in H,\, |\xi| \leq 1. \tag{6.15}$$

To prove the above estimate note that

$$\int_0^1 DP_t\psi_*^{(i)}(w)[\xi]dt := \mathbb{E}\left[\mathcal{K}(\Xi(1))(0)\right],$$

where $\Xi(w) := \int_0^1 \xi(t;w)dt$ and $\xi(t) := D\omega(t;w)[\xi]$. From (6.1) for $s = 1$ there exists $C > 0$ such that

$$|\mathcal{K}(\Xi(w))(0)| \le C\|\Xi(w)\|, \quad \forall\, w \in H.$$

Hence, from (7.9), we conclude that for any $\nu > 0$ there exists $C > 0$ such that

$$\left|\int_0^1 DP_t\psi_*^{(i)}(w)[\xi]dt\right|^2 \le |\xi|^2\mathbb{E}\exp\left\{\nu|\omega(1)|^2 + \frac{\nu}{2e}\int_0^1 \|\omega(s)\|^2 ds\right\}$$

and (6.15) follows from estimate (7.10) formulated below. □

6.3. **Proof of part 2).** After a simple calculation we get

$$D_{ij}(T) := \frac{1}{T}\mathbb{E}\Big[x_i(T)x_j(T)\Big] = D^1_{ij}(T) + D^2_{ij}(T),$$

with

$$D^1_{ij}(T) := \frac{1}{T}\int_0^T \left\langle \mu_0 P_s, \psi_*^{(i)} \int_0^{T-s} P_t\psi_*^{(j)}\, dt\right\rangle ds,$$

$$D^2_{ij}(T) := \frac{1}{T}\int_0^T \left\langle \mu_0 P_s, \psi_*^{(j)} \int_0^{T-s} P_t\psi_*^{(i)}\, dt\right\rangle ds.$$

It suffices only to deal with the limit of $D^1_{ij}(T)$, the other term can be handled in a similar way. We can write that

$$\left|D^1_{ij}(T) - \frac{1}{T}\int_0^T \langle \mu_0 P_s, \psi_*^{(i)}\chi^{(j)}\rangle\, ds\right| = \frac{1}{T}\left|\int_0^T \Big\langle \mu_0 P_s, \psi_*^{(i)}(\chi^{(j)} - \chi^{(j)}_{T-s})\Big\rangle\, ds\right| = R_{ij}(T),$$

where

$$R_{ij}(T) := \left|\int_0^1 \Big\langle \mu_0 P_{sT}, \psi_*^{(i)}(\chi^{(j)} - \chi^{(j)}_{T(1-s)})\Big\rangle\, ds\right|. \tag{6.16}$$

Lemma 6.3. *We have*

$$\lim_{T\to+\infty} R_{ij}(T) = 0. \tag{6.17}$$

Proof. Suppose that p is a positive even integer and q is sufficiently close to 1 so that $q\nu < \nu_0$ and $1/q = 1 - 1/p$, where ν is as in (6.11) and (6.13), while ν_0 is such that (3.7) is in force. Then, we can find a constant $C > 0$ such that

$$|\chi^{(j)}(w) - \chi^{(j)}_{T(1-s)}(w)|^q \le Ce_{\nu_0}(w), \quad \forall\, w \in H \quad \forall\, s \in [0,1],\ T > 0. \tag{6.18}$$

Using Proposition 6.1 and (3.7) we conclude that

$$\lim_{T\to+\infty} \langle \mu_0 P_{sT}, |\chi^{(j)} - \chi^{(j)}_{T(1-s)}|^q\rangle = 0, \quad \forall\, s \in [0,1).$$

Equality (6.17) can be concluded, provided we can substantiate passage to the limit with T under the integral appearing on the right hand side of (6.16). Suppose first that the argument s appearing in the integral satisfies $sT \geq 1$. Using Hölder's inequality, in the same way as it was done in (6.9), and estimates (6.11) and (6.13) the expression under the integral can be estimated by

$$\begin{aligned}&\langle \mu_0, P_{sT}|\psi_*^{(i)}|^p\rangle^{1/p}\langle \mu_0 P_{sT}, |\chi^{(j)} - \chi_{T(1-s)}^{(j)}|^q\rangle^{1/q}\\ &\quad\leq \sup_{t\geq 1}\langle \mu_0, P_t|\psi_*^{(i)}|^p\rangle^{1/p}\langle \mu_0 P_{sT}, |\chi^{(j)} - \chi_{T(1-s)}^{(j)}|^q\rangle^{1/q}.\end{aligned} \tag{6.19}$$

Since $|\psi_*|^p \in C_p^1(V)$ we have $\sup_{t\geq 1}\langle \mu_0, P_t|\psi_*|^p\rangle < +\infty$, thanks to part 2) of Theorem 5.2. As a result the left hand side of (6.19) is bounded for all $s \in [1/T, 1]$. From the Lebesgue dominated convergence theorem we conclude therefore that

$$\lim_{T\to+\infty}\int_{1/T}^{1}\Big\langle \mu_0 P_{sT}, \psi_*^{(i)}(\chi^{(j)} - \chi_{T(1-s)}^{(j)})\Big\rangle\, ds = 0. \tag{6.20}$$

Next we shall prove that there exists $C > 0$ such that

$$\left|\int_0^{1/T}\Big\langle \mu_0 P_{sT}, \psi_*^{(i)}(\chi^{(j)} - \chi_{T(1-s)}^{(j)})\Big\rangle\, ds\right| \leq \frac{C}{T}, \tag{6.21}$$

provided that $T \geq 1$. Indeed, using first the Cauchy–Schwartz inequality and then (6.11), and (6.13) we get that the left hand side can be estimated by

$$C\mathbb{E}\left\{\left\{\int_0^{1/T}|\psi_*(\omega(sT))|^2 ds\right\}^{1/2}\left\{\int_0^{1/T} e_{2\nu}(\omega(sT))ds\right\}^{1/2}\right\}.$$

Applying Hölder's inequality with $q \in (1, 2)$ and $1/p = 1 - 1/q$ we get that this expression can be estimated by

$$\begin{aligned}&C\left\{\mathbb{E}\left\{\int_0^{1/T}|\psi_*(\omega(sT))|^2 ds\right\}^{p/2}\right\}^{1/p}\left\{\mathbb{E}\left\{\int_0^{1/T} e_{2\nu}(\omega(sT))ds\right\}^{q/2}\right\}^{1/q}\\ &\leq C_1\left\{\mathbb{E}\left\{\frac{1}{T}\int_0^1\|\omega(s))\|^2 ds\right\}^{p/2}\right\}^{1/p}\left\{\mathbb{E}\left\{\int_0^{1/T} e_{2\nu}(\omega(sT))ds\right\}\right\}^{1/2}\\ &\leq \frac{C_2}{T}\left\{\mathbb{E}\exp\left\{\nu\int_0^1\|\omega(s))\|^2 ds\right\}\right\}^{p/2} \leq \frac{C_3}{T},\end{aligned}$$

provided $2\nu < \nu_0$. The penulmative inequality follows from (5.1) and assumption (3.7), while the last estimate is a consequence of (7.11) stated below. Thus, (6.21) follows. □

We are left therefore with the problem of finding the limit of

$$S_{ij}(T) = \frac{1}{T}\int_0^T\langle \mu_0 P_s, \psi_*^{(i)}\chi^{(j)}\rangle\, ds \tag{6.22}$$

as $T \to +\infty$. Let $R \geq 1$ be fixed and $\varphi_R\colon \mathbb{R} \to \mathbb{R}$ be a smooth mapping such that $\varphi_R(x) = 1$ for $|x| \leq R$ and $\varphi_R(x) = 0$ for $|x| \geq R+1$. Observe that

$$\hat{\chi}^{(R)}(w) := \chi^{(j)}(w)\varphi_R(|w|^2)$$

belongs to $C_b^1(H)$, and thus also to $C_b^1(V)$. Therefore, $\psi_*^{(i)}\hat{\chi}^{(R)} \in C_1^1(V)$. Denote by $S^{(R)}(T)$ the expression in (6.22) with $\chi^{(j)}$ replaced by $\hat{\chi}^{(R)}$.

Let $\varepsilon > 0$ be arbitrary. Using the same argument as in the proof of Lemma 6.3 one can show that for any $\varepsilon > 0$ there exists a sufficiently large $R \geq 1$ and $T_0 > 0$ so that

$$\left|\frac{1}{T}\int_0^T \left\langle \mu_0 P_s, \psi_*^{(i)}(\chi^{(j)} - \hat{\chi}^{(R)})\right\rangle ds\right| < \frac{\varepsilon}{2}.$$

Likewise, we can choose $R \geq 1$ and $T_0 > 0$ so large that

$$\left|\left\langle \mu_*, \psi_*^{(i)}(\chi^{(j)} - \hat{\chi}^{(R)})\right\rangle\right| < \frac{\varepsilon}{2}.$$

By Corollary 5.4 we have

$$\|P_t(\psi_*^{(i)}\hat{\chi}^{(R)}) - \langle \mu_*, \psi_*^{(i)}\hat{\chi}^{(R)}\rangle\|_\nu \leq Ce^{-\gamma t}\|\psi_*\hat{\chi}^{(R)}\|_2, \quad \forall\, t \geq 0.$$

In consequence we conclude that

$$\lim_{T\to+\infty} S^{(R)}(T) = \langle \mu_*, \psi_*^{(i)}\hat{\chi}^{(R)}\rangle.$$

Hence,

$$\begin{aligned}&\limsup_{T\to+\infty} |S_{ij}(T) - \langle \mu_*, \psi_*^{(i)}\chi^{(j)}\rangle| \\ &\qquad \leq \limsup_{T\to+\infty} |S_{ij}(T) - S^{(R)}(T)| + |\langle \mu_*, \psi_*^{(i)}\hat{\chi}^{(R)}\rangle - \langle \mu_*, \psi_*^{(i)}\chi^{(j)}\rangle| < \varepsilon.\end{aligned}$$

This proves that

$$\lim_{T\to+\infty} S_{ij}(T) = \langle \mu_*, \psi_*^{(i)}\chi^{(j)}\rangle.$$

We have shown therefore part 2) of the theorem with

$$\lim_{T\to+\infty} D_{ij}(T) := \langle \mu_*, \psi_*^{(i)}\chi^{(j)}\rangle + \langle \mu_*, \psi_*^{(j)}\chi^{(i)}\rangle. \qquad \square \tag{6.23}$$

6.4. **Proof of part 3).**

6.4.1. *Reduction to the central limit theorem for martingales.* Note that

$$\frac{1}{\sqrt{T}}\int_0^T \psi_*(\omega(s))\,ds = \frac{1}{\sqrt{T}}M_T + R_T, \tag{6.24}$$

where

$$M_T := \chi(\omega(T)) - \chi(\omega(0)) + \int_0^T \psi_*(\omega(s))\,ds \tag{6.25}$$

and

$$R_T := \frac{1}{\sqrt{T}}\left[\chi(\omega(0)) - \chi(\omega(T))\right].$$

Proposition 6.4. *The process $\{M_T,\, T\geq 0\}$ is a square integrable, two dimensional vector martingale with respect to the filtration $\{\mathcal{F}_T, T\geq 0\}$. Moreover, random vectors R_T converge to 0, as $T\to+\infty$, in the L^1-sense.*

The proof of this result is quite standard and can be found in [17], see Proposition 5.2 and Lemma 5.3.

6.4.2. *Central limit theorem for martingales.* Assume that $\{\mathcal{M}_n,\, n\geq 0\}$ is a zero mean martingale subordinated to a filtration $\{\mathfrak{F}_n,\, n\geq 0\}$ and $\mathcal{Z}_n := \mathcal{M}_n - \mathcal{M}_{n-1}$ for $n\geq 1$, is the respective sequence of martingale differences. Recall that the quadratic variation of the martingale is defined as

$$\langle\mathcal{M}\rangle_n = \sum_{j=1}^n \mathbb{E}\left[\mathcal{Z}_j^2|\mathfrak{F}_{j-1}\right], \quad n\geq 1.$$

The following theorem has been shown in [17], see Theorem 4.1.

Theorem 6.5. *Suppose also that*

M1)

$$\sup_{n\geq 1}\mathbb{E}\mathcal{Z}_n^2 < +\infty, \tag{6.26}$$

M2) *for every $\varepsilon>0$,*

$$\lim_{N\to+\infty}\frac{1}{N}\sum_{j=0}^{N-1}\mathbb{E}\Big[\mathcal{Z}_{j+1}^2,\, |\mathcal{Z}_{j+1}|\geq\varepsilon\sqrt{N}\Big]=0,$$

M3) *there exists $\sigma\geq 0$ such that*

$$\lim_{K\to\infty}\limsup_{\ell\to\infty}\frac{1}{\ell}\sum_{m=1}^{\ell}\mathbb{E}\Big|\frac{1}{K}\mathbb{E}\left[\langle\mathcal{M}\rangle_{mK}-\langle\mathcal{M}\rangle_{(m-1)K}\Big|\mathfrak{F}_{(m-1)K}\right]-\sigma^2\Big|=0, \tag{6.27}$$

M4) *for every $\varepsilon>0$*

$$\lim_{K\to\infty}\limsup_{\ell\to\infty}\frac{1}{\ell K}\sum_{m=1}^{\ell}\sum_{j=(m-1)K}^{mK-1}\mathbb{E}[1+\mathcal{Z}_{j+1}^2,\, |\mathcal{M}_j-\mathcal{M}_{(m-1)K}|\geq\varepsilon\sqrt{\ell K}]=0. \tag{6.28}$$

Then,

$$\lim_{N\to+\infty}\frac{\mathbb{E}\langle\mathcal{M}\rangle_N}{N}=\sigma^2 \tag{6.29}$$

and

$$\lim_{N\to\infty}\mathbb{E}e^{i\theta\mathcal{M}_N/\sqrt{N}}=e^{-\sigma^2\theta^2/2},\quad \forall\,\theta\in\mathbb{R}. \tag{6.30}$$

6.4.3. *Proof of the central limit theorem for $M_T/\sqrt{T}$.* We prove that $M_n/\sqrt{n}$, where $n\geq 1$ is an integer, converge in law to a Gaussian random vector, as $n\to+\infty$. This suffices to conclude that in fact $M_T/\sqrt{T}$ satisfy the central limit theorem. Indeed, let $Z_n:=M_n-M_{n-1}$ for $n\geq 1$. Note that for any $\varepsilon>0$

$$\lim_{N\to\infty}\sup_{T\in[N,N+1)}|M_T/\sqrt{T}-M_N/\sqrt{N}|=0,\quad \mathbb{P}-\text{a.s.} \tag{6.31}$$

For a given $\varepsilon_N>0$ we let

$$A_N:=[\sup_{T\in[N,N+1)}|M_T/\sqrt{T}-M_N/\sqrt{N}|\geq\varepsilon_N].$$

We have

$$\begin{aligned}\mathbb{P}[A_N] &\leq \mathbb{P}[\sup_{T\in[N,N+1)}|M_T-M_N|\geq\varepsilon_N\sqrt{N}/2]+\mathbb{P}[|M_N|[N^{-1/2}-(N+1)^{-1/2}]\geq\varepsilon_N/2]\\ &\leq \frac{C}{N^2\varepsilon_N^4}\mathbb{E}|Z_{N+1}|^4+\frac{C}{N^3\varepsilon_N^2}\sum_{j=1}^N\mathbb{E}|Z_j|^2.\end{aligned}$$

The last inequality follows from the Doob and Chebyshev estimates and the elementary inequality $N^{-1/2}-(N+1)^{-1/2}\leq CN^{-3/2}$ that holds for all $N\geq 1$ and some constant $C>0$. We denote the first and second terms on the right hand side by I_N and II_N, respectively. We claim that there exists $C>0$ such that

$$\mathbb{E}|Z_{N+1}|^4\leq C,\quad\forall\, N\geq 0. \tag{6.32}$$

Indeed, we have

$$\mathbb{E}|Z_{N+1}|^4\leq C\left\{\mathbb{E}|\chi(\omega(N+1))|^4+\mathbb{E}|\chi(\omega(N))|^4+\mathbb{E}\left|\int_N^{N+1}\psi_*(\omega(s))ds\right|^4\right\}.$$

To estimate the first two terms appearing on the right hand side we use (6.13) and then subsequently (5.2). We conclude that all these terms can be estimated by a constant independent of N. The last expectation can be estimated using (6.1) by

$$C\mathbb{E}\left[\int_N^{N+1}\|\omega(s)\|^2ds\right]^2=C\left\langle\mu_0P_N,\mathbb{E}\left[\int_0^1\|\omega(s;\cdot)\|^2ds\right]^2\right\rangle.$$

Applying (7.10) and then again (5.2) we obtain that also this term can be estimated independently of N. Hence

$$I_N\leq\frac{C}{N^2\varepsilon_N^4}.$$

On the other hand, from (6.32) we conclude also that for some constants $C,C_1>0$ independent of N we have

$$II_N=\frac{C}{N^3\varepsilon_N^2}\sum_{k=1}^N\mathbb{E}|Z_k|^2\leq\frac{C_1}{N^2\varepsilon_N^2}.$$

Choosing ε_N tending to 0 sufficiently slowly we can guarantee that

$$\sum_{N\geq 1}\mathbb{P}[A_N]<+\infty,$$

and (6.31) follows from an application of the Borel–Cantelli lemma.

Choose $a\in\mathbb{R}^2$ and let $\mathcal{M}_n:=M_n\cdot a$. Condition M1) obviously holds in light of (6.32). Condition M2) also easily follows from (6.32) and the Chebyshev inequality. Before verifying hypothesis M3) let us introduce some additional notation. For a given probability measure μ on H and a Borel event A write

$$\mathbb{P}_\mu[A]:=\int_H \mathbb{P}[A|\omega(0)=w]\mu(dw).$$

The respective expectation shall be denoted by $\mathbb{E}_\mu$. We write $\mathbb{P}_w$ and $\mathbb{E}_w$ in case of $\mu=\delta_w$. We can write that

$$\frac{1}{K}\mathbb{E}\left[\langle\mathcal{M}\rangle_{mK}-\langle\mathcal{M}\rangle_{(m-1)K}\Big|\mathfrak{F}_{(m-1)K}\right]=\frac{1}{K}\sum_{j=0}^{K-1}P_j\Psi(\omega((m-1)K))$$

with $\Psi(w):=\mathbb{E}_w\mathcal{M}_1^2$. Suppose that $\sigma^2=\langle\mu_*,\Psi\rangle$. Let also $\tilde\Psi(w):=\Psi(w)-\sigma^2$,

$$S_K(w):=\frac{1}{K}\sum_{j=0}^{K-1}P_j\Psi(w)$$

and

$$\tilde S_K(w):=|S_K(w)|-\langle\mu_*,|S_K|\rangle,\quad w\in H.$$

We can rewrite the expression under the limit in (6.27) as being equal to

$$\frac{1}{\ell}\sum_{m=1}^{\ell}\mathbb{E}\Big|\frac{1}{K}\sum_{j=0}^{K-1}P_j\tilde\Psi(\omega((m-1)K))\Big|=\langle\mu_0Q_\ell^K,|S_K|\rangle, \tag{6.33}$$

where

$$Q_\ell^K:=\frac{1}{\ell}\sum_{m=1}^{\ell}P_{(m-1)K}.$$

It is obvious that the second term on the right hand side of (6.33) does not contribute to the limit in hypothesis M3). We prove that

$$\lim_{\ell\to+\infty}\sum_{m=1}^{\ell}\langle\mu_0Q_\ell^K,\tilde S_K\rangle=0. \tag{6.34}$$

Then M3) shall follow upon subsequent applications of (6.34), as $\ell\to+\infty$, and Birkhoff's individual ergodic theorem, as $K\to+\infty$. To prove (6.34) it suffices only to show that the function $S_K(\cdot)$ is continuous on H and for any K fixed there exists a constant $C>0$ such that

$$|S_K(w)|\leq Ce_\nu(w),\quad\forall\, w\in H. \tag{6.35}$$

Equality (6.34) is then a consequence of the fact that measures $\mu_0 Q_\ell^K$ converge weakly to μ_* as $\ell \to +\infty$, and estimate (5.1). Continuity of $S_K(\cdot)$ follows from the fact that $\tilde{\Psi} \in \mathcal{B}_\nu$. On the other hand estimate (6.35) follows from the fact that for any $j \geq 1$ fixed there exists a constant $C > 0$ such that

$$P_j \Psi(w) \leq C e_\nu(w), \quad w \in H. \tag{6.36}$$

The last estimate can be seen as follows

$$\Psi(w) \leq |a|^2 \mathbb{E}_w |M_1|^2 = |a|^2 \sum_{i=1}^{2} \left\{ P_1[\chi^{(i)}]^2(w) + [\chi^{(i)}(w)]^2 + 2\int_0^1 P_s(\tilde{\psi}_*^{(i)} P_{1-s}\chi^{(i)})(w)\, ds \right.$$

$$\left. +2\int_0^1 ds \int_0^s P_{s'}(\tilde{\psi}_*^{(i)} P_{s-s'} \tilde{\psi}_*^{(i)})(w)\, ds' + 2(\chi^{(i)} P_1 \chi^{(i)})(w) + 2\chi^{(i)}(w) \int_0^1 P_s \tilde{\psi}_*^{(i)}(w)\, ds \right\}. \tag{6.37}$$

Using estimates (5.1) and (6.13) we conclude that for any $\nu > 0$ there exists a constant $C > 0$ such that

$$\Psi(w) \leq C e_\nu(w), \quad \forall\, w \in H.$$

Hence, using again (5.1), we conclude (6.36). This ends the proof of hypothesis M3).

Finally we verify condition M4). For that purpose it suffices only to prove that

$$\lim_{K \to +\infty} \limsup_{\ell \to +\infty} \frac{1}{K} \sum_{j=0}^{K-1} \langle \mu_0 Q_\ell^K, G_{\ell,j} \rangle = 0,$$

where

$$G_{\ell,j}(w) := \mathbb{E}_w \left[1 + |Z_{j+1}|^2, |M_j| \geq \varepsilon \sqrt{\ell K} \right].$$

The latter follows if we show that

$$\limsup_{\ell \to +\infty} \langle \mu_0 Q_\ell^K, G_{\ell,j} \rangle = 0, \quad \forall\, j = 0, \ldots, K-1. \tag{6.38}$$

From the Markov inequality we obtain

$$\mathbb{P}_w \left[|M_j| \geq \varepsilon \sqrt{\ell K} \right] \leq \frac{\mathbb{E}_w |M_j|}{\varepsilon \sqrt{\ell K}} \leq I_1 + I_2,$$

where

$$I_1 := \frac{1}{\varepsilon \sqrt{\ell K}} \sum_{i=1}^{2} \mathbb{E}_w |\chi^{(i)}(\omega(j)) - \chi^{(i)}(w)|$$

and

$$I_2 := \frac{1}{\varepsilon \sqrt{\ell K}} \sum_{i=1}^{2} \mathbb{E}_w \left| \int_0^j \tilde{\psi}_*^{(i)}(\omega(s)) ds \right|.$$

Using (6.13) we conclude that

$$I_1 \leq \frac{C_1 e_\nu(w)}{\varepsilon \sqrt{\ell K}}.$$

On the other hand, we have

$$I_2 \le \frac{C_2}{\varepsilon\sqrt{\ell K}}\mathbb{E}_w \int_0^j \|\omega(s)\| ds$$

and from (7.11) we get that

$$I_2 \le \frac{C_3 e_\nu(w)}{\varepsilon\sqrt{\ell K}}.$$

Summarizing, we have shown that for any $R > 0$,

$$\sup_{|w|\le R} \mathbb{P}_w\Big[|M_j| \ge \varepsilon\sqrt{\ell K}|\Big] \le \frac{C}{\sqrt{\ell K}}. \tag{6.39}$$

Furthermore,

$$\begin{aligned} &\sup_{|w|\le R} \mathbb{E}_w\Big[|Z_{j+1}|^2, |M_j| \ge \varepsilon\sqrt{\ell K}\Big] \\ &\le\ 2\sum_{i=1}^2 \Bigg\{\sup_{|w|\le R} \mathbb{E}_w\Big\{\big[\chi^{(i)}(\omega(j+1)) - \chi^{(i)}(\omega(j))\big]^2, |M_j| \ge \varepsilon\sqrt{\ell K}\Big\} \\ &+ \sup_{|w|\le R} \mathbb{E}_w\Big\{\Big[\int_j^{j+1} \tilde\psi_*^{(i)}(\omega(s))ds\Big]^2, |M_j| \ge \varepsilon\sqrt{\ell K}\Big\}\Bigg\} \\ &\le C \sup_{t\in[0,K]}\sup_{|w|\le R} \mathbb{E}_w\Big[e_\nu(\omega(t)), |M_j| \ge \varepsilon\sqrt{\ell K}\Big] \end{aligned} \tag{6.40}$$

for some constant C independent of ℓ. The above argument shows that

$$\lim_{\ell\to+\infty}\sup_{|w|\le R} |G_{\ell,j}(w)| = 0.$$

To obtain (6.38) it suffices only to prove that for $\delta > 0$ as in H3) we have

$$\limsup_{\ell\to+\infty}\langle \mu_0 Q_\ell^K, G_{\ell,j}^{1+\delta/2}\rangle < +\infty, \quad \forall\, K \ge 1,\ 0 \le j \le K-1. \tag{6.41}$$

Note that

$$\langle \mu_0 Q_\ell^K, G_{\ell,j}^{1+\delta/2}\rangle \le \mathbb{E}_{\mu_0 Q_\ell^K}(1 + |Z_{j+1}|^2)^{1+\delta/2}. \tag{6.42}$$

This however is a consequence of (5.1). Thus condition M4) follows.

Summarizing, we have shown that

$$\lim_{n\to+\infty} \exp\left\{\frac{ia\cdot M_N}{\sqrt{N}}\right\} = \exp\left\{-\frac{1}{2}\sum_{i,j=1}^2 D_{ij}a_i a_j\right\},$$

where

$$D_{ij} := \left\langle \mu_*, \mathbb{E}\left\{\prod_{p=i,j}\left[\chi^{(p)}(\omega(1;w)) - \chi^{(p)}(w) + \int_0^1 \tilde\psi_*^{(p)}(\omega(s;w))\,ds\right]\right\}\right\rangle.$$

After a somewhat lengthy, but straightforward calculation, using stationarity of μ_* and the fact that

$$\left\langle \mu_*, \left[P_s\chi^{(i)} - \chi^{(i)} + \int_0^s P_{s'}\tilde{\psi}_*^{(i)}\, ds' \right] \tilde{\psi}_*^{(j)} \right\rangle = 0, \quad \forall\, s \geq 0$$

we conclude that D_{ij} coincides with the expression on the right hand side of (6.23). □

7. Proof of the results from section 5

7.1. **Proof of Theorem 5.1.** Part 3) is a direct consequence of parts 1) and 2). The invariance of μ_* under $\mathfrak{s}_*$ follows from Proposition 4.7.

7.1.1. *Proof of part 1).* Suppose that $\omega(t) := \omega(t; w)$. From (7.10) to conclude that for $\nu \in (0, \nu_0]$, where $\nu_0 = 1/(4\|Q\|)$, there exists a constant $C > 0$ such that

$$\mathbb{E}\exp\left\{\nu|\omega(n+1)|^2\right\} \leq C\mathbb{E}\exp\left\{q\nu|\omega(n)|^2\right\}, \quad \forall\, n \geq 0. \tag{7.1}$$

Let $q = e^{-1/2}$. The right hand side can be further estimated using Jensen's inequality

$$C\mathbb{E}\exp\left\{q\nu|\omega(n)|^2\right\} \leq C\left(\mathbb{E}\exp\left\{\nu|\omega(n)|^2\right\}\right)^q \leq C^{1+q}\left(\mathbb{E}\exp\left\{q\nu|\omega(n-1)|^2\right\}\right)^q.$$

Iterating this procedure we conclude that for any $n \geq 0$

$$\begin{aligned} \mathbb{E}\exp\left\{\nu|\omega(n+1)|^2\right\} &\leq C^{1+q+\ldots+q^n}\left\{\exp\left\{q^{n+1}\nu|\omega(0)|^2\right\}\right\}^{1/q^{n+1}} \\ &\leq C^{1/(1-q)}\exp\left\{\nu|w|^2\right\}. \end{aligned} \tag{7.2}$$

Therefore (cf. part 3) of Lemma A.1 of [9]) we have the following.

Lemma 7.1. *There exists a constant $C > 0$ such that*

$$\mathbb{E}\exp\left\{\nu|\omega(t;w)|^2\right\} \leq C\exp\left\{\nu|w|^2\right\}, \quad \forall\, t \geq 0,\ \nu \in (0, \nu_0],\ w \in H. \tag{7.3}$$

The above lemma obviously implies (5.1).

7.1.2. *A stability result of Hairer and Mattingly.* In our proof we use Theorems 3.4 and 3.6 of [9], which we recall below. Suppose that $(\mathcal{H}, |\cdot|)$ is a separable Hilbert space with a stochastic flow $\Phi_t\colon \mathcal{H} \times \Omega \to \mathcal{H}$, $t \geq 0$, i.e. a family of C^1-class random mappings of $\mathcal{H}$ defined over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that satisfies $\Phi_t(\Phi_s(x;\omega);\omega)) = \Phi_{t+s}(x;\omega)$ for all $t, s \geq 0$, $x \in \mathcal{H}$ and $\mathbb{P}$ a.s. $\omega \in \Omega$. We assume that P_t and $P_t(x, \cdot)$, $x \in \mathcal{H}$, are transition semigroup and a family of transition probabilities corresponding to the flow, i.e.

$$P_t\phi(x) = \int \phi(y)P_t(x, dy) = \mathbb{E}\phi(\Phi_t(x)), \quad \forall \phi \in B(\mathcal{H}),\ x \in \mathcal{H}.$$

Here $B(\mathcal{H})$ is the space of Borel and bounded functions on $\mathcal{H}$. The dual semigroup acting on a Borel probability measure μ shall be denoted by μP_t. We adopt the following hypotheses on the flow.

Assumption 1. There exists a measurable function $V\colon \mathcal{H} \to [1, +\infty)$ and two increasing continuous functions $V_*, V^*\colon [0, +\infty) \to [1, +\infty)$ that satisfy

1)
$$V_*(|x|) \le V(x) \le V^*(|x|), \quad \forall\, x \in \mathcal{H},$$
and $\lim_{a\to+\infty} V_*(a) = +\infty$,

2) there exist $C > 0$ and $\kappa_1 > 1$ such that
$$aV^*(a) \le CV_*^{\kappa_1}(a), \quad \forall\, a \ge 0,$$

3) there exist $\kappa_0 < 1$, $C > 0$ and a decreasing function $\alpha\colon [0,1] \to [0,1]$ with $\alpha(1) < 1$ such that
$$\mathbb{E}\left[V^\kappa(\Phi_t(x))\,(1 + |D\Phi_t(x)[h]|)\right] \le CV^{\alpha(t)\kappa}(x), \quad \forall\, x, h \in \mathcal{H},\ |h| = 1,$$
and $t \in [0,1]$, $\kappa \in [\kappa_0, \kappa_1]$. Here $D\Phi_t(x)[h]$ denotes the Fréchet derivative at x in the direction h.

Assumption 2. There exist $C > 0$ and $\kappa_2 \in [0,1)$ such that for any $\varepsilon \in (0,1)$ one can find $C(\varepsilon), T(\varepsilon) > 0$, for which

$$|DP_t\phi(x)| \le CV^{\kappa_2}(x)\left\{C(\varepsilon)\left[P_t(|\phi|^2)(x)\right]^{1/2} + \varepsilon\left[P_t(|D\phi|^2)(x)\right]^{1/2}\right\}, \tag{7.4}$$

for all $x \in \mathcal{H}$, $t \ge T(\varepsilon)$.

Introduce now the following family of metrics on $\mathcal{H}$. For $\kappa \ge 0$ and $x, y \in \mathcal{H}$ we let
$$\mathrm{d}_\kappa(x,y) := \inf_{c\in\Pi(x,y)} \int_0^1 V^\kappa(c(t))|\dot c(t)|dt,$$
where the infimum extends over the set $\Pi(x,y)$ consisting of all C^1 regular paths $c\colon [0,1] \to \mathcal{H}$ such that $c(0) = x$, $c(1) = y$. In the special case of $\kappa = 1$ we set $\mathrm{d} = \mathrm{d}_1$. For two Borel probability measures μ_1, μ_2 on $\mathcal{H}$ denote by $\mathcal{C}(\mu_1,\mu_2)$ the family of all Borel measures on $\mathcal{H}\times\mathcal{H}$ whose marginals on the first and second coordinate coincide with μ_1, μ_2 respectively. We denote also by
$$\mathrm{d}(\mu_1,\mu_2) := \sup\left[|\langle\mu_1,\phi\rangle - \langle\mu_2,\phi\rangle| : \mathrm{Lip}(\phi) \le 1\right].$$
Here $\mathrm{Lip}(\phi)$ is the Lipschitz constant of $\phi\colon \mathcal{H} \to \mathbb{R}$ in the metric $\mathrm{d}(\cdot,\cdot)$. By $\mathcal{P}_1(\mathcal{H},\mathrm{d})$ we denote the space of all Borel, probability measures μ on $\mathcal{H}$ satisfying $\int_{\mathcal{H}} \mathrm{d}(x,0)\mu(dx) < +\infty$.

Let $A \subset \mathcal{H}\times\mathcal{H}$ be Borel measurable. For a given $t \ge 0$ and $x, y \in \mathcal{H}$ denote
$$\mathcal{P}_t(x,y;A) = \sup\left[\mu[A]\colon \mu \in \mathcal{C}(P_t(x,\cdot), P_t(y,\cdot))\right].$$

Assumption 3. Given any $\kappa \in (0,1)$ and $\delta, R > 0$ there exists $T_0 > 0$ such that for any $T \ge T_0$ there exists $a > 0$ for which
$$\inf_{|x|,|y|\le R} \mathcal{P}_T(x,y;\Delta_{\delta,\kappa}) \ge a.$$
Here,
$$\Delta_{\delta,\kappa} := [(x,y) \in \mathcal{H}\times\mathcal{H}\colon \mathrm{d}_\kappa(x,y) < \delta], \quad \forall\, \kappa, \delta > 0.$$

Theorem 7.2. *Suppose that Assumptions 1, 2, 3 stated above are in force. Then the following are true:*

1) *there exist* $C,\gamma>0$ *such that*

$$\mathrm{d}(\mu_1P_t,\mu_2P_t)\le C\mathrm{e}^{-\gamma t}\mathrm{d}(\mu_1,\mu_2),\quad \forall\mu_1,\mu_2\in\mathcal{P}_1(\mathcal{H},\mathrm{d}),\tag{7.5}$$

2) *there exists a unique probability measure* $\mu_*\in\mathcal{P}_1(\mathcal{H},\mathrm{d})$ *invariant under* $\{P_t,\, t\ge 0\}$, *i.e.* $\mu_*=\mu_*P_t$ *for all* $t\ge 0$,
3) *we have*

$$\|P_t\phi-\langle\mu_*,\phi\rangle\|_{\mathrm{Lip}}\le C\mathrm{e}^{-\gamma t}\|\phi-\langle\mu_*,\phi\rangle\|_{\mathrm{Lip}},\quad \forall\,\phi\in C^1(\mathcal{H}),\, t\ge 0.\tag{7.6}$$

Here

$$\|\phi\|_{\mathrm{Lip}}:=\sup_{x\neq y}\frac{|\phi(x)-\phi(y)|}{\mathrm{d}(x,y)}+|\langle\mu_*,\phi\rangle|.$$

7.1.3. *Proof of part 2).*

Verification of Assumption 1. Denote $\Phi_t(w;W):=\omega(t;w,W)$, where W is the cylindrical Wiener process appearing in (4.1). Let

$$\xi(t;w,\xi):=D\Phi_t(w)[\xi],\quad \xi\in H.\tag{7.7}$$

In what follows we suppress w and ξ in our notation when their values are obvious from the context. The following result holds.

Proposition 7.3. *For any* $\nu>0$ *there exists* $C>0$ *such that for any* $w,\xi\in H$ *we have*

$$|\xi(t)|\le|\xi|\exp\left\{\nu\int_0^t\|\omega(s)\|^2ds+Ct\right\}\tag{7.8}$$

and

$$\left\{\int_0^t\|\xi(s)\|^2ds\right\}^{1/2}\le|\xi|\exp\left\{\nu\int_0^t\|\omega(s)\|^2ds+Ct\right\},\quad\forall\,t\ge 0,\quad \mathbb{P}-\text{a.s.}\tag{7.9}$$

In addition, there exist $\nu_0,C_1>0$ *such that*

$$\mathbb{E}\exp\left\{\nu|\omega(t)|^2+\frac{\nu}{2e}\int_0^t\|\omega(s)\|^2ds\right\}\le C_1\exp\left\{\nu|w|^2e^{-t/2}\right\},\quad\forall\,t\in[0,1],\,\nu\in[0,\nu_0]\tag{7.10}$$

and

$$\mathbb{E}\exp\left\{\nu\sup_{t\ge0}\left[|\omega(t)|^2+\int_0^t\|\omega(s)\|^2ds-t\mathrm{tr}\,Q^2\right]\right\}\le Ce_\nu(w),\quad\forall\,\nu\in[0,\nu_0].\tag{7.11}$$

Proof. Note that $\xi(t)$ satisfies a (non-stochastic) equation

$$\begin{aligned}\partial_t\xi(t)&=\Delta\xi(t)-\eta(t)\cdot\nabla\xi(t)-\mathcal{K}(\xi(t))\cdot\nabla\omega(t)\\&+\eta(t,0)\cdot\nabla\xi(t)+\mathcal{K}(\xi(t))(0)\cdot\nabla\omega(t),\qquad \xi(0)=\xi\in H.\end{aligned}\tag{7.12}$$

Hence,

$$\partial_t|\xi(t)|^2=-2\|\xi(t)\|^2-2\langle\mathcal{K}(\xi(t))\cdot\nabla\omega(t),\xi(t)\rangle+2\langle\mathcal{K}(\xi(t))(0)\cdot\nabla\omega(t),\xi(t)\rangle.$$

Using (A.5) and (A.6) (for $r = 1/2$) we conclude that for some deterministic $C > 0$,

$$\begin{aligned}\partial_t|\xi(t)|^2 &\le -2\|\xi(t)\|^2 + C|\xi(t)|_{1/2}\|\omega(t)\||\xi(t)|\\ &\le -2\|\xi(t)\|^2 + \nu\|\omega(t)\|^2|\xi(t)|^2 + \frac{C^2}{4\nu}|\xi(t)|^2_{1/2}.\end{aligned}$$

An application of the Gagliardo–Nirenberg inequality (2.2) with $s = 1$, $\beta = 1/2$ yields

$$|\xi(t)|_{1/2} \le C\|\xi(t)\|^{1/2}|\xi(t)|^{1/2}$$

for some constant $C > 0$. In consequence, there exist $C, C_1 > 0$ such that

$$\begin{aligned}\partial_t|\xi(t)|^2 &\le -\|\xi(t)\|^2 + \nu\|\omega(t)\|^2|\xi(t)|^2 + \frac{C^2}{2\cdot 4^3\nu}|\xi(t)|^2\\ &\le -\|\xi(t)\|^2 + (\nu\|\omega(t)\|^2 + C_1)|\xi(t)|^2.\end{aligned} \tag{7.13}$$

Estimate (7.8) follows upon an application of Gronwall's inequality. In addition, from (7.8) and (7.9) we conclude that there exists $C > 0$ such that

$$\begin{aligned}\int_0^t \|\xi(s)\|^2 ds &\le |\xi|^2 + \int_0^t (\nu\|\omega(s)\|^2 + C_1)|\xi(s)|^2 ds\\ &\le |\xi|^2 + |\xi|^2\int_0^t (\nu\|\omega(s)\|^2 + C)\exp\left\{\nu\int_0^s \|\omega(u)\|^2 du + Cs\right\} ds\\ &\le |\xi|^2 \exp\left\{\nu\int_0^t \|\omega(s)\|^2 dt + Ct\right\}.\end{aligned}$$

This ends the proof of (7.9).

Estimates (7.10) and (7.11) can be found in [9], see (5.2) for the first one, while the second one is contained in part 1) of Lemma 4.10 of ibid. A minor modification of the argument is required, due to the fact that equation (4.2) contains also the expression corresponding to bilinear form $B_1(\cdot)$.$\square$

Define $V(w) := V_*(|w|) = V^*(|w|) = e^{\nu|w|^2}$. Assumption 1 of Theorem 7.2 is a consequence of Proposition 7.3.

7.2. **Verification of Assumption 2.** Suppose that $\Psi\colon H \to \mathcal{H}$ is a Borel measurable function. Given an $(\mathcal{F}_t)$-adapted process $g\colon [0,\infty)\times\Omega \to H$ satisfying $\mathbb{E}\int_0^t |g_s|^2 ds < +\infty$ for each $t \ge 0$ we denote by $\mathcal{D}_g\Psi(\omega(t))$ the Malliavin derivative of $\Psi(\omega(t))$ in the direction of g; that is

$$\mathcal{D}_g\Psi(\omega(t;w)) := \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left[\Psi(\omega(t;w,W+\varepsilon g)) - \Psi(\omega(t;w,W))\right],$$

where the limit is understood in the $L^2(\Omega,\mathcal{F},\mathbb{P};\mathcal{H})$ sense. Recall that $\omega_g(t;w) := \omega(t;w,W+g)$ solves the equation

$$\begin{aligned}d\omega_g(t;w) &= [\Delta\omega_g(t) - B_0(\omega_g(t;w)) + B_1(\omega_g(t;w))]dt + QdW(t) + Qg(t)dt,\\ \omega(0;w) &= w \in H.\end{aligned} \tag{7.14}$$

Directly from the definition of the Malliavin derivative we conclude *the chain rule:* suppose that $\Psi \in C_b^1(H;\mathcal{H})$ then

$$\mathcal{D}_g \Psi(\omega(t;w)) = D\Psi(\omega(t;w))[D(t)], \tag{7.15}$$

with $D(t;w,g) =: \mathcal{D}_g\omega(t;w)$, $t \geq 0$. In addition, the *integration by parts formula* holds, see Lemma 1.2.1, p. 25 of [25]. Suppose that $\Psi \in C_b^1(H)$ then

$$\mathbb{E}[\mathcal{D}_g\Psi(\omega(t;w))] = \mathbb{E}\left[\Psi(\omega(t;w))\int_0^t \langle g(s), dW(s)\rangle\right]. \tag{7.16}$$

In particular, one can easily show that when $H = \mathcal{H}$ and $\Psi = I$, where I is the identity operator, the Malliavin derivative of $\omega(t;w)$ exists and the process $D(t;w,g)$ (we omit writing w and g when they are obvious from the context), solves the linear equation

$$\begin{aligned} \frac{dD}{dt}(t) &= \Delta D(t) - \eta(t)\cdot\nabla D(t) - \delta k(t)\cdot\nabla\omega(t) \\ &\quad + \eta(t,0)\cdot\nabla D(t) + \delta k(t,0)\cdot\nabla\omega(t) + Qg(t), \\ D(0) &= 0. \end{aligned} \tag{7.17}$$

Here $\delta k(t) := \mathcal{K}(D(t))$. Denote $\rho(t;w,\xi) := \xi(t) - \mathcal{D}_g\omega(t;w)$. We have the following.

Proposition 7.4. *For any $\nu, \gamma > 0$ there exists a constant $C > 0$ such that for any given $w, \xi \in H$ one can find an $(\mathcal{F}_t)$-adapted H-valued process $g(t) = g(t,w,\xi)$ that satisfies*

$$\sup_{|\xi|\leq 1} \mathbb{E}\,|\rho(t;w,\xi)|^2 \leq Ce_\nu(w)e^{-\gamma t}, \quad \forall\, t \geq 0, \tag{7.18}$$

and

$$\sup_{|\xi|\leq 1}\int_0^\infty \mathbb{E}\,|g(s,w,\xi)|^2 ds \leq Ce_\nu(w), \quad \forall\, w \in H. \tag{7.19}$$

Proof. The argument can be adapted directly from the proof of Proposition 4.11 from [8]. Estimate (7.18) follows from (4.13) of [8] (modulo minor modification, due to the presence of the form $B_1(\cdot)$ in equation (4.2)). Estimate (7.19) follows from the estimate appearing in the display following (4.13) in [8] and Lemma A.1 of ibid. □

We use the above result to verify Assumption 2. We have

$$DP_t\phi(w)[\xi] = \mathbb{E}\left\{ D\phi(\omega(t;w))[D(t)]\right\} + \mathbb{E}\left\{D\phi(\omega(t;w))[\rho(t;w,\xi)]\right\}.$$

Using the chain rule, see (7.15), the right hand side can be rewritten as

$$\begin{aligned} &\mathbb{E}\left\{\mathcal{D}_g\phi(\omega(t;w))\right\} + \mathbb{E}\left\{D\phi(\omega(t;w))[\rho(t;w,\xi)]\right\} \\ &= \mathbb{E}\left\{\phi(\omega(t;w))\int_0^t\langle g(s), \mathrm{d}W(s)\rangle\right\} + \mathbb{E}\left\{D\phi(\omega(t;w))[\rho(t;w,\xi)]\right\}. \end{aligned}$$

The last equality follows from integration by parts formula (7.16). We have

$$\left|\mathbb{E}\left\{\phi(\omega(t;w))\int_0^t\langle g(s), dW(s)\rangle\right\}\right| \leq \left(P_t|\phi|^2(w)\right)^{1/2}\left(\mathbb{E}\int_0^\infty |g(s)|^2 ds\right)^{1/2}$$

and

$$|\mathbb{E}\,\{D\phi(\omega(t;w))[\rho(t;w,\xi)]\}|\le\left(P_t|D\phi|^2(w)\right)^{1/2}\left(\mathbb{E}\,|\rho(t;w,\xi)|^2\right)^{1/2}.$$

Hence, by (7.19) and (7.18), given $\kappa_2\in(0,1)$, $\nu>0$, the corresponding $V(w)=e_\nu(w)$ and $\varepsilon\in(0,1)$, we conclude estimate (7.4) with T_0, $C(\varepsilon)$, such that

$$\left(\mathbb{E}\int_0^\infty |g(s)|^2ds\right)^{1/2}\le C(\varepsilon)V^{\kappa_2}(w)$$

and

$$\sup_{|\xi|\le 1}\sup_{t\ge T_0}\left\{\mathbb{E}\,|\rho(t;w,\xi)|^2\right\}^{1/2}\le\varepsilon V^{\kappa_2}(w).$$

7.3. **Assumption 3.** To verify this assumption consider the solution $y(t;w)$, $t\ge 0$, to the deterministic equation

$$\frac{dy(t)}{dt}=\Delta y(t)+B(y(t)),\qquad t\ge 0,$$

with the initial condition $y(0)=w$. Then

$$\lim_{t\to+\infty}\sup_{|w|\le R}|y(t;w)|=0,\quad \forall\, R>0.$$

Fix $\delta>0$ and $R>0$. Let $T_0>0$ be such that

$$\sup_{|w|\le R}\mathrm{d}_\kappa(y(T;w),0)\le\delta/4,\quad\forall\, T\ge T_0.$$

Since

$$W_{\Delta,Q}(t):=\int_0^t \mathrm{e}^{\Delta(t-s)}QdW(s)$$

is a centered Gaussian random element in the Banach space $C([0,T];V)$ with the uniform norm

$$\|f\|_{\infty,T}:=\sup_{t\in[0,T]}\|f(t)\|,\quad f\in C([0,T];V),$$

its topological support is a closed linear subspace (see e.g. [30]). Thus, in particular, 0 belongs to the support of its law and for any $\varrho>0$, $\mathbb{P}(F_\varrho)>0$, where

$$F_\varrho=\{\pi\in\Omega\colon\|W_{A,Q}(\pi)\|_{\infty,T}<\varrho\}.$$

Choose $\varrho_0>0$ such that

$$\mathrm{d}_\kappa(\omega(T;w_i)(\pi),y(T;w_i))|\le\delta/4\qquad\text{for all }\pi\in F_{\varrho_0},\ i=1,2\text{ and }|w|\le R,$$

and set $a:=\mathbb{P}(F_{\varrho_0})>0$. For any $|w_1|,|w_2|\le R$ we have

$$\mathcal{P}_T(w_1,w_2;\Delta_{\delta,\kappa})\ge\mathbb{P}\left[\pi\in\Omega\colon \mathrm{d}_\kappa(\omega(T;w_i)(\pi),y(T;w_i))|\le\delta/4,\ i=1,2\right]\ge\mathbb{P}(F_{\varrho_0})=a,$$

and thus we have finished verification of Assumption 3. $\square$

7.4. **Proof of Theorem 5.2.**

7.4.1. *Proof of part 1).* Let us fix an arbitrary $T>0$ and define $\zeta(t):=|\omega(t)|^2+t\|\omega(t)\|^2$ and $\operatorname{tr} Q_1:=\sum_{k\in\mathbb{Z}^2_*}|k|^2q_k^2$. By Itô's formula we have

$$d\zeta(t)=\left[\operatorname{tr} Q^2+t\operatorname{tr} Q_1-2t|\omega(t)|_2^2-\|\omega(t)\|^2+2t\langle B(\omega(t)),\Delta\omega(t)\rangle\right]dt+dM_t \tag{7.20}$$

and

$$dM_t:=2\langle QdW(t),(I+t\Delta)\omega(t)\rangle.$$

According to (A.5) there exist $C,C_1>0$ such that

$$|\langle B_0(\omega),\Delta\omega\rangle|\le C|\omega|_{1/2}\|\omega\||\omega|_2\le\frac{1}{4}|\omega|_2^2+C_1|\omega|^4,\quad\forall\,\omega\in H_2.$$

To estimate the respective bilinear form corresponding to $B_1(\cdot)$ we use the following estimates, see Proposition 6.1 of [2]. Suppose that $s_1,s_2,s_3\ge 0$ such that $s_1+s_2+s_3>1$ and $s_1>1$. Then, there exists $C>0$ such that

$$|\langle B_1(h,\omega_1),\omega_2)\rangle|\le C|h|_{s_1-1}|\omega_1|_{1+s_2}|\omega_2|_{s_3},\quad\forall\,(h,\omega_1,\omega_2)\in H_{s_1-1}\times H_{1+s_2}\times H_{s_3}. \tag{7.21}$$

From (7.21) with $s_1=3/2$, $s_2=s_3=0$ it follows that

$$|\langle B_1(\omega),\Delta\omega\rangle|\le C|\omega|_{1/2}\|\omega\||\omega|_2,\quad\forall\,\omega\in H_2.$$

With these inequalities we conclude that

$$|\langle B(\omega),\Delta\omega\rangle|\le\frac{1}{2}|\omega|_2^2+C_1|\omega|^4,\quad\forall\,\omega\in H_2.$$

From here on we proceed as in the proof of Lemma A.3 of [23] and conclude from (7.20) that

$$\zeta(t)\le|w|^2+t\operatorname{tr} Q^2+\frac{t^2\operatorname{tr} Q_1}{2}+C\int_0^t s|\omega(s)|^4ds+U(t), \tag{7.22}$$

where $U(0)=0$ and

$$dU(t)=-(t|\omega(t)|_2^2+\|\omega(t)\|^2)dt+dM_t.$$

Since

$$U(t)\le M_t-(\alpha/2)\langle M\rangle_t$$

for some sufficiently small $\alpha>0$ we conclude from the exponential martingale inequality that

$$\mathbb{P}[\sup_{t\in[0,T]}U(t)\ge K]\le \mathrm{e}^{-\alpha K},\quad\forall\,K>0.$$

This, of course, implies that $\mathbb{E}\exp\left\{\alpha'\sup_{t\in[0,T]}U(t)\right\}<+\infty$ for any $\alpha'\in(0,\alpha)$. From (7.11) we get

$$\mathbb{E}\exp\left\{\nu\sup_{t\in[0,T]}|\omega(t)|^2\right\}\le Ce_\nu(w),$$

which in turn implies that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|\omega(t)|^{4N}\right]\le C|w|^{4N}.$$

Summarizing, the above consideration we obtain from (7.22) that for any $T > 0$ and $N \geq 0$ there exists a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{s\in[0,T]} \zeta^{2N}(s)\right] \leq C\left(|w|^{4N} + 1\right). \tag{7.23}$$

We conclude therefore the proof of part 1) of Theorem 5.2.

7.4.2. *Proof of part 2).* Suppose that $\phi \in C^1_N(V)$. Then, $P_t\phi(w)$ is well defined thanks to the already established estimate (5.4). In addition, we have

$$e_{-\nu}(w)|P_t\phi(w)| \leq \|\phi\|_N e_{-\nu}(w)(1 + \mathbb{E}\|\omega(t; w)\|^N) \leq C\|\phi\|_N, \quad \forall\, w \in H. \tag{7.24}$$

To deal with $DP_t\phi(w)[\xi]$ we use the following:

Lemma 7.5. *Suppose that $\{\xi(t),\, t \geq 0\}$ is defined by (7.7). Then, for any $t, \nu > 0$ there exists $C > 0$ such that*

$$\|\xi(t)\|^2 \leq C\|\xi\|^2 \exp\left\{\nu \int_0^t \|\omega(s; w)\|^2 ds + Ct\right\}, \quad \forall\, t \geq 0,\, w \in H,\, \xi \in V,\, \mathbb{P} - \text{a.s.} \tag{7.25}$$

Proof. This estimate can be established analogously to the corresponding bound obtained in Lemma B.1 of [23] (with $\alpha = 0$). Minor modifications needed to account for the term corresponding to $B_1(\cdot)$ present no difficulty and we leave them to a reader. □

Concerning the estimates of $|DP_t\phi(w)[\xi]|$ we can write that

$$\begin{aligned} e_{-\nu}(w)|DP_t\phi(w)[\xi]| &= e_{-\nu}(w)\left|\mathbb{E}\left[(D\phi)(\omega(t; w))[\xi(t)]\right]\right| \qquad (7.26)\\ &\leq \|\phi\|_N e_{-\nu}(w)\mathbb{E}\left[(1 + \|\omega(t; w)\|^N)\|\xi(t)\|\right]\\ &\leq C\|\phi\|_N e_{-\nu}(w)\left\{\mathbb{E}(1 + \|\omega(t; w)\|)^{2N}\right\}^{1/2}\left\{\mathbb{E}\|\xi(t)\|^2\right\}^{1/2}, \quad \forall\, w \in H. \qquad (7.27) \end{aligned}$$

By the already proved part 1) of the theorem and Lemma 7.5 we obtain that the utmost right hand side is less than, or equal to

$$C_1\|\xi\|\|\phi\|_N e_{-\nu}(w)(1 + |w|^{4N})\mathbb{E}\exp\left\{\frac{\nu}{2}\int_0^t \|\omega(s; w)\|^2 ds + C_1 t\right\} \leq C_2\|\xi\|\|\phi\|_N.$$

Hence

$$e_{-\nu}(w)\|DP_t\phi(w)\| \leq C_2\|\phi\|_N$$

and thus we have finished the proof of part 2) of Theorem 5.2.

Appendix A. Existence of the Markov, Feller family

Proof of Theorem 4.2. Given $N \in \mathbb{N}$, denote by Π_N the orthogonal projection of H into $H_N := \operatorname{span}\{e_k,\, 0 < |k| \leq N\}$. Consider the following finite dimensional Itô stochastic differential equation

$$\begin{aligned} &d\omega^{(N)}(t) = [\Delta\omega^{(N)}(t) - B_0^{(N)}(\omega^{(N)}(t)) - B_1^{(N)}(\omega^{(N)}(t))]dt + Q^{(N)}dW(t),\\ &\omega^{(N)}(0) = w^{(N)} \in H, \end{aligned} \tag{A.1}$$

with $W^{(N)}(t) := \Pi_N W(t)$, $Q^{(N)} := \Pi_N Q$, and

$$B_0^{(N)}(\omega) := \Pi_N B_0(\omega), \quad B_1^{(N)}(\omega) := \Pi_N B_1(\omega), \quad \omega \in H_N.$$

The local existence and uniqueness of solution to (A.1) follows from a result for finite dimensional S.D.E.-s. By Itô's formula we get the estimate

$$\mathbb{E}\left\{|\omega^{(N)}(T)|^2 + \frac{1}{2}\int_0^T \|\omega^{(N)}(t)\|^2 dt\right\} \le |w^{(N)}|^2 + \|Q^{(N)}\|^2_{L_{(HS)(H,H)}} T \tag{A.2}$$

From this we conclude that the sequence $\{\omega^{(N)}(t),\, t \in [0,T]\}$, $N \ge 1$ is compact in $L^2(\Omega, \mathcal{E}_T)$. In addition,

$$\omega^{(N)}(t) = \mathrm{e}^{\Delta t} w^{(N)} - \int_0^t \mathrm{e}^{\Delta(t-s)} B_0^{(N)}(\omega^{(N)}(s))ds + \int_0^t \mathrm{e}^{\Delta(t-s)} B_1^{(N)}(\omega^{(N)}(s))ds + \int_0^t \mathrm{e}^{\Delta(t-s)} Q^{(N)} dW(s).$$

Any weak limiting point satisfies therefore (4.2). To show uniqueness we need the following.

Lemma A.1. *There exists a constant $C > 0$ such that for all $w_0, w_1 \in H$, and $t \ge 0$,*

$$|\omega(t; w_0) - \omega(t; w_1)| \le |w_0 - w_1| \exp\left\{C \int_0^t \|\omega(s; w_0)\|^2 ds\right\}, \qquad \mathbb{P} - a.s. \tag{A.3}$$

Proof. Let $\rho(t) := \omega(t; w_1) - \omega(t; w_0)$ and $r(t) := \mathcal{K}(\rho(t))$. From (7.14) we conclude

$$\frac{d}{dt}|\rho(t)|^2 = -2\|\rho(t)\|^2 - 2\langle (r(t)\cdot\nabla)\omega(t; w_0), \rho(t)\rangle + 2\langle (r(t,0)\cdot\nabla)\omega(t; w_0), \rho(t)\rangle. \tag{A.4}$$

To deal with the second term on the right hand side we use the following estimate. Suppose that $v = \mathcal{K}(h)$. Then, for any $r > 0$ there exists a constant $C > 0$ such that

$$|\langle (v\cdot\nabla) f, g\rangle| \le C\|f\| |g|_r |h|, \quad \forall\, f \in V,\ g \in H_r,\ h \in H \tag{A.5}$$

and

$$|\langle (v\cdot\nabla) f, g\rangle| \le C\|f\| |g| |h|_r, \quad \forall\, g \in H,\ f \in V,\ h \in H_r, \tag{A.6}$$

see e.g. (6.10) of [2]. With these two inequalities in mind we conclude from (A.4) that

$$\begin{aligned}\frac{d}{dt}|\rho(t)|^2 \quad &\le -2\|\rho(t)\|^2 + C\|\omega(t; w_0)\| |\rho(t)|_{1/2} |\rho(t)| \\ &\le -2\|\rho(t)\|^2 + C_1\|\omega(t; w_0)\|^2 |\rho(t)|^2 + 2\|\rho(t)\|^2.\end{aligned}$$

By Gronwall's inequality we conclude then (A.3). □

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