HOPF IMAGES IN LOCALLY COMPACT QUANTUM GROUPS

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Oświadczam, że niniejsza rozprawa została napisana przeze mnie samodzielnie.

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Niniejsza rozprawa jest gotowa do oceny przez recenzentów.

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Streszczenie

Motywem przewodnim rozprawy jest zagadnienie generowania podgrupy kwantowej ustalonej grupy kwantowej lokalnie zwartej. We właściwym dla badania grup kwantowych sformułowaniu dualnym, miast o podgrupie kwantowej generowanej przez dany zbiór kwantowy mówimy ściśle o *obrazie Hopfa morfizmu*. Terminy te powinny być uważane za tożsame, lecz ze względu na charakterystykę teorii grup kwantowych, będziemy się starali używać terminu *obraz Hopfa morfizmu*.

Grupy kwantowe przyciągnęły uwagę, gdy S. L. Woronowicz zaproponował, by używać C^*-algebraicznego formalizmu do ich badania (zob. [62, 68]). Jednym z głównych osiągnięć teorii jest jednoczesne rozszerzenie dualności Pontrjagina i dualności Tannaki-Krajna na istotnie większą klasę obiektów, zawierającą zarówno grupy abelowe (jak w klasycznej dualności Pontrjagina), jak i grupy zwarte (jak w dualności Tannaki-Krajna, która proponuje alternatywny opis dualności, w której obiektem dualnym jest kategoria zamiast przestrzeni), jak wyjaśniono w [65]. Wyniki te rozszerzono dalej do ogólnych lokalnie zwartych grup kwantowych w [39, 54], czyniąc tym samym istotny postęp w obszarze abstrakcyjnej analizy harmonicznej: istnieje dobrze zdefiniowany obiekt dualny dowolnej lokalnie zwartej grupy (niekoniecznie zwartej bądź abelowej), który jest po prostu grupą kwantową lokalnie zwartą. Co więcej, teoria ta jest zamknięta na branie obiektów dualnych.

Teoria ta, choć bardzo owocna z perspektywy analitycznej, wciąż wymaga dopracowania z perspektywy teoriogrupowej. Teoriogrupowa natura grup kwantowych nie została zbadana dokładnie i dopiero niedawno poczyniono w tym kierunku pewne postępy. Na przykład, pojęcie podgrupy ([22]) i homomorfizmu ([43]) zostały dokładnie opisane zaledwie kilka lat temu, a na przykład relacja między produktem półprostym oraz krótkimi ciągami dokładnymi została opisana bardzo niedawno ([36, 35]). To dość zaskakujące, że teoriogrupowa natura grup kwantowych pozostawała na uboczu tak długo. Aktualnie jednak stała się bardzo aktywnym polem badań.

Grupy kwantowe zostały stworzone w celu uchwycenia pewnego rodzaju kwantowych symetrii ukrytych w modelach fizycznych. Teoria grup kwantowych, choć bardzo atrakcyjna z perspektywy matematycznej, wciąż nie jest gotowa do zastosowań w fizyce. W modelach mechaniki kwantowej, obserwable modelowane są przy pomocy operatorów w przestrzeni Hilberta. Ich komutowanie może być uchwycone przy pomocy klasycznych grup. Niemniej, dla zastosowań praktycznych, nie można ograniczać się do układów operatorów komutujących (ani żadnego rodzaju restrykcje dotyczące komutowania nie powinny być nakładane), zaś to teoria grup kwantowych potrafi uchwycić symetrie zachowującą daną *miarę nieprzemienności* danego układu operatorów, zob. np. [1] i referencje tamże. Mamy nadzieję, że dalsze rozwijanie matematycznych aspektów teorii grup kwantowych pozwoli na ich zastosowanie w modelach mechaniki kwantowej bądź innych gałęziach współczesnej fizyki teoretycznej.

Klasycznie, zbiory generujące są przydatne do niektórych rozumowań indukcyjnych i w badaniu pewnego rodzaju zachowań w nieskończoności. Używa się ich do zdefiniowania grafów Cayleya (które dla grup kwantowych dyskretnych zostały wprowadzone przez Vergnioux w [58]) i bezpośrednio związane z funkcjami długości na grupach, które pozwalają na naturalne partycjonowanie grupy w większe i większe, lecz skończone (czy też zwarte) kawałki. Używa się ich do badania pewnych własności aproksymacyjnych, jak własność Haagerupa czy K-średniowalność (zob np. [21, 27] oraz [56, 57], odpowiednio). Spodziewamy się, że pogłębienie rozumienia pojęcia zbiorów generujących w kontekście lokalnie zwartych grup kwantowych może okazać się owocne i otworzyć możliwość lokalnego opisu tych własności.

Jednym z najbardziej spektakularnych zastosowań zbiorów generujących jest teoria Kestena

spacerów losowych na grupach dyskretnych. W teorii tej, stan układu w chwili zero jest opisywany przy pomocy rozkładu probabilistycznego na skończonym zbiorze generującym (i nawet najprostszy rozkład jednostajny wystarcza do uzyskania istotnych wyników). Dynamika układu opisywana jest potęgami splotowymi tegoż rozkładu, zaś zachowanie asymptotyczne dynamiki pozwala uchwycić pewne nietrywialne własności grupy, np. to, czy jest (bądź nie) średniowalna (zob. oryginlna praca Kestena [37], choć dziś wynik ten jest uważany za klasyczny i znajduje się w wielu podręcznikach dotyczących metod probabilistycznych badania grup). Spodziewamy się, że nasz opis zbiorów generujących dla grup kwantowych pozwoli otworzyć nową perspektywę dla badań podobnych zagadnień na grupach kwantowych lokalnie zwartych. Należy tutaj podkreślić, że spacery losowe na grupach kwantowych były już badane ([28]), co pozwoliło uzyskać niejeden satysfakcjonujący wynik.

Badania zbiorów generujących mogą być także użyteczne w klasyfikacji podgrup ustalonej grupy kwantowej: każda z nich jest generowana przez pewien kwantowy podzbiór. Wysoka ranga opisu struktury kraty podgrup kwantowych danej grupy kwantowej nie podlega dyskusji. Zadanie to postawiono explicite w [7] w przypadku kwantowych grup permutacji S_n^+ i związane było z klasycznym rozumieniem podgrup grupy permutacji: skoro są to symetrie zbioru n-elementowego bez struktury, jej podgrupy odpowiadają symetriom zachowującym pewną (geometryczną bądź kobimnatoryczną) strukturę tej przestrzeni. Badania kwantowych symetrii grafów (m.in. [4, 11, 16]) oparte były na wskazaniu podgrup kwantowych, które zachowywały pewną strukturę klasyczną. Dokładny opis struktury podgrup kwantowych grup permutacji pozwoliłby na zidentyfikowanie pewnego rodzaju *kwantowych struktur* zbioru n-elementowego, których to struktur symetrie byłby opisywane przez te podgrupy kwantowe.

Rozprawa zawiera dokładny opis pojęcia podgrupy kwantowej generowanej przez podzbiór kwantowy (tj. obrazu Hopfa ustalonego morfizmu). Rozważamy pytania w duchu tych zaanonsowanych na wstępie, lecz kluczowym składnikiem pracy jest opis procedury generowania samej w sobie. To dokładny opis procedury generowania pozwala uzyskać wyniki dotyczące własności zbiorów generujących. Materiał jest podzielony na cztery rozdziały.

Rozdział pierwszy zawiera, głównie choć nie jedynie, przygotowania potrzebne do czytania rozprawy. Zebraliśmy rozmaite wyniki występujące w literaturze dotyczące przestrzeni Banacha i ich przekształceń (część pierwsza), teorii C^*-algebr i algebr von Neumanna (część druga) oraz teorii grup kwantowych (część trzecia). Rozbudowana prezentacja mogłaby być bardziej skondensowana, lecz zależało nam na zarysowaniu szerokiego obrazu. Szczegółowy opis niektórych elementów teorii grup kwantowych był również pomocny dla wprowadzania obiektów rozważań w bardziej naturalny i logiczny sposób. Dzięki niemu wygodniej również było wprowadzić oznaczenia. W części 1.2.2 uwzględniliśmy pewne wyniki wraz z dowodami, których nie udało nam się znaleźć w literaturze w formie, w której ich będziemy potrzebować. Ich wersja jest na pewno znana ekspertom i nie przypisujemy sobie ich odkrycia. Istotnie, główny wynik części 1.2.2 można znaleźć w Appendiksie książki [10] w przypadku reprezentacji grup, zaś my używaliśmy ich w kontekście reprezentacji C^*-algebr. Jednak nawet w przypadku reprezentacji grup dowód umieszczony w części 1.2.2 jest krótszy i łatwiejszy niż ten zawarty w książce [10].

Rozdział drugi zawiera kluczowe elementy rozprawy. Koncepcja obrazu Hopfa jest zaprezentowana i badana dogłębnie. W pierwszej części rozdziału podajemy dokładne sformułowanie problemu istnienia obrazu Hopfa. W części 2 podajemy konstrukcję i kilka pierwszych własności algebr von Neumanna, które grają w tej konstrukcji istotną rolę. Kończymy tę część podając dowód własności uniwersalnej obrazu Hopfa. Część trzecia poświęcona jest porównaniu naszej konstrukcji obrazu Hopfa z innymi, znanymi w literaturze, pojęciami odpowiadających generowaniu grupy kwantowej. W szczególności, konfrontujemy naszą konstrukcję z:

1. konstrukcją algebraiczną obrazu Hopfa dla *-algebr Hopfa á la Banica i Bichon;

2. konstrukcją C^*-algebraiczną obrazu Hopfa dla zwartych grup kwantowych á la Skalski i Sołtan;

3. teorio-reprezentacyjnym opisem obrazu Hopfa zwartej macierzowej grupy kwantowej á la Brannan, Collins i Vergnioux oraz

4. pojęciem (skończonego) zbioru generującego dla dyskretnej grupy kwantowej á la Izumi i Vergnioux.

W rozdziale 3 zebraliśmy rozmaite wyniki dotyczące konstrukcji obrazu Hopfa. Większość z nich jest sformułowanych dla lokalnie zwartych grup kwantowych, lecz część z nich byliśmy w stanie udowodnić jedynie przy pewnych dodatkowych założeniach. Motywem przewodnim tych wyników jest *perspektywa lokalna*, którą zapewnia pojęcie obrazu Hopfa. Formułujemy również pewne kryteria pozwalające stwierdzić, czy dany zbiór jest generującym. Dalej podajemy pierwsze przykłady kwantowych zbiorów generujących. Część z nich otrzymujemy dzięki przeinterpretowaniu istniejących wyników z literatury przy pomocy technik i wyników rozdziału 2. Formułujemy również własność (FAG) dotyczącą pewnej specjalnej roli podgrupy charakterów w procedurze generowania. Własność ta ma charakter równoważności, w której jedna z implikacji jest stosunkowo prosta i wystarczy do uzyskania pewnych konkluzji w rozdziale czwartym. Implikację przeciwna nie zachodzi w pełnej ogólności z powodów analitycznych, lecz formułujemy pewne warunki, przy których analityczne przeszkody przestają grać rolę. Jednak grupa kwantowa posiada własność (FAG), jesteśmy w stanie uzyskać kilka ciekawych wniosków.

W rozdziale czwartym podajemy również pierwszy prawdzwie kwantowy przykład zbioru generującego: kwantowe ciągi rosnące $\mathbb{I}_{2,4}$ generują kwantową grupę permutacji S_4^+, co daje odpowiedź na pytanie Skalskiego i Sołtana z [49]. Do odpowiedzi na to pytanie używamy własności badanej w rozdziale 3. Dalej podajemy pewne dodatkowe wyniki dotyczące kwantowej grupy permutacji S_4^+. Nie wszystkie z nich są bezpośrednio związane z zagadnieniem obrazu Hopfa, lecz dają lepszy opis tej grupy kwantowej. Bazując na wcześniejszej pracy Baniki i Bichona ([5]), jesteśmy w stanie:

1. podać grupę automorfizmów S_4^+;
2. sklasyfikować włożenia $O_{-1}(2) \subset S_4^+$;
3. sklasyfikować włożenia $A_5^\tau \subset S_4^+$;

Powodem, dla którego koncentrujemy się na włożeniach $O_{-1}(2)$ i A_5^τ w S_4^+ jest fakt wynikający z rezultatów otrzymanych w [5]: są to maksymalne podgrupy właściwe. Dzięki uzyskanej klasyfikacji włożeń podgrup $O_{-1}(2)$ i A_5^τ w S_4^+ jesteśmy w stanie pokazać, że S_4^+ nie ma własności (FAG), tj. (FAG) jest faktycznie własnością grupy kwantowej, nie stwierdzeniem o wszystkich grupach kwantowych. Pokazujemy również, jak przy pomocy podanego kryterium można łatwo wywnioskować hiperliniowość grupy kwantowej $\widehat{S_4^+}$.

Summary

The leitmotiv of the thesis is the notion of generation of a quantum subgroup of a given locally compact quantum group. The precise formulation in the spirit of the theory of quantum groups should be contravariant, then instead of speaking on quantum subgroup generated by a quantum subset we speak on the *Hopf image of a morphism*. These terms should be considered equivalent, but the characteristics of the theory of quantum groups is so that the term *Hopf image of a morphism* is usually more convenient.

Quantum groups attracted attention when S. L. Woronowicz proposed a C^*-algebraic formalism to study them (see [62, 68]). One of the main achievements in the field was the simultaneous extension of both classical Pontrjagin duality and Tannaka-Krein duality to a substantially wider class of objects, containing both abelian groups (like in classical Pontrjagin duality) and compact groups (as in Tannaka-Krein duality, which proposes an alternative description of duality, in which the dual object is a category rather than a space), as explained in [65]. This was further generalized to locally compact quantum groups in [39, 54] making a step forward in abstract harmonic analysis: there is now a good notion of a dual object for any locally compact group (not necessarily compact or abelian), which is simply a locally compact quantum group. Moreover, the theory is closed under taking duals.

As the theory turned out to be fruitful from analytic point of view, there is still a desire to find some more group-theoretic treatment of quantum groups. This area is not explored very thoroughly in the literature and it is only recently that some development in this direction has been made. For instance, the notions of subgroup ([22]) and of homomorphisms ([43]) were examined just several years ago, whereas the relation of semidirect product and short exact sequences has been covered just recently ([34, 35]). It is surprising that the group-theoretic nature of quantum groups has remained on sidelines for such a long time, but currently it is becoming a very active area of research.

Quantum groups were designed in order to capture a kind of quantum symmetry hidden in some physical models. The theory, although very attractive from the mathematical perspective, is still not ready for physical applications. In models of quantum mechanics, observables are represented by operators in Hilbert spaces. Their commutation relations can be captured by symmetries measured by classical groups. However, for practical purposes one should not restrict attention to tuples of commuting operators (nor should any commutation relations be imposed) and it is the quantum group theory that can capture the symmetries of a system preserving the *amount of non-commutativity* of a given tuple, see e.g. [1] and references therein. We hope that developing the mathematical side of the theory of locally compact quantum groups can eventually render the theory to be applicable in models of quantum mechanics or some other branches of modern theoretical physics.

Classically, generating sets are useful to proceed with various induction arguments and in study of certain type of behavior-at-infinity. They are used to define Cayley graphs (which were defined for discrete quantum groups by R. Vergnioux [58]), closely related to distances on groups, which create a natural way of partitioning the group into bigger and bigger but finite parts. Those are often used in order to study certain approximation properties, e.g. the Haagerup property or K-amenability (see e.g. [21, 27] and [56, 57], respectively). We expect that deepening the understanding of generating sets in the context of locally compact quantum groups may turn fruitful and open a more local approach to the study of these objects.

One of the most significant of applications of generating sets is Kesten's approach to random walks on discrete groups. The state of the system at time zero is usually described by certain probability distribution on a finite generating set (even the simplest one, uniform distribution, leads to major achievements). Its dynamics is described by taking convolution powers of the probability distribution at time zero and the asymptotic behavior of this dynamics can capture various nontrivial properties of the group itself, e.g. whether or not it is amenable (see the original article of H. Kesten [37], but this result is now considered classical and is contained in various textbooks on probabilistic methods on groups). We expect that our study of generating sets for quantum groups can give a new perspective for the study of these questions for quantum groups. Let us remark here that the study of random walks on quantum groups has been already undertaken (e.g. [28]) and turned to be fruitful.

The study of quantum generating sets may also be used to classify subgroups of a given quantum group: each of them needs to be generated by some quantum subset. The importance of studying the structure of the lattice of subgroups of a given quantum group is commonly agreed upon. The questions posed in [7] on the aforementioned lattice in case of S_n^+, the quantum permutation group, were related to the classical understanding of subgroups of permutations: if these are symmetries of n-point space, its subgroups are related to symmetries of certain (combinatorial or geometric) structures of n-point space. The study of quantum symmetries of graphs (see, e.g., [4, 11, 16]) amounted to identifying those quantum permutations which preserved certain classical structure. The precise description of the lattice of subgroups of quantum permutation groups would shed some light on identifying some kind of *quantum structures* of an n-point space, whose symmetries are described by these quantum subgroups.

The thesis contains a thorough study of the notion of generation of a quantum subgroup (i.e. a Hopf image of a given morphism). We address several of the questions discussed above in specific examples, but central to our thesis is the procedure of generation itself. It is this construction that allow to prove some specific properties of generating sets. The material is organized as follows.

Chapter 1 contains, mainly but not only, the preliminaries needed to read the thesis. Namely, we collected some of the results from the literature that concern the theory of transformations of Banach, and in particular Hilbert, spaces (Section 1.1), the theory of C^*-algebras and von Neumann algebras (Section 1.2) and the theory of quantum groups (Section 1.3). Parts of the exposition are not necessary, but we included it in order to provide a broader picture. It was also more convenient to provide a more elaborate description of the elements of the theory of quantum groups in order to present somewhat more logical introduction of the objects under consideration. It was also more convenient from the perspective of establishing the notation. In Section 1.2.2 we included some results not present in the litature in the form we need, so we provided the proofs. We do not claim any originality, the results were probably known to the experts. Indeed, the key result of Section 1.2.2 can be found in the Appendix of the book [10] in the case of group representations, whereas we needed these results in the context of representations of C^*-algebras. Even in the case of group representations, the proof we provided is shorter and simpler than the one given in [10].

Chapter 2 is central to our thesis. There the concept of Hopf image is presented and studied rigorously. We proceed in steps. In Section 2.1, the precise formulation of the problem of existence is presented. In Section 2.2, we provide the construction and some properties of the von Neumann algebras in play that are needed in further study. We finish Section 2.2 by presenting the proof of the universal property of Hopf image. Section 2.3 is dedicated to comparison of our construction of Hopf image to other notions of Hopf images and other notions of generation of quantum groups. In particular, we concentrate on:

1. the algebraic construction of Hopf image for Hopf *-algebras á la Banica and Bichon;
2. the C^*-algebraic version of it for Compact Quantum Groups á la Skalski and Sołtan;
3. a representation-theoretic approach to Hopf image for Compact Matrix Quantum Groups á la Brannan, Collins and Vergnioux and
4. the notion of (finite) generating set for discrete quantum groups á la Vergnioux.

In Chapter 3 we collected various general results concerning the construction of Hopf image. Most of them are formulated in the context of locally compact quantum groups, but some of them we were able to prove only under some additional assumptions. The leitmotiv of these results is the *local perspective*, which the notion of Hopf image allows. We provide also some criteria for a subset to be generating. Later on, we present first examples of quantum generating sets. Some of them are obtained by reinterpreting existing results from the literature and results of Chapter 2. We also formulate Property (FAG) concerning a special role of subgroup of characters in the procedure of generation. Property (FAG) has the form of equivalence of two conditions, one of the implications is easily obtained and is enough to conclude some results in Chapter 4. There are some analytic obstacles for other implications to hold and the Property is phrased so that these analytic obstacles vanish. This property, if true, it enables to draw some very interesting conclusions.

In Chapter 4 we provide the first genuinly quantum example of a generating set: quantum increasing sequences $\mathbb{I}_{2,4}$ generate the whole quantum permutation group S_4^+, which answers a question of Skalski and Sołtan from [49]. To answer this question we use the established part of Property (FAG). We further present some additional results on quantum permutation group S_4^+. Not all of them are directly related to the notion of Hopf image, but they provide a better description of the quantum group. Building on previous work of Banica and Bichon ([5]), we were able to

1. classify the automorphisms of S_4^+;
2. classify the embeddings $O_{-1}(2) \subset S_4^+$;
3. classify the embeddings $A_5^\tau \subset S_4^+$;

The reason we are interested in the embeddings of quantum groups $O_{-1}(2)$ and A_5^τ into S_4^+ is because from the results of [5] it follows that they are maximal proper subgroups. Thanks to the obtained classification of embeddings of $O_{-1}(2)$ and A_5^τ into S_4^+ we show that the quantum group S_4^+ lacks Property (FAG), i.e. the two conditions studied are not equivalent in full generality. We finish with showing how our criterion enables us to show that $\widehat{S_4^+}$ is hyperlinear in an elementary way.

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Chapter 1

Preliminaries

1.1 Banach and Hilbert spaces and their transformations

1.1.1 Banach and Hilbert spaces

All the vector spaces appearing in the thesis will be complex. If X is a Banach space and $T: \subset X \to X$ is a linear operator, we denote by $\mathcal{D}(T) \subseteq X$ its domain. The operator T is called closed, if $\{(x, Tx) : x \in \mathcal{D}(T)\} \subseteq X \times X$ is closed for the product topology. T is bounded if and only if $\mathcal{D}(T) = X$ and T is closed – this is one of the corollaries of the famous Banach-Steinhaus Theorem (also known as Uniform Boundedness Principle).

All Hilbert spaces $\mathcal{H}$ will be endowed with an inner product $\langle\cdot|\cdot\rangle: \mathcal{H} \times \mathcal{H} \to \mathbb{C}$, which is sesquilinear form that is **linear in the right variable**. Given $\xi, \eta \in \mathcal{H}$, the functional $\langle\xi|\cdot|\eta\rangle \in \mathsf{B}(\mathcal{H})_*$ will be denoted $\omega_{\xi,\eta}$ and if $\xi = \eta$ we shorten this notation to $\omega_\xi = \omega_{\xi,\xi}$. We will also freely use the bra-ket notation, so that in particular $|\xi\rangle\langle\xi|$ is a rank-one orthogonal projection onto $\mathbb{C}\xi$, whenever $\|\xi\| = 1$.

For a C^*-algebra A we denote by A_{sa} and A_+ the sets of self-adjoint elements and of all positive elements of A, respectively. We will always denote by $\mathsf{K}(\mathcal{H})$ the C^*-algebra of compact operators on the Hilbert space $\mathcal{H}$ and by $\mathsf{B}(\mathcal{H})$ the algebra of all bounded operators on $\mathcal{H}$.

The symbol $\otimes_{alg}$ will denote the (algebraic) tensor product of vector spaces, the symbol $\otimes$ is reserved for completed ones. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces. We denote by $\mathcal{H}_1 \otimes \mathcal{H}_2$ a unique Hilbert space obtained by completion of $\mathcal{H}_1 \otimes_{alg} \mathcal{H}_2$. Similarily, if $\mathsf{A}_1, \mathsf{A}_2$ are C^*-algebras, then $\mathsf{A}_1 \otimes \mathsf{A}_2$ denotes their minimal tensor product (a.k.a. spatial tensor product).

By σ we mean the tensor flip map (whatever the objects are): the unique extension of the map $\mathsf{A} \otimes \mathsf{B} \ni x \otimes y \overset{\sigma}{\longmapsto} y \otimes x \in \mathsf{B} \otimes \mathsf{A}$.

1.1.2 One-parameter group of transformations and analytic generator

In this section we recall some of the standard knowledge about one-parameter groups of transformations and their analytic generators. We refer to [17] for proofs and explanations. Let (X, F) be a dual pair of Banach spaces (that is, $F \subseteq X^*$ is a closed subspace and we use the standard pairing). The F-topology on X is the weakest linear topology which makes the pairing continuous. We assume that the pair (X, F) satisfies the following property: the convex hull of every relatively F-compact subset of X is relatively F-compact. In most of the cases we will be interested only in the case (X, X^*) or (X^*, X) (with the canonical embedding $X \subseteq X^{**}$), which satisfy this property.

Let Ω be a locally compact space, μ a complex regular Borel measure on Ω with variation $|\mu|$. Let $f: \Omega \to X$ be a function such that: for all $\varepsilon > 0$ and all $K \subseteq \Omega$ compact one can find $L \subseteq K$ compact such that $\mu(K \setminus L) < \varepsilon$, $f\restriction_L$ is F-continuous and such that

$$\Omega \ni \omega \mapsto |f(\omega)|$$

has $|\mu|$-integrable majorant. Then there exists a unique $x_f \in X$ such that for all $\varphi \in F$ we have

$$\int_\Omega \langle f(\omega)|\varphi\rangle \mathrm{d}\mu(\omega) = \langle x_f, \varphi\rangle.$$

This unique element will be denoted as

$$x_f = F - \int_\Omega f(\omega)\mathrm{d}\mu(\omega).$$

Above, F-continuous means that all the maps $\omega \mapsto \langle f(\omega)|\varphi\rangle$ are continuous ($\varphi \in F$). If $\Omega \subseteq \mathbb{C}^n$, then a function $f\colon \Omega \to X$ is F-regular, if it is F-continuous and analytic in the interior Ω^o.

Let $(U_t)_{t\in\mathbb{R}}$ be a one-parameter group of transformations, i.e. $U_0 = \mathrm{id}$ and $U_s U_t = U_{s+t}$, that is pointwise-F-continuous: for every $x \in X$, the map $t \mapsto U_t x\colon \mathbb{R} \to X$ is F-continuous. For $t_1 \leq 0 \leq t_2$ real, consider the sets

$$\mathcal{D}(t_1, t_2) = \{x \in X : it \mapsto U_t x \text{ has an } F\text{-regular extension on } t_1 \leq \Re(z) \leq t_2\}$$

If $x \in \mathcal{D}(t_1, t_2)$, then the F-regular extension of $it \mapsto U_t x$ is unique, call it F_x. Denote

$$\mathcal{D} = \bigcap_{\substack{t_1\leq 0\\ t_2\geq 0}} \mathcal{D}(t_1, t_2),$$

one forms the analytic extension of U_t at $z \in \mathbb{C}$, denoted B_z, by $\mathcal{D}(B_z) = \mathcal{D}(\Re(z), 0)$ if $\Re(z) \leq 0$ and $\mathcal{D}(B_z) = \mathcal{D}(0, \Re(z))$ if $\Re(z) \geq 0$, then put $B_z x = F_x(z)$. Then B_1 is the analytic generator of $(U_t)_{t\in\mathbb{R}}$. It recovers the group $(U_t)_{t\in\mathbb{R}}$ completely:

$$U_t x = F - \lim_{\substack{z\to it\\ 0<\Re(z)<1}} \frac{\sin(\pi z)}{\pi} F - \int_0^\infty \lambda^{z-1}(\lambda + B_1)^{-1} B_1 x \mathrm{d}\lambda$$

where $x \in \mathcal{D}(B_1)$. In particular, if $Y \subseteq X$ is a closed subspace that is preserved by the whole one-parameter group $(U_t)_{t\in\mathbb{R}}$, observe that the analytic generator of $(U_t \restriction_Y)_{t\in\mathbb{R}}$ is precisely $B_1 \restriction_Y$. Indeed, by uniqueness of the F-regular extension in the definition of $\mathcal{D}(t_1, t_2)$ one sees that for $y \in Y \cap \mathcal{D}(-\infty, \infty)$ the vector $B_1 y$ is independent of whether we consider $(U_t)_{t\in\mathbb{R}}$ or $(U_t \restriction_Y)_{t\in\mathbb{R}}$. In particular, $B_1 y \in Y$.

Let us end with the remark that an element $B_{1/2}$ is sometimes called an analytic generator of a one-parameter group $(U_t)_{t\in\mathbb{R}}$. This shall cause no confusion, as from the above integral formula it is clear that one recovers the group $(U_{t/2})_{t\in\mathbb{R}}$ from the generator $B_{1/2}$, which is essentially the same group of operators. In the theory of locally compact quantum groups, this viewpoint is more common.

1.2 C^*-algebras

1.2.1 Category $\mathfrak{C}^*\mathfrak{alg}$

Let A, B be two C^*-algebras and let $\varphi\colon \mathsf{A} \to \mathsf{B}$ be a *-homomorphism. Recall that then:

1. $\|\varphi(a)\| \leq \|a\|$ for all $a \in \mathsf{A}$ ([24, Proposition 1.3.7])
2. $\varphi(\mathsf{A}) \subset \mathsf{B}$ is closed ([29, Theorem 4.1.9]).

We will need the following classical result (established firstly in [18]).

Theorem 1.1 (Cohen's factorization theorem)**.** *Let V be a non-degenerate left module over a Banach algebra A and let $v \in V$. Assume A has a bounded left approximate unit. Then there exists $w \in V$ and $a \in \mathsf{A}$ such that $v = aw$.*

In fact we will use this Theorem only when A is a C^*-algebra.

Recall that any C^*-algebra can be unitized in various ways. We stick to one particular unitization (a maximal one, in a sense): by means of the multiplier algebra. Let us recall how this algebra is obtained.

A double centralizer of a C^*-algebra A is a pair (L, R) of linear maps on A such that $aL(b) = R(a)b$ for all $a, b \in \mathsf{A}$ (note these maps are autimatically bounded). The set of double centralizers of A can be given a C^*-algebra structure: firstly, when (L, R) is a double centralizer, then in fact $L, R \in \mathsf{B}(\mathsf{A})$ (treating A as a Banach space) and it is easy to check that $\|L\| = \|R\|$. The vector space structure is obvious, composition is defined as $(L, R) \circ (L', R') = (L \circ L', R' \circ R)$. The involution is defined as follows: $(L, R)^* = (R^\#, L^\#)$, where $L^\#(a) = L(a^*)^*$ (and likewise for R), the map $(\mathrm{id}, \mathrm{id})$ is the unit among double centralizers. With the norm inherited from $\mathsf{B}(\mathsf{A})$ it is a C^*-algebra. It contains A, because for any $a \in \mathsf{A}$ the left multiplication map $L_a(b) = ab$ and the right multiplication map $R_a(b) = ba$ constitute a double centralizer (L_a, R_a). Thus we obtained the C^*-algebra $\mathsf{M}(\mathsf{A})$ of double centralizers.

Equivalently, we may pick a faithful non-degenerate representation $\pi\colon \mathsf{A} \to \mathsf{B}(\mathcal{H})$ and look at the set

$$\mathsf{M}(\mathsf{A}) = \{T \in \mathsf{B}(\mathcal{H}) | \forall_{a \in \mathsf{A}}\ T\pi(a) \in \pi(\mathsf{A}) \text{ and } \pi(a)T \in \pi(\mathsf{A})\}.$$

It may be checked that up to isomorphism, the above set does not depend on the choice of π.

The multiplier algebra is equipped with the strict topology: the topology induced by the family of seminorms $\mathsf{M}(\mathsf{A}) \ni T \mapsto \|Ta\|$ and $\mathsf{M}(\mathsf{A}) \ni T \mapsto \|aT\|$ for all $a \in \mathsf{A}$. For an excellent treatment of multiplier algebras, we refer to [13, Chapter VI]

We now turn to defining the category $\mathfrak{C}^*\mathfrak{alg}$. Its objects are C^*-algebras, and given $\mathsf{A}, \mathsf{B} \in \mathrm{Ob}(\mathfrak{C}^*\mathfrak{alg})$, a morphism $\varphi \in \mathrm{Mor}_{\mathfrak{C}^*\mathfrak{alg}}(\mathsf{A}, \mathsf{B}) = \mathrm{Mor}(\mathsf{A}, \mathsf{B})$ is by definition a non-degenerate *-homomorphism to the multiplier algebra of B, $\varphi\colon \mathsf{A} \to \mathsf{M}(\mathsf{B})$. Here, non-degenerate means that the set

$$\{\varphi(a)b | a \in \mathsf{A}, b \in \mathsf{B}\}$$

is linearly dense in B. By Cohen' factorization theorem, this is equivalent to saying that the above set is equal to B, as B is a left module over A by means of Φ: $a.b = \Phi(a)b \in \mathsf{B}$. Any morphism $\varphi \in \mathrm{Mor}(\mathsf{A}, \mathsf{B})$ can be uniquely extended to $\overline{\varphi}\colon \mathsf{M}(\mathsf{A}) \to \mathsf{M}(\mathsf{B})$, so that given $\varphi \in \mathrm{Mor}_{\mathfrak{C}^*\mathfrak{alg}}(\mathsf{A}, \mathsf{B})$ and $\psi \in \mathrm{Mor}_{\mathfrak{C}^*\mathfrak{alg}}(\mathsf{B}, \mathsf{C})$ we can set $\psi \circ \varphi \in \mathrm{Mor}_{\mathfrak{C}^*\mathfrak{alg}}(\mathsf{A}, \mathsf{C})$ to be equal to the map $\overline{\psi} \circ \varphi\colon \mathsf{A} \to \mathsf{M}(\mathsf{C})$.

Let us remark that $\mathsf{M}(\mathsf{K}(\mathcal{H})) = \mathsf{B}(\mathcal{H})$ and we will treat representations $\pi\colon \mathsf{A} \to \mathsf{B}(\mathcal{H})$ as morphisms $\pi \in \mathrm{Mor}(\mathsf{A}, \mathsf{K}(\mathsf{A}\mathcal{H}))$ (observe that by Cohen's factorization theorem $\mathsf{A}\mathcal{H}$ is automatically closed).

C^*-algebras can be viewed as a non-commutative generalization of topology. Indeed, with the above definition of $\mathfrak{C}^*\mathfrak{alg}$, the (by now classical) Gelfand-Naimark theory ensures us that $\mathfrak{C}^*\mathfrak{alg}$ is antiequivalent to $\mathfrak{LCTop}$, the category of locally compact Hausdorff topological spaces endowed with continuous maps as morphisms. More specifically, the following theorem adresses part of this antiequivalence:

Theorem 1.2. *Let X, Y be locally compact Hausdorff spaces and let $\mathsf{B} = C_0(X)$ and $\mathsf{A} = C_0(Y)$. Then:*

1. *any continuous $\phi\colon X \to Y$ defines a morphism $\Phi \in \mathrm{Mor}(\mathsf{A}, \mathsf{B})$ via*

$$\Phi(f) = f \circ \phi \qquad (f \in \mathsf{A}); \tag{1.1}$$

2. *for any $\Phi \in \mathrm{Mor}(\mathsf{A}, \mathsf{B})$ there exists a unique $\phi\colon X \to Y$ such that (1.1) holds;*

Fixing Φ and ϕ linked by (1.1), we have:

3. *the image of Φ is contained in B if and only if ϕ is a proper map,*
4. *ϕ has dense image if and only if Φ is injective,*
5. *ϕ is injective if and only if Φ has strictly dense range.*

Furthermore, there are several natural constructions one can do to produce a new C^*-algebras out of a family of given ones. Let us fix a family $(\mathsf{A})_{n\in N}$ and its concrete realizations $\mathsf{A}_n \subset \mathsf{B}(\mathcal{H}_n)$ (the inclusion is an element of $\mathrm{Mor}(\mathsf{A}_n, \mathsf{K}(\mathcal{H}_n))$, to be precise). Without loss of generality we can assume the indexing set N is well-ordered. Following [14], we indicate:

1. The direct sum $\bigoplus_{n\in N} \mathsf{A}_n$ is defined as follows. Let $v_n \in \mathsf{B}(\mathcal{H}_n, \bigoplus_{n\in N} \mathcal{H}_n)$ be an isometry sending $\mathcal{H}_n$ to the respective direct summand of $\bigoplus_{n\in N} \mathcal{H}_n$, then $\bigoplus_{n\in N} \mathsf{A}_n$ is by definition the C^*-algebra generated by $\{v_n a_n v_n^* : n \in N, a_n \in \mathsf{A}_n\} \subseteq \mathsf{B}(\bigoplus_{n\in N} \mathcal{H}_n)$.

2. the spatial/minimal tensor product $\bigotimes_{n\in N} \mathsf{A}_n$ is defined as follows (for the sake of our usage we restrict to the case $N = \{1, 2, \dots, M\} \subseteq \mathbb{N}$ finite). For $a \in \mathsf{A}_n$ we denote by $(a)_n$ the elemenent $\bigotimes_{i<n} \mathbb{1}_{\mathcal{H}_i} \otimes a \bigotimes_{i>n} \mathbb{1}_{\mathcal{H}_i}$. Then $\bigotimes_{n\in N} \mathsf{A}_n$ is by definition the C^*-algebra generated by $\{(a)_n : n \in N, a \in \mathsf{A}_n\} \subseteq \mathsf{B}(\bigotimes_{n\in N} \mathcal{H}_n)$.

3. the full free product $*_{n\in N}\mathsf{A}_n$, which is the universal C^*-algebra generated by copies of A_n with no additional relations.

4. the free product $*_{n\in N}\mathsf{A}_n$, provided that all A_n are unital, the free product is a quotient of the full free product by the ideal generated by $\{\mathbb{1}_{\mathsf{A}_n} - \mathbb{1}_{\mathsf{A}_m} : n, m \in N\}$. This construction is often called *the free product amalgamated over* $\mathbb{C}\mathbb{1}$ and has natural representation in a certain kind of Fock-type construction.

We also use the following notation. Let $\mathsf{A}, \mathsf{B} \subseteq \mathsf{B}(\mathcal{H})$ be norm-closed sets of operators. We define the products as

$$\mathsf{A} \circ \mathsf{B} = \mathrm{span}_{\mathbb{C}}\{ab : a \in \mathsf{A}, b \in \mathsf{B}\} \tag{1.2}$$

and

$$\mathsf{A} \cdot \mathsf{B} = \mathrm{span}_{\mathbb{C}}\{ab : a \in \mathsf{A}, b \in \mathsf{B}\}^{-\|\cdot\|} = (\mathsf{A} \circ \mathsf{B})^{-\|\cdot\|}. \tag{1.3}$$

Observe that $\cdot$ is associative.

1.2.2 Weak containment of representations

Let A be a C^*-algebra, let $\pi \in \mathrm{Mor}(\mathsf{A}, \mathsf{K}(\mathcal{H}))$ be a representation.

Definition 1.3 ([24, 2.3.1&2.3.2])**.** We say that π is irreducible if the following equivalent conditions hold:

1. if $\mathcal{K} \subseteq \mathcal{H}$ is a closed subspace and $\pi(\mathsf{A})\mathcal{K} \subseteq \mathcal{K}$, then $\mathcal{K} = \{0\}$ or $\mathcal{K} = \mathcal{H}$;

2. $\mathbb{C}\mathbb{1} = \pi(\mathsf{A})' \subseteq \mathsf{B}(\mathcal{H})$;

3. every non-zero vector in $\mathcal{H}$ is cyclic.

For every $\xi \in \mathcal{H}$, the (positive) functional $\phi \in \mathsf{A}^*_+$ given by $\phi = \omega_\xi \circ \pi$ will be called a *positive form on* A *associated with the representation* π. If $\|\xi\|= 1$, then this ϕ will be called a *state on* A *associated with the representation* π. If $\mathcal{K} \subset \mathcal{H}$ is a $\pi(A)$-invariant closed subspace, then we call (every representation unitarily equivalent to) $\pi\restriction_{\mathcal{K}}$ a subrepresentation of π and write $\pi\restriction_{\mathcal{K}} \subset \pi$. Let S be a set of representations, so that for $s \in S$ there is a representation $\pi_s \colon \mathsf{A} \to \mathsf{B}(\mathcal{H}_s)$.

Definition 1.4 ([24, 3.4.4&3.4.5])**.** We say that π is weakly contained in S, and write $\pi \prec S$, if the following equivalent conditions hold:

1. $\ker(\pi) \supseteq \bigcap_{s\in S} \ker(\pi_s)$;

2. every positive form on A associated with the representation π is a weak*-limit of linear combinations of positive forms associated with the representations π_s (for some $s \in S$);

3. every state on A associated with the representation π is a weak*-limit of states which are linear combinations of positive forms associated with the representations π_s (for some $s \in S$).

If the set $S = \{\rho\}$ contains a single reprentation, we write $\pi \prec \rho$ instead of $\pi \prec \{\rho\}$. As we are unable to find appropriate reference we are going to present here the proof of the following

Theorem 1.5. *Let A be a C^*-algebra and let π, ρ be representations of A. Assume that π is finite dimensional and that ρ is irreducible. If $\rho \prec \pi$, then $\rho \subset \pi$.*

Let us remark here that in the group case this is [10, Corollary F.2.9]. Here we will present an alternative short proof relying on condition (i) of the Definition of weak containment and the classical Wedderburn's Theorem.

Proof. Let us denote $\mathsf{A}_\pi = \pi(\mathsf{A}) \subseteq \mathsf{B}(\mathcal{H}_\pi)$. As π is finite dimensional, from Wedderburn's Theorem we know that $\mathsf{A}_\pi = \oplus_{j=1}^{J} M_{n_j}(\mathbb{C})$.

Similarly, let us denote $\mathsf{A}_\rho = \rho(\mathsf{A}) \subseteq \mathsf{B}(\mathcal{H}_\rho)$. From condition (i) of weak containment we know that there exists a surjective *-homomorphism $\phi \colon \mathsf{A}_\pi \to \mathsf{A}_\rho$ given by

$$\phi(\pi(a)) = \rho(a), \tag{1.4}$$

and in particular A_ρ is finite dimensional and hence ρ is fintie dimensional. Indeed, $\mathsf{A}_\rho \subseteq \mathsf{B}(\mathcal{H}_\rho)$ is WOT-closed, hence $\mathsf{A}_\rho = \mathsf{A}_\rho'' = \mathsf{B}(\mathcal{H})$ because ρ is irreducible. Conseqnently $\mathsf{B}(\mathcal{H}_\rho)$ is finite dimensional.

Again, as ρ is finite dimensional, $\mathsf{A}_\rho = \oplus_{k=1}^{K} M_{m_k}(\mathbb{C})$ by Wedderburn's Theorem. But on the other hand, as ρ is irreducible and as $\left(\oplus_{k=1}^{K} M_{m_k}(\mathbb{C})\right)' = \oplus_{k=1}^{K} \mathbb{C}\mathbb{1}_{M_{m_k}(\mathbb{C})}$, it follows that $K = 1$ and hence $\mathsf{A}_\rho = M_m(\mathbb{C})$.

But as matrix algebras are simple, $\phi \restriction_{M_{n_j}(\mathbb{C})}$ is either zero map or an isomorphism. Assume that for at least one j it is non-zero: otherwise we evidently have $\rho = 0$, which clearly satisfies $\rho \subseteq \pi$.

On the other hand, it is known that if a map $M_k(\mathbb{C}) \to M_m(\mathbb{C})$ is a surjective unital *-homomorphism, then necessarily $k = m$, and hence for at least one j the matrix summand $M_{n_j}(\mathbb{C})$ is isomorphically mapped to $M_m(\mathbb{C})$. Let e be the central projection associated to this matrix summand. Then (1.4) shows that $\pi(\cdot)e$ is equivalent to ρ. □

1.2.3 C^*-algebras generated by unbounded elements

In this section we recall elements of the theory of C^*-algebras generated by elements which do not necessarily belong to it. The exposition is based on [66].

Let A be a C^*-algebra. Recall that $a \in \mathsf{M}(\mathsf{A})$ is called strictly positive if $a \geq 0$ in the natural sense and $a\mathsf{A} \subseteq \mathsf{A}$ is dense. In such case we write $a > 0$. For $a, b \in \mathsf{M}(\mathsf{A})$ we say that b strictly dominates a, and write $b > a$ for this, if $b - a > 0$.

Now fix a faithful, non-degenerate representation $\rho \in \mathrm{Mor}(\mathsf{A}, \mathsf{K}(\mathcal{H}))$, we identify A with $\rho(\mathsf{A}) \subseteq \mathsf{B}(\mathcal{H})$. Recall that if T is a closed, densely defined operator acting on $\mathcal{H}$, then its z-transform is given by

$$z_T = T(\mathbb{1} + T^*T)^{-\frac{1}{2}}.$$

Recall that T is bounded if and only if $\|z_T\| < 1$, whereas $\|z_T\| \leq 1$ holds always. The z-transform z_T contains full information of T or, in other words, the z-transform $T \mapsto z_T$ is an invertible map. We have

$$T = z_T(\mathbb{1} - z_T^* z_T)^{-\frac{1}{2}}.$$

We say that T is affiliated with A (and write $T\eta\mathsf{A}$) if $z_T \in \mathsf{M}(\mathsf{A})$ and $z_T^* z_T < \mathbb{1}$. The set of elements affiliated with A is denoted A^η.

Let $\pi \in \mathrm{Mor}(\mathsf{A}, \mathsf{K}(\mathcal{K}))$ be another representation of A. As noted in Section 1.2.1, π extends uniquely to $\mathsf{M}(\mathsf{A})$. Even more is true, one can extend π to A^η: if $T\eta\mathsf{A}$, the element $\pi(z_T)$ is well defined and one can perform the inverse z-transform to get a closed, densely defined operator which will be denoted $\pi(T)$.

Although we will not use this notion, let us complete the picture. If $\phi \in \mathrm{Mor}(\mathsf{A}, \mathsf{B})$ is a morphisms of C^*-algebras, then for $T \eta \mathsf{A}$ one can show that in fact $\phi(T)\eta\mathsf{B}$ and hence ϕ extends to a map $\mathsf{A}^\eta \to \mathsf{B}^\eta$. The composition of morphisms in the sense described in Section 1.2.1 extends to affiliated elements.

Definition 1.6 ([66, Definition 4.1])**.** Let A and B be C^*-algebras and let $T \in (\mathsf{B} \otimes \mathsf{A})^\eta$. We say that A is generated by T if and only if for any Hilbert space $\mathcal{H}$, any representation $\rho \in \mathrm{Mor}(\mathsf{C}, \mathsf{K}(\mathcal{H}))$ and any representation $\pi \in \mathrm{Mor}(\mathsf{A}, \mathsf{K}(\mathcal{H}))$ we have

$$\Big(\Big((\mathrm{id} \otimes \pi)T\Big)\eta(\mathsf{B} \otimes \rho(\mathsf{C}))\Big) \implies \Big(\pi \in \mathrm{Mor}(\mathsf{A}, \rho(\mathsf{C}))\Big)$$

Lemma 1.7 ([22, Lemma 1.4])**.** *Let A and B be C^*-algebras with $\mathsf{B} \subseteq \mathsf{B}(\mathcal{H})$ non-degenerately represented in a Hilbert space $\mathcal{H}$. Let $T \in \mathsf{M}(\mathsf{B} \otimes \mathsf{A})$ be unitary and define*

$$S = \{(\omega \otimes \mathrm{id})(T) : \omega \in \mathsf{B}(\mathcal{H})_*\} \subset \mathsf{M}(\mathsf{A}).$$

If $S \subset \mathsf{A}$ and S generates A (as a subset of the C^*-algebra A) then $T \in \mathsf{M}(\mathsf{B} \otimes \mathsf{A})$ generates A.

1.2.4 Von Neumann algebras

The particular family of C^*-algebras, called von Neumann algebras (and sometimes also W^*-algebras), is of great significance in the theory of quantum groups. The facts we recall are standard and can be found in various monographs on von Neumann algebras, we based this part on [14, 29, 30, 47, 51, 52]. Let us recall that for a set of operators on a Hilbert space $S \subseteq \mathsf{B}(\mathcal{H})$ one defines its commutant as

$$S' = \bigcap_{x \in S} \{y \in \mathsf{B}(\mathcal{H}) : ys = sy\}.$$

Furthermore, one can consider various topologies on $\mathsf{B}(\mathcal{H})$, not only the norm-topology. We now recall the convergences in various topologies that are of interest. Let $(T_n)_{n \in N} \subseteq \mathsf{B}(\mathcal{H})$ be a net of operators (N being a directed set) and $T \in \mathsf{B}(\mathcal{H})$ a specified operator.

1. SOT (strong operator topology): $T_n \to T$ in SOT if and only if for every $\xi \in \mathcal{H}$ we have $T_n \xi \to T\xi$ in norm.

2. S*OT (strong* operator topology): $T_n \to T$ in S*OT if and only if for every $\xi \in \mathcal{H}$ we have that both $T_n \xi \to T\xi$ and $T_n^* \xi \to T^* \xi$ in norm.

3. WOT (weak operator topology): $T_n \to T$ in WOT if and only if for every $\xi \in \mathcal{H}$ we have $T_n \xi \to T\xi$ weakly, i.e. for all $\eta \in \mathcal{H}$ we have $\langle T_n \xi | \eta \rangle \to \langle T\xi | \eta \rangle$.

4. σ-SOT (ultrastrong operator topology): $T_n \to T$ if whenever $(\xi_k)_{k \geq 1} \subseteq \mathcal{H}$ is a sequence of vectors such that $\sum_{k \geq 1} \|\xi_k\|^2 < \infty$, then $\sum_{k \geq 1} \|(T_n - T)\xi_k\|^2 \to 0$. Equivalently, σ-SOT can be obtained as restriction of the SOT on the space $\mathsf{B}(\mathcal{H} \otimes \ell^2)$ to $\mathsf{B}(\mathcal{H})$ embedded via $\mathsf{B}(\mathcal{H}) \ni T \mapsto 1 \otimes \mathbb{1}_{\ell^2} \in \mathsf{B}(\mathcal{H} \otimes \ell^2)$.

5. σ-WOT (ultraweak operator topology) is defined analogously: it is the restriction to $\mathsf{B}(\mathcal{H}) \subseteq \mathsf{B}(\mathcal{H} \otimes \ell^2)$ of WOT of the latter space.

Theorem 1.8. *Let $\mathbb{1} \in \mathsf{M} \subseteq \mathsf{B}(\mathcal{H})$ be a *-subalgebra. Then the following are equivalent:*

- $\mathsf{M} = \mathsf{M}''$*;*
- M *is closed in SOT;*
- M *is closed in WOT;*
- M *is closed in σ-SOT;*

- M *is closed in σ-WOT.*

The equivalent conditions of the above Theorem are usually taken as the definition of a (concrete) von Neumann algebra. However, there is also an abstract characterisation, known as Sakai's theorem:

Theorem 1.9 (Sakai's theorem)**.** *Let M be a C^*-algebra. Then M is a von Neumann algebra if and only if it is a dual Banach space. Furthermore, the predual space is unique.*

It also worth mentioning that σ-WOT topology on a von Neumann algebra M is precisely the weak* topology of M (as a Banach space dual of its predual M_*). Moreover, σ-WOT and WOT coincide on bounded sets, and likewise for σ-SOT and SOT.

The right morphisms between von Neumann algebras $\mathsf{M} \to \mathsf{N}$ are normal unital *-homomorphisms: normal means just that they are obtained as dual of maps of predual spaces $\mathsf{N}_* \to \mathsf{M}_*$. The most important concepts from the theory of von Neumann algebras exploited in the theory of locally compact quantum groups are the tensor products of von Neumann algebras, and in particular the so calles slice maps, and the theory of weights. Let $i = 1, 2$ and $\mathsf{M}_i \subset \mathsf{B}(\mathcal{H}_i)$ be two von Neumann algebras. There are natural inclusions $\mathsf{M}_1, \mathsf{M}_2 \subseteq \mathsf{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$; we denote by $\mathsf{M}_1 \overline{\otimes} \mathsf{M}_2$ the von Neumann algebra generated by images of these two inclusions. We have the following result:

Theorem 1.10 ([30, Theorem 11.2.16])**.** $\left(\mathsf{M}_1 \overline{\otimes} \mathsf{M}_2\right)' = \mathsf{M}_1' \overline{\otimes} \mathsf{M}_2'$

Let also $\mathsf{N}_i \subset \mathsf{M}_i$ for $i = 1, 2$ be von Neumann subalgebras. Then one has the following useful corollary

Corollary 1.11 ([31, 12.4.36])**.** *$x \in \mathsf{N}_1 \overline{\otimes} \mathsf{N}_2$ if and only if $(\mathrm{id} \otimes \omega_2)(x) \in \mathsf{N}_1$ and $(\omega_1 \otimes \mathrm{id})(x) \in \mathsf{N}_2$ for all $\omega_k \in (\mathsf{M}_k)_*$, $k = 1, 2$.*

Lemma 1.12. *Let $\mathsf{A} \subseteq \mathsf{B}(\mathcal{H}), \mathsf{B} \subseteq \mathsf{B}(\mathcal{K})$ be concrete C^*-algebras and let $\mathsf{M} = \mathsf{A}''$ and $\mathsf{N} = \mathsf{B}''$. Let $T \in \mathsf{M}(\mathsf{A} \otimes \mathsf{B}) \subseteq \mathsf{M} \overline{\otimes} \mathsf{N}$ and let $(\omega_n)_{n \in N} \subseteq \mathsf{B}(\mathcal{H})_*$ be a net of normal functionals such that $\omega_n \to \omega \in \mathsf{B}(\mathcal{H})^*$ in the weak*-topology. Then $x_n = (\omega_n \otimes \mathrm{id})T \to (\omega \otimes \mathrm{id})T = x$ in σ-WOT of N.*

Proof. Let $\mu \in \mathsf{N}_*$, we need to show that $\mu(x_n) \to \mu(x)$, as σ-WOT is the same as weak*-topology in N. We have

$$\mu(x_n) = (\omega_n \otimes \mu)(T) = \omega_n((\mathrm{id} \otimes \mu)T). \tag{1.5}$$

Now as $t = (\mathrm{id} \otimes \mu)T \in \mathsf{M}$ and $\omega_n \to \omega$ in weak* topology of M^*, we have that $\omega_n(t) \to \omega(t)$. But this is equivalent to

$$\mu(x_n) \to \omega(t) = (\omega \otimes \mu)(T) = \mu(\omega \otimes \mathrm{id})T)$$

which finishes the proof by σ-WOT closedness of N. □

Let $\psi \colon \mathsf{M}_+ \to [0, \infty]$ be a weight, that is positively homogenous, additive map. We have the sets

$$\mathcal{M}_\psi = \{x \in \mathsf{M}_+ : \psi(x) < +\infty\}$$

and

$$\mathcal{N}_\psi = \{x \in \mathsf{M} : \psi(x^* x) < +\infty\}.$$

We say that ψ is *semi-finite*, if $\mathcal{N}_\psi$ is WOT-dense in M. We say that ψ is *faithful* if for any $x \in \mathsf{M}_+$ we have that $\psi(x) = 0 \iff x = 0$. Lastly, we say that ψ is normal whenever given a net of positive elements $(x_n)_{n \in N} \subset \mathsf{M}_+$ we have $\psi(\sup_{n \in N} x_n) = \sup_{n \in N} \psi(x_n)$. In what follows we essentially work only with normal semi-finite faithful (n.s.f.) weights.

Of course, given a state $\varphi \in M^*$ and a positive scalar $\lambda \geq 0$ the formula $\psi = \lambda \varphi \restriction_{M_+}$ defines a weight (normal if and only if $\varphi \in M_*$). Moreover, if $\mathbb{1} \in \mathcal{M}_\psi$, then there exists a positive scalar $\lambda \geq 0$ and a state $\varphi \in M^*$ such that $\psi = \lambda \varphi \restriction_{M_+}$. One can perform the GNS-construction for φ and get the triple $(\mathcal{H}_\varphi, \pi_\varphi, \Omega_\varphi)$.

The technical obstacles one has to meet when performing a similar construction for a non-bounded weights are tractable. Then, the role of Ω_φ has to be understood correctly. In fact what is crucial is the non-necessarily closed subspace $\mathsf{M}\Omega_\varphi \subseteq \mathcal{H}_\varphi$. If ψ is a weight, then one can endow the space $\mathcal{N}_\psi$ with a seminorm (norm only if ψ is faithful) $\|x\|_2 = \|x\|_{2,\psi} = \psi(x^*x)^{\frac{1}{2}}$ and as usual perform Hausdorff completion to get the Hilbert space $\mathcal{H}_\psi$. The map identifying $x \in \mathcal{N}_\psi$ with its image in $\mathcal{H}_\psi$ will be called η_ψ. The representation $\pi\colon \mathsf{M} \to \mathsf{B}(\mathcal{H})$ induced from left multiplication is well defined and we call the resulting triple $(\mathcal{H}_\psi, \eta_\psi, \pi_\psi)$ the GNS-triple for the weight ψ. If ψ is semifinite, normal or faithful, then the resulting representation π_ψ is non-degenerate, normal or faithful (i.e. injective as a map $\pi_\psi\colon \mathsf{M} \to \mathsf{B}(\mathcal{H}_\psi)$), respectively.

1.3 Quantum Groups

1.3.1 Locally compact quantum groups

In this part we fix a locally compact quantum group $\mathbb{G}$ in the sense of Kustermans-Vaes [39]. The study of this object can be undertaken from various perspectives (apart from very simple cases, one needs to take into account several of them). The (incomplete) list of objects one uses to study $\mathbb{G}$ consists of:

1. The von Neumann algebra $L^\infty(\mathbb{G})$, endowed with a coproduct $\Delta_\mathbb{G}$ and n.s.f. weights $\varphi^\mathbb{G}, \psi^\mathbb{G}$ satisfying the left- and right-invariance conditions (called the left and right Haar weights, respectively);

2. The reduced C^*-algebra $C_0(\mathbb{G})$, endowed with the same structure as above;

3. The universal C^*-algebra $C_0^u(\mathbb{G})$,

4. The Kac-Takesaki operator $\mathrm{W}^\mathbb{G} \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$ and its universal companions: $\mathbb{W}^\mathbb{G} \in \mathsf{M}(C_0^u(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$, $\mathbb{V}\!\!\!\mathbb{W}^\mathbb{G} \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0^u(\mathbb{G}))$ and $\mathbb{W}\!\!\!\mathbb{W}^\mathbb{G} \in \mathsf{M}(C_0^u(\widehat{\mathbb{G}}) \otimes C_0^u(\mathbb{G}))$;

We will sketch the elements of their theory: from the very beginning we (may) assume that the von Neumann algebra $L^\infty(\mathbb{G})$ is represented in $L^2(\mathbb{G})$, the GNS-space of the right-invariant weight $\psi^\mathbb{G}$ (in fact one can start with an abstract von Neumann algebra M and represent it via the GNS for ψ, the isomorphic copy is then denoted $L^\infty(\mathbb{G})$ and it turns out that it acts standardly on $L^2(\mathbb{G})$). The GNS-triple is denoted $(L^2(\mathbb{G}), \pi_\psi, \eta_\psi)$. Then one constructs the Kac-Takesaki operator (which is a multiplicative unitary) so that for $x, y \in \mathcal{N}_\psi = \{z \in L^\infty(\mathbb{G})\colon \psi(z^*z) < \infty\}$ one has

$$\mathrm{W}^\mathbb{G}(\eta_\psi(x) \otimes \eta_\psi(y)) = (\eta_\psi \otimes \eta_\psi)(\Delta(x)\mathbb{1} \otimes y).$$

The Kac-Takesaki operator, seen as $\mathrm{W}^\mathbb{G} \in \mathsf{B}(L^2(\mathbb{G}))\overline{\otimes} L^\infty(\mathbb{G})$, implements the coproduct:

$$\Delta(x) = \mathrm{W}^\mathbb{G}(x \otimes \mathbb{1})(\mathrm{W}^\mathbb{G})^* \tag{1.6}$$

Using Kac-Takesaki operator, one defines

$$C_0(\mathbb{G}) = \{(\omega \otimes \mathrm{id})\mathrm{W}^\mathbb{G} : \omega \in \mathsf{B}(L^2(\mathbb{G}))_*\}^{-\|\cdot\|}.$$

Then one can show that $\mathrm{W}^\mathbb{G} \in \mathsf{M}(\mathsf{K}(L^2(\mathbb{G})) \otimes C_0(\mathbb{G}))$ and that the coproduct restricts to a morphism $\Delta_\mathbb{G} \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes C_0(\mathbb{G}))$.

Subsequently, one defines the dual quantum group $\widehat{\mathbb{G}}$ by the formula $C_0(\widehat{\mathbb{G}}) = \{(\mathrm{id} \otimes \omega)\mathrm{W}^\mathbb{G} : \omega \in \mathsf{B}(L^2(\mathbb{G}))_*\}^{-\|\cdot\|}$ and $L^\infty(\widehat{\mathbb{G}}) = C_0(\widehat{\mathbb{G}})''$. Then $\mathrm{W}^\mathbb{G} \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$ and one can endow $C_0(\widehat{\mathbb{G}})$ with the coproduct:

$$\Delta_{\widehat{\mathbb{G}}}(x) = \sigma(\mathrm{W}^\mathbb{G})^*(x \otimes \mathbb{1})\sigma(\mathrm{W}^\mathbb{G}) = \sigma\left((\mathrm{W}^\mathbb{G})^*(\mathbb{1} \otimes x)\mathrm{W}^\mathbb{G}\right). \tag{1.7}$$

Then the Tomita-Takesaki theory enables us to construct the dual weights $\varphi^{\widehat{\mathbb{G}}}$, $\psi^{\widehat{\mathbb{G}}}$ (it essentially follows from appropriate construction of a full left Hilbert algebra, on which the von Neumann algebra $L^\infty(\widehat{\mathbb{G}})$ acts).

The semifiniteness of $\varphi^{\mathbb{G}}$, $\psi^{\mathbb{G}}$ is sufficient to construct a (in principle, unbounded) operator $S^{\mathbb{G}}$ with domain $\mathcal{D}(S^{\mathbb{G}})$ such that the elements of the form $(\omega\otimes\mathrm{id})\mathrm{W}^{\mathbb{G}}$ form a core for $S^{\mathbb{G}}$ and $S^{\mathbb{G}}((\omega\otimes\mathrm{id})\mathrm{W}^{\mathbb{G}}) = (\omega\otimes\mathrm{id})((\mathrm{W}^{\mathbb{G}})^*)$. The modular elements of the Haar weights $\nabla_\varphi, \nabla_\psi$ define a one-parameter groups of automorphisms $\sigma_t^\varphi, \sigma_t^\psi$ by the formulas $x \xmapsto{\sigma_t^\varphi} \nabla_\varphi^{it} x \nabla_\varphi^{-it}$ and $x \xmapsto{\sigma_t^\psi} \nabla_\psi^{it} x \nabla_\psi^{-it}$. One also has the scaling group $\tau_t^{\mathbb{G}}$. It can be constructed in several different ways. Possibly the easiest one is to use the modular element of the dual weight $\nabla_{\widehat{\psi}}$, it implements the scaling group by $\tau_t^{\mathbb{G}}(x) = \nabla_{\widehat{\psi}}^{it} x \nabla_{\widehat{\psi}}^{-it}$. When working in the setting of manageable multiplicative unitaries (which is more general than the setting of Kustermans and Vaes), one can also implement $\tau^{\mathbb{G}}$ by means of the operator Q appearing in the definition of manageability, see [42].

With the scaling group in hand, one is able to construct the polar decomposition of the antipode $S^{\mathbb{G}} = R^{\mathbb{G}} \circ \tau^{\mathbb{G}}_{\frac{i}{2}} = \tau^{\mathbb{G}}_{\frac{i}{2}} \circ R^{\mathbb{G}}$, where $\tau^{\mathbb{G}}_{\frac{i}{2}}$ is the analytic generator of the group $\tau_t^{\mathbb{G}}$ and $R^{\mathbb{G}}$ is a surjective isometry $C_0(\mathbb{G}) \to C_0(\mathbb{G})$, which is antimultiplicative. This in particular means that $R^{\mathbb{G}}(\mathcal{D}(\tau^{\mathbb{G}}_{\frac{i}{2}})) = \mathcal{D}(\tau^{\mathbb{G}}_{\frac{i}{2}})$. Moreover, $R^{\mathbb{G}} \circ R^{\mathbb{G}} = \mathrm{id}$ and hence $S^{\mathbb{G}} \circ S^{\mathbb{G}} = \tau_i^{\mathbb{G}}$. It is known that the analytic generator $\tau^{\mathbb{G}}_{\frac{i}{2}}$ is a closed linear mapping with domain $\mathcal{D}(\tau^{\mathbb{G}}_{\frac{i}{2}})$, such that $\mathcal{D}(\tau^{\mathbb{G}}_{\frac{i}{2}})$ is a WOT-dense subalgebra $L^\infty(\mathbb{G})$ and $\tau^{\mathbb{G}}_{\frac{i}{2}}$ is multiplicative. Moreover, $\tau^{\mathbb{G}}_{\frac{i}{2}}(a)^* \in \mathcal{D}(\tau^{\mathbb{G}}_{\frac{i}{2}})$ if $a \in \mathcal{D}(\tau^{\mathbb{G}}_{\frac{i}{2}})$, cf. [67, Section 1].

The duality is a mean to study representation theory and topology of a locally compact quantum group in a uniform way. A representation of a locally compact quantum group $\mathbb{G}$ can be described by a unitary element $U \in \mathsf{M}(\mathsf{K}(\mathcal{H}_U) \otimes C_0(\mathbb{G}))$ or, alternatively, by a unitary element $U \in \mathsf{B}(\mathcal{H}_U)\overline{\otimes}L^\infty(\mathbb{G})$, called a corepresentation, which satisfies the corepresentation condition:

$$(\mathrm{id}\otimes\Delta_{\mathbb{G}})U = U_{12}U_{13} \tag{1.8}$$

Out of two corepresentations $U \in \mathsf{M}(\mathsf{K}(\mathcal{H}_U)\otimes C_0(\mathbb{G}))$ and $V \in M(\mathsf{K}(\mathcal{H}_V)\otimes C_0(\mathbb{G}))$ one can form two new corepresentations: the direct sum and tensor product. The direct sum is obtained by viewing $U \in M(\mathsf{K}(\mathcal{H}_U)\otimes C_0(\mathbb{G})) \subseteq M(\mathsf{K}(\mathcal{H}_U\oplus\mathcal{H}_V)\otimes C_0(\mathbb{G}))$ by means of the spatial map $\mathcal{H}_U \subseteq \mathcal{H}_U \oplus \mathcal{H}_V$ (and similarly for $V \in M(\mathsf{K}(\mathcal{H}_V)\otimes C_0(\mathbb{G})) \subseteq M(\mathsf{K}(\mathcal{H}_U\oplus\mathcal{H}_V)\otimes C_0(\mathbb{G}))$). Then the direct sum is defined as

$$U \oplus V := U + V \in M(\mathsf{K}(\mathcal{H}_U\oplus\mathcal{H}_V)\otimes C_0(\mathbb{G})). \tag{1.9}$$

Similarly, one can define the tensor product

$$U \mathbin{\bigcirc\!\!\!\!\!\top} V = U_{13}V_{23} \in M(\mathsf{K}(\mathcal{H}_U)\otimes\mathsf{K}(\mathcal{H}_V)\otimes C_0(\mathbb{G})) \cong M(\mathsf{K}(\mathcal{H}_U\otimes\mathcal{H}_V)\otimes C_0(\mathbb{G})). \tag{1.10}$$

The viewpoint $U \in \mathsf{B}(\mathcal{H}_U)\overline{\otimes}L^\infty(\mathbb{G})$ enables us to make sense of the following crucial observation, which is contained in [67, Theorem 1.6]. Let $\omega \in \mathsf{B}(\mathcal{H}_U)_*$ be normal state. Then

$$(\omega\otimes\mathrm{id})U \in \mathcal{D}(S^{\mathbb{G}}) \quad \text{and} \quad S^{\mathbb{G}}((\omega\otimes\mathrm{id})U) = (\omega\otimes\mathrm{id})(U^*). \tag{1.11}$$

The dual was constructed by means of a representation of the algebra $L^1(\mathbb{G}) = L^\infty(\mathbb{G})_*$. In principle it is not a *-algebra and one way to overtake this problem one introduces the algebra $L^1_\#(\mathbb{G})$, then one is able to construct the universal completion $C_0^u(\widehat{\mathbb{G}})$ and a universal corepresentation $\mathbb{W}^{\mathbb{G}} \in \mathsf{M}(C_0^u(\widehat{\mathbb{G}})\otimes C_0(\mathbb{G}))$, see [38]

There is also a shorter way to construct the universal corepresentation, which we recall. By taking the direct sum of appropriately chosen representations of $\mathbb{G}$ (for a chosen dense set $(\omega_n)_n$ in the unit ball of $L^1(\mathbb{G})$, take a family of corepresentations U_n approaching the supremum of the norms $(\mathrm{id}\otimes\omega_n)U$, where the supremum is taken over corepresentations U of $\mathbb{G}$), one constructs an element $\mathbb{W}^{\mathbb{G}} \in \mathsf{M}(K(\oplus_n\mathcal{H}_n)\otimes C_0(\mathbb{G}))$. Taking the slices with normal functionals over the second leg and completing the outcome in norm (inherited from $\mathsf{B}(\oplus_n\mathcal{H}_n)$, one gets a C^*-algebra $C_0^u(\widehat{\mathbb{G}})$

and then $\mathbb{W}^{\mathbb{G}} \in \mathsf{M}(C_0^u(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$. One can show that the coproduct "lifts" to the universal level, i.e. there exists $\Delta_{\mathbb{G}}^u \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{G}) \otimes C_0^u(\mathbb{G}))$. Then one has a bijection between corepresentations of $C_0(\mathbb{G})$ and representations of the C^*-algebra $C_0^u(\widehat{\mathbb{G}})$:

Theorem 1.13 ([38, Proposition 5.2])**.** *Let unitary $U \in \mathsf{M}(\mathsf{K}(\mathcal{H}_U) \otimes C_0(\mathbb{G}))$ be a corepresentation. Then there exists a unique morphism $\phi_U \in \mathrm{Mor}(C_0^u(\widehat{\mathbb{G}}), \mathsf{K}(\mathcal{H}_U))$ such that*

$$(\phi_U \otimes \mathrm{id})\mathbb{W}^{\mathbb{G}} = U.$$

Conversely, given a C^*-algebra B, its representation in a Hilbert space $\rho \in \mathrm{Mor}(\mathsf{B}, \mathsf{K}(\mathcal{H}))$ and morphism $\phi \in \mathrm{Mor}(C_0^u(\widehat{\mathbb{G}}), \mathsf{B})$), the unitary $U_{\phi,\rho} = (\rho \circ \phi \otimes \mathrm{id})\mathbb{W}^{\mathbb{G}} \in \mathsf{M}(\mathsf{K}(\mathcal{H}_U) \otimes C_0(\mathbb{G}))$ is a corepresentation of $C_0(\mathbb{G})$

In what follows, the term *representation of* $\mathbb{G}$ will be used to describe both the unitary corepresentations of $C_0(\mathbb{G})$ and the morphisms $\phi \in \mathrm{Mor}(C_0^u(\widehat{\mathbb{G}}), \mathsf{K}(\mathcal{H}))$. Sometimes it is also convenient to utilize the notion of an antirepresentation. These are usually described by unitary elements $V \in \mathsf{M}(\mathsf{K}(\mathcal{H})) \otimes C_0(\mathbb{G}))$ such that V^* is a corepresentation, the corresponding ϕ_V is then antimorphism (so that $\phi_V \circ R$ is a morphism). As the link between representations of $\mathbb{G}$ and antirepresentations of $\mathbb{G}$ is so direct, we will occasionally call antirepresentations of $\mathbb{G}$ *representations of* $\mathbb{G}$, this shall cause no confusion.

Performing the same trick for $\widehat{\mathbb{G}}$ one gets $\mathcal{W}^{\mathbb{G}} \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0^u(\mathbb{G}))$. The reducing morphism $\Lambda_{\mathbb{G}} \colon C_0^u(\mathbb{G}) \to C_0(\mathbb{G})$ (and a similar for $\widehat{\mathbb{G}}$) connects these operators to the standard Kac-Takesaki operator via

$$(\mathrm{id} \otimes \Lambda_{\mathbb{G}})\mathcal{W}^{\mathbb{G}} = \mathrm{W}^{\mathbb{G}} = (\Lambda_{\widehat{\mathbb{G}}} \otimes \mathrm{id})\mathbb{W}^{\mathbb{G}}.$$

Moreover, one can do these liftings simultaneously and get the universal Kac-Takesaki operator $\boldsymbol{\mathbb{W}}^{\mathbb{G}} \in \mathsf{M}(C_0^u(\widehat{\mathbb{G}}) \otimes C_0^u(\mathbb{G}))$ such that $(\mathrm{id} \otimes \Lambda_{\mathbb{G}})\boldsymbol{\mathbb{W}}^{\mathbb{G}} = \mathbb{W}^{\mathbb{G}}$ and $(\Lambda_{\widehat{\mathbb{G}}} \otimes \mathrm{id})\boldsymbol{\mathbb{W}}^{\mathbb{G}} = \mathcal{W}^{\mathbb{G}}$.

These operators are bicharacters, in particular

$$(\Delta_{\widehat{\mathbb{G}}}^u \otimes \mathrm{id})(\boldsymbol{\mathbb{W}}^{\mathbb{G}}) = \boldsymbol{\mathbb{W}}_{23}^{\mathbb{G}}\boldsymbol{\mathbb{W}}_{13}^{\mathbb{G}} \tag{1.12}$$

and

$$(\mathrm{id} \otimes \Delta_{\mathbb{G}}^u)(\boldsymbol{\mathbb{W}}^{\mathbb{G}}) = \boldsymbol{\mathbb{W}}_{12}^{\mathbb{G}}\boldsymbol{\mathbb{W}}_{23}^{\mathbb{G}} \tag{1.13}$$

and similarly for the semireduced versions (as $(\Lambda_{\mathbb{G}} \otimes \Lambda_{\mathbb{G}}) \circ \Delta_{\mathbb{G}}^u = \Delta_{\mathbb{G}} \circ \Lambda_{\mathbb{G}}$ and $(\Lambda_{\widehat{\mathbb{G}}} \otimes \Lambda_{\widehat{\mathbb{G}}}) \circ \Delta_{\widehat{\mathbb{G}}}^u = \Delta_{\widehat{\mathbb{G}}} \circ \Lambda_{\widehat{\mathbb{G}}}$).

Moreover, in [43, Proposition 4.4] the following form of pentagonal-like equation was established:

$$\boldsymbol{\mathbb{W}}_{13} = \mathbb{W}_{12}^*\mathcal{W}_{23}\mathbb{W}_{12}\mathcal{W}_{23}^* \in \mathsf{M}(C_0^u(\widehat{\mathbb{G}}) \otimes \mathsf{K}(L^2(\mathbb{G})) \otimes C_0^u(\mathbb{G})) \tag{1.14}$$

If no confusion arises (e.g. when there is a single quantum group in consideration), we will drop the superscript ${}^{\mathbb{G}}$ and if using the structure for the dual group $\widehat{\mathbb{G}}$, the respective objects will be decorated with the hat $\widehat{\cdot}$, e.g. the scaling group $\tau_t^{\widehat{\mathbb{G}}}$ will be written by $\widehat{\tau}_t$. This is especially convenient, because the scaling group and the unitary antipode also have their universal counterparts, so in particular $\widehat{\tau}_t^u$ will denote the scaling group of the C^*-algebra $C_0^u(\widehat{\mathbb{G}})$ and R^u will denote the unitary antipode of $C_0^u(\mathbb{G})$, given a fixed locally compact quantum group $\mathbb{G}$.

The scaling groups and unitary antipodes are compatible with the reduction morphisms in the following sense:

$$\tau_t \circ \Lambda_{\mathbb{G}} = \Lambda_{\mathbb{G}} \circ \tau_t^u \qquad \text{and} \qquad R \circ \Lambda_{\mathbb{G}} = \Lambda_{\mathbb{G}} \circ R^u \tag{1.15}$$

and similarly for $\widehat{\mathbb{G}}$. Moreover, the scaling groups and unitary antipodes are compatible with the Kac-Takesaki operators in the following sense:

$$(\widehat{\tau}_t^u \otimes \tau_t^u)\boldsymbol{\mathbb{W}} = \boldsymbol{\mathbb{W}} \qquad \text{and} \qquad (\widehat{R}^u \otimes R^u)\boldsymbol{\mathbb{W}} = \boldsymbol{\mathbb{W}} \tag{1.16}$$

For the above, we refer to [50, Proposition 39, Lemma 40 and Proposition 42]. Moreover, the following holds:

$$(R^u \otimes R^u) \circ \Delta^u \circ R^u = \sigma \circ \Delta^u \tag{1.17}$$

There is yet another very useful result, established in [43, Theorem 2.6] (even in a more general form concerning modular multiplicative unitaries)

Theorem 1.14 ([43, Theorem 2.6]). *Let $a, b \in \mathsf{M}(\mathsf{K}(L^2(\mathbb{G})) \otimes \mathsf{D})$ for some C^*-algebra D. Then $W^{\mathbb{G}}_{12} a_{13} = b_{23} W^{\mathbb{G}}_{12}$ if and only if $a = b \in \mathbb{C}\mathbb{1} \otimes \mathsf{M}(\mathsf{D})$*

One can then easily conclude that the comultiplication is ergodic:

Corollary 1.15 ([43, Corollary 2.9]). *Let $a \in \mathsf{M}(C_0(\mathbb{G}))$. Then $\Delta(a) = b \otimes \mathbb{1}$ for some $b \in \mathsf{M}(C_0(\mathbb{G}))$ if and only if $\Delta(a) = \mathbb{1} \otimes c$ for some $c \in \mathsf{M}(C_0(\mathbb{G}))$ if and only if $a = b = c \in \mathbb{C}\mathbb{1}$. More generally, if B is a C^*-algebra and $c \in \mathsf{M}(C_0(\mathbb{G}) \otimes \mathsf{B})$, then $(\Delta_{\mathbb{G}} \otimes \mathrm{id})c \in \mathsf{M}(C_0(\mathbb{G}) \otimes \mathbb{C}\mathbb{1} \otimes \mathsf{B})$ if and only if $(\Delta_{\mathbb{G}} \otimes \mathrm{id})c \in \mathsf{M}(\mathbb{C}\mathbb{1} \otimes C_0(\mathbb{G}) \otimes \mathsf{B})$ if and only if $c \in \mathbb{C}\mathbb{1} \otimes \mathsf{M}(\mathsf{D})$.*

This phenomenon has also its von Neumann algebraic counterpart, which may be phrased slightly differently.

Proposition 1.16 ([42, Proposition 4.7]). $L^\infty(\widehat{\mathbb{G}})' \cap L^\infty(\mathbb{G}) = \mathbb{C}\mathbb{1}$

1.3.2 Compact quantum groups

Although compact quantum groups are special case of locally compact quantum groups, its axiomatization is way simpler, so we recall it. The rudiments of their theory is contained in [63, 68].

A unital C^*-algebra A endowed with a *-homomorphism $\Delta \colon \mathsf{A} \to \mathsf{A} \otimes \mathsf{A}$ (the minimal tensor product of C^*-algebras) satisfying the coassociativity condition: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ is called a *Woronowicz algebra*, if the *cancellation laws* (also known as *Podleś condition*) holds:

$$(\mathbb{1} \otimes \mathsf{A}) \cdot \Delta(\mathsf{A}) = \mathsf{A} \otimes \mathsf{A} = (\mathsf{A} \otimes \mathbb{1}) \cdot \Delta(\mathsf{A}).$$

The Woronowicz algebra A can be always endowed with a unique state $h \in \mathsf{A}^*$, called the *Haar state*, which is left and right invariant:

$$(\mathrm{id} \otimes h)\Delta = (h \otimes \mathrm{id})\Delta = h(\cdot)\mathbb{1}.$$

Such an algebra correspond to a *compact quantum group* $\mathbb{G}$ by abstractly extending the Gelfand-Naimark duality: $\mathsf{A} = C(\mathbb{G})$, the algebra of continuous functions on $\mathbb{G}$. This algebra always contains a unique dense Hopf *-subalgebra (i.e. the coproduct map Δ restricts to this *-subalgebra), denoted $\mathrm{Pol}(\mathbb{G})$; it is spanned by matrix coefficients of unitary representations of $\mathbb{G}$. This Hopf *-algebra can have, a priori, a plethora of different C^*-completions: the reduced one (corresponding to GNS-representation associated to the Haar state), denoted $C^r(\mathbb{G})$, completion in the norm given by $\mathsf{A} = C(\mathbb{G})$ (i.e. A itself) and the universal C^*-norm, denoted $C^u(\mathbb{G})$ need not coincide, in general. For further discussion on this topic, see e.g. [40]. In any case, there are always quotient maps

$$C^u(\mathbb{G}) \to C(\mathbb{G}) \to C^r(\mathbb{G})$$

where $C(\mathbb{G})$ denotes a general C^*-completion. In case the quotient map $\Lambda_{\mathbb{G}} \colon C^u(\mathbb{G}) \to C^r(\mathbb{G})$ is injective, we call $\mathbb{G}$ coamenable and declare that $C^u(\mathbb{G}) = C(\mathbb{G}) = C^r(\mathbb{G})$ and $\Lambda_{\mathbb{G}} = \mathrm{id}$. In a typical situation the quantum group is defined either by $C^u(\mathbb{G})$ or $C^r(\mathbb{G})$ and the respective Hopf-structures, and typically the superscript ^{r} is redundant: if there is no need to distinguish between the reduced and intermediate completions, one uses the symbol $C(\mathbb{G})$ to denote the reduced C^*-algebra of continuous functions on $\mathbb{G}$. Then $L^\infty(\mathbb{G})$, the WOT-closure of $C(\mathbb{G})$ in the GNS-representation, is a locally compact quantum group in the sense of Kustermans and Vaes. Moreover, the Haar state is faithful when restricted to the Hopf *-algebra $\mathrm{Pol}(\mathbb{G})$. Equivalently, one can say that that the map $\Lambda_{\mathbb{G}} \restriction_{\mathrm{Pol}(\mathbb{G})}$, seen as

$$C^u(\mathbb{G}) \supseteq \mathrm{Pol}(\mathbb{G}) \xrightarrow{\Lambda_{\mathbb{G}} \restriction_{\mathrm{Pol}(\mathbb{G}}} \mathrm{Pol}(\mathbb{G}) \subseteq C^r(\mathbb{G})$$

is an isomorphism of Hopf *-algebras. To simplify things, we identify these Hopf *-algebras and say that

$$\Lambda_{\mathbb{G}} \restriction_{\mathrm{Pol}(\mathbb{G})} = \mathrm{id}_{\mathrm{Pol}(\mathbb{G})} . \tag{1.18}$$

Still, the corepresentation theory of all these Woronowicz algebras is the same. Moreover, the Hopf *-algebra $\mathrm{Pol}(\mathbb{G})$ is the unique dense Hopf *-subalgebra of $C(\mathbb{G})$ (and all the other versions of C^*-algebra of continuous functions of $\mathbb{G}$), see [9, Theorem 5.1]. Moreover, because the corepresentation theory of $L^\infty(\mathbb{G})$ is the same as the one of $\mathrm{Pol}(\mathbb{G})$, there is a unique weak*-dense Hopf *-algebra $\mathcal{A} \subseteq L^\infty(\mathbb{G})$: the canonical one $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$. These are in fact three independent statements (with non-trivial intersection) in this paragraph, so let us express them directly.

Theorem 1.17 ([68, Theorems 2.2 & 2.6(2)]). *Let $C(\mathbb{G})$ be a Woronowicz algebra (i.e. any version of the C^*-algebra of continuous functions on the compact quantum group $\mathbb{G}$) and let $\mathrm{Pol}(\mathbb{G})$ be the set of all linear combinations of matrix elements of all finite dimensional unitary corepresentations of $C(\mathbb{G})$, i.e. elements of the form $(\omega_{\xi,\eta} \otimes \mathrm{id})U$ for a unitary corepresentation $U \in \mathsf{B}(\mathcal{H}_U) \otimes C(\mathbb{G})$ and $\xi, \eta \in \mathcal{H}_U$ with $\dim(\mathcal{H}_U) < \infty$. Then $\mathrm{Pol}(\mathbb{G})$ is a dense *-subalgebra of $C(\mathbb{G})$ and $\Delta(\mathrm{Pol}(\mathbb{G})) \subseteq \mathrm{Pol}(\mathbb{G}) \otimes_{alg} \mathrm{Pol}(\mathbb{G})$. Moreover, $(\mathrm{Pol}(\mathbb{G}), \Delta \restriction_{\mathrm{Pol}(\mathbb{G})})$ is a Hopf *-algebra, the antipode and counit are given by linear extensions of the maps $S((\omega_{\xi,\eta} \otimes \mathrm{id})U) = ((\omega_{\eta,\xi} \otimes \mathrm{id})U)^*$ and $\varepsilon((\omega_{\xi,\eta} \otimes \mathrm{id})U) = \langle \xi | \eta \rangle$. Furthermore, one has $\mathrm{Pol}(\mathbb{G}) = \{x \in C^r(\mathbb{G}) : \Delta_{\mathbb{G}}(x) \in C^r(\mathbb{G}) \otimes_{alg} C^r(\mathbb{G})\}$*

Theorem 1.18 ([9, Theorem 5.1]). *Let $C(\mathbb{G})$ be a Woronowicz algebra and let $\mathcal{A} \subseteq C(\mathbb{G})$ be a Hopf *-algebra. Then $\mathcal{A}$ is dense in $C(\mathbb{G})$ if and only if $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$.*

Theorem 1.19. *Let $L^\infty(\mathbb{G})$ be the von Neumann algebra of bounded measurable functions on $\mathbb{G}$ and let $\mathcal{A} \subseteq L^\infty(\mathbb{G})$ be a Hopf *-algebra. Then $\mathcal{A}$ is WOT-dense if and only if $\mathcal{A} = \mathrm{Pol}(\mathbb{G})$*

Let us stress that the first two results concern arbitrary Woronowicz algebras, whereas the last result is valid only in the von Neumann algebra $L^\infty(\mathbb{G})$. From Theorem 1.19 one can deduce Theorem 1.18 only in the case of the reduced version of the C^*-algebra of continuous functions on $\mathbb{G}$ (with the aid of Theorem 1.17).

Moreover, the representation theory of $\mathbb{G}$, and hence the form of $\widehat{\mathbb{G}}$, is fully understood: all representations of $\mathbb{G}$ are of type I (i.e. decompose into direct sums of irreducible representations), the irreducible representations are finite dimensional. Hence $C_0(\widehat{\mathbb{G}}) = \bigoplus_{\alpha \in Irr(\mathbb{G})} M_{\dim(\alpha)}(\mathbb{C})$, where $Irr(\mathbb{G})$ denotes the set of equivalence classes of irreducible representations of $\mathbb{G}$. Because of the *discreteness* of the right hand side we will write $c_0(\widehat{\mathbb{G}})$ for this C^*-algebra. Similarly, we will write $\ell^\infty(\widehat{\mathbb{G}})$ for the von Neumann algebra $L^\infty(\widehat{\mathbb{G}})$, so that in particular $\ell^\infty(\widehat{\mathbb{G}}) = \prod_{\alpha \in Irr(\mathbb{G})} M_{\dim(\alpha)}(\mathbb{C})$. Observe that in particular $\ell^\infty(\widehat{\mathbb{G}})$ is atomic (every non-zero projection majorizes a non-zero minimal projection) and of type I_{fin} as a von Neumann algebra.

The most studied examples are the *compact matrix quantum groups*: $\mathbb{G}$ is a compact matrix quantum group if the Woronowicz algebra $C(\mathbb{G})$ can be given a *fundamental* corepresentation $U \in M_n(C(\mathbb{G})) = \mathsf{M}(\mathsf{K}(\mathbb{C}^n) \otimes C(\mathbb{G}))$: denoting $U_{i,j} = (\omega_{e_i,e_j} \otimes \mathrm{id})U$ for a fixed basis $(e_i)_{1 \le i \le n} \subset \mathbb{C}^n$, orthonormal with respect to the standard inner product on $\mathbb{C}^n$, we ask for:

$$\Delta(U_{i,j}) = \sum_{k=1}^{n} U_{i,k} \otimes U_{k,j}$$

and

$$\langle \{U_{i,j} : 1 \le i, j \le n\} \rangle = \mathrm{Pol}(\mathbb{G})$$

where $\langle X \rangle$ denotes the *-algebra generated by elements of $X \subseteq C(\mathbb{G})$.

1.3.3 Homomorphisms of quantum groups

The thorough description of homomorphisms between quantum groups was given in [43], let us recall the main points. Fix two locally compact quantum groups $\mathbb{G}$ and $\mathbb{H}$. A homomorphism $\mathbb{H} \to \mathbb{G}$ can be equivalently described by three objects:

Quantum group homomorphisms morphisms $\varphi \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$, which intertwine the coproducts:

$$(\varphi \otimes \varphi) \circ \Delta^u_{\mathbb{G}} = \Delta^u_{\mathbb{H}} \circ \varphi$$

Bicharacters unitary elements $V \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{H}))$, which are (anti)corepresentations on both legs:

$$(\Delta_{\widehat{\mathbb{G}}} \otimes \mathrm{id})V = V_{23}V_{13} \quad \text{and} \quad (\mathrm{id} \otimes \Delta_{\mathbb{H}})V = V_{12}V_{13} \tag{1.19}$$

Moreover, they satisfy $(\widehat{R}^{\mathbb{G}} \otimes R^{\mathbb{H}})V = V$ and $(\tau_t^{\widehat{\mathbb{G}}} \otimes \tau_t^{\mathbb{H}})V = V$.

Right quantum group morphisms morphisms $\rho \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes C_0(\mathbb{H}))$ satisfying

$$(\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \rho = (\mathrm{id} \otimes \rho) \circ \Delta_{\mathbb{G}} \quad \text{and} \quad (\mathrm{id} \otimes \Delta_{\mathbb{H}}) \circ \rho = (\rho \otimes \mathrm{id}) \circ \rho \tag{1.20}$$

Moreover, they satisfy $(\mathrm{id} \otimes \rho)\mathrm{W}^{\mathbb{G}} = \mathrm{W}^{\mathbb{G}}_{12}V_{13}$, where V is the corresponding bicharacter.

Moreover, each homomorphism $\mathbb{H} \to \mathbb{G}$ has its dual homomorphism $\widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$. It can be described as follows. If $\varphi \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$ is a quantum group homomorphism, then there exists a unique $\widehat{\varphi} \in \mathrm{Mor}(C_0^u(\widehat{\mathbb{H}}), C_0^u(\widehat{\mathbb{G}}))$, these maps are linked via

$$(\mathrm{id} \otimes \varphi)\mathbb{W}^{\mathbb{H}} = (\widehat{\varphi} \otimes \mathrm{id})\mathbb{W}^{\mathbb{G}}. \tag{1.21}$$

Equivalently, if $V \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{H}))$ is a bicharacter representing a homomorphism $\mathbb{H} \to \mathbb{G}$, then $\widehat{V} = \sigma(V^*) \in \mathsf{M}(C_0(\mathbb{H}) \otimes C_0(\widehat{\mathbb{G}}))$ is a bicharacter representing the dual homomorphism $\widehat{\mathbb{G}} \to \widehat{\mathbb{H}}$.

Let us also stress that bicharacters and right quantum group morphisms are equally good studied in the von Neumann algebraic context, so that a unitary $V \in L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\mathbb{H})$ satisfying (1.19) and normal *-homomorphism $\rho\colon L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{H})$ satisfying (1.20) also describe a homomorphism of quantum groups $\mathbb{H} \to \mathbb{G}$. Right quantum group morphisms in the von Neumann algebraic context are in fact normal extensions of the respective maps in the C^*-algebraic context: they are implemented by V by the formula $\rho(x) = V(x \otimes \mathbb{1})V^*$. Let us note that condition on the right in (1.20) correspond to ρ being a right action of $\mathbb{H}$ on $L^\infty(\mathbb{G})$. These correspond to the natural actions by right shifts.

Last, but not least, it is worth mentioning that sometimes one deals with morphisms between non-necessarily universal completions that intertwine the relevant coproducts. It also follows from the results of [43] that such morphisms can always be lifted to the universal level:

Proposition 1.20. *Let $\mathbb{G}_1, \mathbb{G}_2$ be locally compact quantum groups and let let $C_0^t(\mathbb{G}_i)$ denote two transitional C^*-algebras associated to $\mathbb{G}_i$, i.e. the composition*

$$C_0^u(\mathbb{G}_i) \xrightarrow{\Lambda^1_{\mathbb{G}_i}} C_0^t(\mathbb{G}_i) \xrightarrow{\Lambda^2_{\mathbb{G}_i}} C_0(\mathbb{G}_i)$$

is precisely the canonical reducing morphsism $\Lambda_{\mathbb{G}_i}$ $(i = 1, 2)$. Let $\varphi \in \mathrm{Mor}(C_0^t(\mathbb{G}_1), C_0^t(\mathbb{G}_2))$ be a morphism that intertwines coproducts. Then there exists a morphism $\varphi^u \in \mathrm{Mor}(C_0^u(\mathbb{G}_1), C_0^u(\mathbb{G}_2))$ that intertwines the coproducts and such that $\varphi \circ \Lambda^1_{\mathbb{G}_1} = \Lambda^1_{\mathbb{G}_2} \circ \varphi^u$.

1.3.4 Closed quantum subgroups

For an excellent account on the notion of a closed quantum subgroup we refer to [22], here we collected numerous useful results from that article. We are concerned with the situation when a homomorphism $\mathbb{H} \to \mathbb{G}$ (described by a quantum group homomorphism $\varphi \in \mathrm{Mor}(C_0^u(\mathbb{H}), C_0^u(\mathbb{G}))$, a bicharacter $V \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{H})$ and a right quantum group morphism $\rho \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes C_0(\mathbb{H})))$ of locally compact quantum groups identifies $\mathbb{H}$ with a closed quantum subgroup of $\mathbb{G}$. There are two competing definitions of closedness in this context.

Definition 1.21. We say that $\mathbb{H} \subset \mathbb{G}$ is a Woronowicz-closed quantum subgroup if the following equivalent conditions hold:

1. the map $\Lambda_{\mathbb{H}} \circ \varphi$ satisfies $(\Lambda_{\mathbb{H}} \circ \varphi)(C_0^u(\mathbb{G})) = C_0(\mathbb{H})$;
2. the quantum group homomorphism φ satisfies $\varphi(C_0^u(\mathbb{G})) = C_0^u(\mathbb{H})$;
3. the bicharacter V generates $C_0(\mathbb{H})$ (in the sense of Section 1.2.3);
4. the right quantum group morphism ρ is strongly non-degenerate:
$$\overline{\rho(C_0(\mathbb{G}))(C_0(\mathbb{G}) \otimes \mathbb{1})} = C_0(\mathbb{G}) \otimes C_0(\mathbb{H}).$$

Definition 1.22. We say that $\mathbb{H} \subset \mathbb{G}$ is a Vaes-closed quantum subgroup if the following equivalent conditions hold:

1. the bicharacters V and $W^{\mathbb{H}}$, viewed as representations of $\mathbb{H}$, are quasiequivalent, i.e. there exists a Hilbert space $\mathcal{L}$ such that denoting the trivial representation on $\mathcal{L}$ by $1_{\mathcal{L}}$ one has that $1_{\mathcal{L}} \oplus V$ is equivalent to $1_{\mathcal{L}} \oplus \mathrm{W}^{\mathbb{H}}$.
2. $\{(\mathrm{id} \otimes \omega)\widehat{V} : \omega \in L^1(\mathbb{H})\} = \{(\mathrm{id} \otimes \omega)\mathrm{W}^{\widehat{\mathbb{H}}} : \omega \in L^1(\mathbb{H})\}$.
3. there exists a normal, injective *-homomorphism $\gamma\colon L^\infty(\widehat{\mathbb{H}}) \to L^\infty(\widehat{\mathbb{G}})$ such that $\Lambda_{\widehat{\mathbb{G}}} \circ \widehat{\varphi} = \gamma \circ \Lambda_{\widehat{\mathbb{H}}}$
4. there exists a normal, injective *-homomorphism $\gamma\colon L^\infty(\widehat{\mathbb{H}}) \to L^\infty(\widehat{\mathbb{G}})$ such that $V = (\gamma \otimes \mathrm{id})\mathrm{W}^{\mathbb{H}}$
5. there exists a normal, injective *-homomorphism $\gamma\colon L^\infty(\widehat{\mathbb{H}}) \to L^\infty(\widehat{\mathbb{G}})$ such that $(\gamma \otimes \gamma) \circ \Delta_{\widehat{\mathbb{H}}} = \Delta_{\widehat{\mathbb{G}}} \circ \gamma$

The normal *-homomorphisms γ appearing in (3), (4), (5) are the same maps (as the notation suggests). Moreover, $\gamma \restriction_{C_0(\widehat{\mathbb{H}})} \in \mathrm{Mor}(C_0(\widehat{\mathbb{H}}), C_0(\widehat{\mathbb{G}}))$. One can also show that

$$\Lambda_{\mathbb{G}} \circ \widehat{\varphi} = \gamma \circ \Lambda_{\mathbb{H}} \tag{1.22}$$

and hence, by (1.21), in particular we have

$$(\mathrm{id} \otimes \varphi)\mathbb{W}^{\mathbb{G}} = (\gamma \otimes \mathrm{id})\mathbb{W}^{\mathbb{H}} \tag{1.23}$$

Theorem 1.23. *If $\mathbb{H}$ is a Vaes-closed quantum subgroup of $\mathbb{G}$, then $\mathbb{H}$ is a Woronowicz-closed quantum subgroup of $\mathbb{G}$.*

It is not clear whether the two definitions are equivalent in full generality. Nonetheless, in the most interesting cases the equivalence holds.

Theorem 1.24. *Let $\mathbb{H}$ be a Woronowicz-closed quantum subgroup of $\mathbb{G}$. Assume moreover that either*

1. *$\mathbb{G}$ is discrete, or*
2. *$\mathbb{H}$ is compact, or*
3. *$\mathbb{H}$ is classical, or*
4. *$\mathbb{H}$ is dual to classical.*

Then $\mathbb{H}$ is a Vaes-closed quantum subgroup of $\mathbb{G}$.

Moreover, there is a way to ensure that a given von Neumann algebra correspond to a Vaes-closed quantum subgroup. For later use, let us recall that $\mathsf{M} \subseteq L^\infty(\mathbb{G})$ is called *invariant* if $\Delta(\mathsf{M}) \subseteq \mathsf{M}\overline{\otimes}\mathsf{M}$ ([53]).

Theorem 1.25 ([2, Proposition 10.5])**.** *Let $\mathsf{M} \subseteq L^\infty(\mathbb{G})$ be a von Neumann subalgebra. Then the following conditions are equivalent:*

1. *the algebra is invariant, preserved by the unitary antipode: $R(\mathsf{M}) = \mathsf{M}$ and by the scaling group: for all $t \in \mathbb{R}$ one has $\tau_t(\mathsf{M}) = \mathsf{M}$.*
2. *there exists a locally compact quantum group $\mathbb{H}$ such that $\mathsf{M} = L^\infty(\mathbb{H})$. Moreover, its dual is a Vaes-closed subgroup of the dual: $\widehat{\mathbb{H}} \subset \widehat{\mathbb{G}}$.*

It seems that the appropriate definition of a closed subgroups is the one given by Vaes-closedness. In what follows, whenever we speak of closed quantum subgroups, we mean Vaes-closed quantum subgroups.

Remark 1.26. It is an open problem to settle whether the assumptions of Theorem 1.25 are optimal in full generality, i.e. whether the invariance under the action of the scaling group and unitary antipode follow from M being an invariant von Neumann algebra. In the case of $\mathbb{G}$ classical or dual to classical group it is also enough to assume that M is invariant, as originaly shown in [53, Theorem 2 & Theorem 6]. In the case $\mathbb{G}$ is compact, it is also enough to assume invariance only, as we will shown in the following Proposition. In the consecutive Section we will see that in case $\mathbb{G}$ is discrete, then it is enough to assume that M is invariant. It seems to be very hard to come up with an example for which these assumptions would be indeed necessary.

Proposition 1.27. *Let $\mathsf{M} \subseteq L^\infty(\mathbb{G})$ be an invariant von Neumann subalgebra. If $\mathbb{G}$ is compact, then M is invariant under the unitary antipode R ant the scaling group τ_t.*

Proof. Let $h: L^\infty(\mathbb{G}) \to \mathbb{C}$ be the Haar state. Let us denote $h_{\mathsf{M}}: \mathsf{M} \to \mathbb{C}$ its restriction to the subalgebra M. Then clearly h_{M} is a left- and right-invariant state, hence $\mathsf{M} \cong L^\infty(\mathbb{H})$ for some compact quantum group $\mathbb{H}$ (compactness instead of only local compactness is a consequence of boundedness of h_{M}). This means that $\widehat{\mathbb{H}} \subseteq \widehat{\mathbb{G}}$ is a Vaes-closed quantum subgroup and the map γ is the inclusion $\mathsf{M} \subseteq L^\infty(\mathbb{G})$, in particular, it describes a homomorphism $\widehat{\mathbb{H}} \to \widehat{\mathbb{G}}$. These always preserve the unitary antipodes and scaling groups, cf. Section 1.3.3, and so do their duals, hence M is preserved by R and τ. ☐

For later use, let us also analyze the notion of subgroup in the case of compact quantum groups. Let $\mathbb{G}, \mathbb{H}$ be compact quantum groups and assume that $\pi: C^u(\mathbb{G}) \to C^u(\mathbb{H})$ is a morphism identifying $\mathbb{H}$ as a closed quantum subgroup of $\mathbb{G}$. Let us denote $\pi_{alg} := \pi\restriction_{\mathrm{Pol}(\mathbb{G})}: \mathrm{Pol}(\mathbb{G}) \to C^u(\mathbb{H})$. Observe that in fact the range of π_{alg} is contained in $\mathrm{Pol}(\mathbb{H})$. Indeed, consider the map $\Lambda_{\mathbb{H}} \circ \pi_{alg}: \mathrm{Pol}(\mathbb{G}) \to C^r(\mathbb{H})$ and observe that its range is contained in $\mathrm{Pol}(\mathbb{H})$ (cf. last part of Theorem 1.17). Then from (1.18) it follows that $\pi_{alg}(\mathrm{Pol}(\mathbb{G})) \subseteq \mathrm{Pol}(\mathbb{H})$. Now, as π is surjective and $\mathrm{Pol}(\mathbb{G}) \subseteq C^u(\mathbb{G})$ is dense, it follows that $\pi_{alg}(\mathrm{Pol}(\mathbb{G}) \subseteq C^u(\mathbb{H})$ is dense. But at the same time $\pi_{alg}(\mathrm{Pol}(\mathbb{G}))$ is a Hopf *-algebra, so by Theorem 1.18 we conclude that $\pi_{alg}(\mathrm{Pol}(\mathbb{G})) = \mathrm{Pol}(\mathbb{H})$.

Conversly, if we have a Hopf *-algebra surjection $\pi: \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{H})$, we can always see as $\pi: \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{H}) \subseteq C^u(\mathbb{H})$. Then using the universal property of $C^u(\mathbb{G})$ (and the fact that $C^u(\mathbb{H})$ is a C^*-algebra), we can extend π to a map $\overline{\pi}: C^u(\mathbb{G}) \to C^u(\mathbb{H})$. Firstly, from density of $\mathrm{Pol}(\mathbb{H}) \subseteq C^u(\mathbb{H})$ we see that the image of $\overline{\pi}$ is dense. But as the range of $\overline{\pi}$ is closed, we see that $\overline{\pi}$ is automatically a surjection. It intertwines the coproducts on a norm-dense sets, hence everywhere. We summarize this in the following:

Theorem 1.28. *The following two categories of compact quantum groups are equivalent:*

1. *the category $\mathfrak{CQG} - \mathfrak{Hopf}$ consisting of Hopf *-algebras coming from compact quantum groups as objects and with Hopf *-algebra surjections as morphisms and*

2. *the category* $\mathfrak{CQG} - \mathfrak{univ}$ *consisting of universal* C^**-algebras associated to compact quantum groups as objects and with surjective* **-homomorphisms intertwining the coproducts as morphisms.*

The equivalence is given by:

1. $\mathrm{Pol}(\mathbb{G}) \mapsto C^u(\mathbb{G})$ *and* $\pi \mapsto \overline{\pi}$ *defines a functor* $\mathfrak{CQG} - \mathfrak{Hopf} \to \mathfrak{CQG} - \mathfrak{univ}$ *and*

2. $C^u(\mathbb{G}) \mapsto \mathrm{Pol}(\mathbb{G})$ *and* $\pi \mapsto \pi_{alg}$ *defines a functor* $\mathfrak{CQG} - \mathfrak{univ} \to \mathfrak{CQG} - \mathfrak{Hopf}$.

and these two functors are mutually inverse.

1.3.5 Woronowicz-Tannaka-Krein duality

The exposition of this part is based on [44, 65]. The approach of [44] is more modern and we stick to the terminology used therein. The objective of this part is to provide a simple proof of a result from [45] that can clarify certain steps of constructions presented in Chapter 2. The key result of [65] establishes a bijective correspondence between compact quantum groups and a structure that is nowadays called rigid C^*-tensor category endowed with a fiber functor (in [65] these were called concrete monoidal W^*-categories). Not getting much into the details, by a result of MacLane it is in fact enough to consider only strict categories. Let then $\mathcal{C}$ be a small category (that is, the class of objects $\mathrm{Ob}(\mathcal{C})$ is in fact a set). To clarify notation, we drop the subscript $_\mathcal{C}$ when denoting parts of the structure related to $\mathcal{C}$, e.g. $\mathrm{Mor} = \mathrm{Mor}_\mathcal{C}$ etc.

Definition 1.29. We say that $\mathcal{C}$ is a *strict* C^**-tensor category*, if:

1. for all objects U, V, the set $\mathrm{Mor}(U, V)$ is a Banach space. The map
$$\mathrm{Mor}(V, W) \times \mathrm{Mor}(U, V) \ni (s, t) \mapsto st \in \mathrm{Mor}(U, W)$$
is bilinear and $\|st\| \leq \|s\|\|t\|$.

2. there exists a contravariant functor $*\colon \mathcal{C} \to \mathcal{C}$ such that
   (a) it is identity on all objects, so if $t \in \mathrm{Mor}(U, V)$, then $t^* \in \mathrm{Mor}(V, U)$
   (b) $\|t^*t\| = \|t\|^2$ for any $t \in \mathrm{Mor}(U, V)$. In particular, $\mathrm{End}(U) = \mathrm{Mor}(U, U)$ is a C^*-algebra.
   (c) for every $t \in \mathrm{Mor}(U, V)$, the element $t^*t \in \mathrm{End}(U)$ is positive.

3. there exists a bilinear functor $\mathcal{C} \times \mathcal{C} \ni (U, V) \xmapsto{\otimes} U \otimes V \in \mathcal{C}$, which is associative: $(U \otimes V) \otimes W = U \otimes (V \otimes W)$, and an object 1, called the unit object, satisfying $U \otimes 1 = U = 1 \otimes U$.

4. $(s \otimes t)^* = s^* \otimes t^*$

5. $\mathcal{C}$ has direct sums, i.e. given objects U, V there exists an object W and isometries $u \in \mathrm{Mor}(U, W)$ and $v \in \mathrm{Mor}(V, W)$ (i.e. $u^*u = \mathbb{1}_{\mathrm{End}(U)}$ and $v^*v = \mathbb{1}_{\mathrm{End}(V)}$) such that $uu^* + vv^* = \mathbb{1}_{\mathrm{End}(W)}$. We denote $W = U \oplus V$.

6. $\mathcal{C}$ has subobjects: for every projection $p \in \mathrm{End}(U)$ there exists an object V and an isometry $v \in \mathrm{Mor}(V, U)$ such that $vv^* = p$.

7. the unit object 1 is simple, i.e. $\mathrm{End}(1) = \mathbb{C}$

8. the category is small,

A canonical example of a strict C^*-tensor category is the category $Hilb_f$, whose objects are finite dimensional (complex) Hilbert spaces and the morphisms $\mathrm{Mor}(\mathcal{H}, \mathcal{K}) = \mathsf{B}(\mathcal{H}, \mathcal{L})$ are all linear maps $\mathcal{H} \to \mathcal{K}$. To be more rigorous, one needs to choose a set of Hilbert spaces that contains at least one Hilbert space of each dimension (treated as a cardinal number), this set then can be taken as objects of $Hilb_f$. The adjoint $*$ is a hermitian conjugate of an operator, and the tensor

$\otimes$ corresponds to the canonical tensor product of Hilbert spaces. Strictness corresponds to fixing identifications $(\mathcal{H}\otimes\mathcal{K})\otimes\mathcal{L} = \mathcal{H}\otimes(\mathcal{K}\otimes\mathcal{L})$ (in fact one needs to be a little careful at this point, but it is not a major difficulty). The unit object is the one-dimensional Hilbert space $\mathbb{C}$. The direct sum is the well-known direct sum of Hilbert spaces, and subobjects correspond to subspaces.

The other example comes from the representation theory of a compact quantum group $\mathbb{G}$. The category $\mathrm{Rep}(\mathbb{G})$ is defined as follows: the objects of $\mathrm{Rep}(\mathbb{G})$ are the (isomorphism classes of) finite dimensional corepresentations. Given two corepresentations $U \in \mathsf{B}(\mathcal{H}_U) \otimes C(\mathbb{G})$ and $V \in \mathsf{B}(\mathcal{H}_V) \otimes C(\mathbb{G})$, the morphisms are intertwiners:

$$\mathrm{Mor}(U, V) = \{t \in \mathsf{B}(\mathcal{H}_U, \mathcal{H}_V) : (t \otimes \mathbb{1})U = V(t \otimes \mathbb{1})\},$$

the direct sum tensor product are the ones appearing in (1.9) and (1.10), respectively. The unit object is the trivial representation of $\mathbb{G}$.

Definition 1.30. Let U be an object of a strict C^*-tensor category $\mathcal{C}$. We say that U has a conjugate, if there exists an object $\bar{U}$ of $\mathcal{C}$ and morphisms $r \in \mathrm{Mor}(1, \bar{U}\otimes U)$ and $\bar{r} \in \mathrm{Mor}(1, U\otimes\bar{U})$ such that $(\bar{r}^* \otimes \mathrm{id}_U)(\mathrm{id}_U \otimes r) = \mathrm{id}_U$ and $(r^* \otimes \mathrm{id}_{\bar{U}})(\mathrm{id}_{\bar{U}} \otimes \bar{r}) = \mathrm{id}_{\bar{U}}$. In such situation $\bar{U}$ is called *the* conjugate object (to U) and the pair $(r, \bar{r})$ is called a solution of the conjugate equations. If every object of $\mathcal{C}$ has a conjugate, then we say that $\mathcal{C}$ is *rigid*.

In what follows we will deal only with strict rigid C^*-tensor categories. In this situation, the next definition has a particularly simple form (without assuming rigidity, the definition can still be phrased).

Definition 1.31. A tensor functor $F\colon \mathcal{C} \to \mathit{Hilb}_f$ (i.e. functor preserving the whole linear and tensor structure) is called a *fiber functor.*

There is a canonical example of a fiber functor: the functor $F_{\mathbb{G}}\colon \mathrm{Rep}(\mathbb{G}) \to \mathit{Hilb}_f$ given by $F_{\mathbb{G}}(U) = \mathcal{H}_U$, the carrier Hilbert space of U. The celebrated Woronowicz-Tannaka-Krein Theorem states that this is in fact the only possible example.

Theorem 1.32 (Woronowicz-Tannaka-Krein duality)**.** *Let $\mathcal{C}$ be a strict, rigid C^*-tensor category and $F\colon \mathcal{C} \to \mathit{Hilb}_f$ a fiber functor. Then there exists a unique (up to isomorphism) compact quantum group $\mathbb{G}$ and a (unitary) tensor equivalence $E\colon \mathrm{Rep}(\mathbb{G}) \to \mathcal{C}$ such that $F \circ E$ is naturally (unitarily) tensor equivalent to $F_{\mathbb{G}}$.*

Remark 1.33. A simple, essentially category-free proof of Theorem 1.32 was recently given in [41].

Lemma 1.34 ([65, Proposition 2.2(3)])**.** *Assume $U, V \in \mathcal{C}$ are such that $U \subset V$ and V has a conjugate $\bar{V}$. Then U has a conjugate.*

Theorem 1.35 ([45])**.** *Let $\mathbb{G}$ be a compact quantum group, M a von Neumann algebra and $\gamma\colon \mathsf{M} \hookrightarrow \ell^\infty(\widehat{\mathbb{G}})$ an injective normal *-homomorphism. Assume that $\gamma(\mathsf{M}) \subseteq \ell^\infty(\widehat{\mathbb{G}})$ is invariant. Then $\mathsf{M} = \ell^\infty(\widehat{\mathbb{H}})$ for some compact quantum group $\mathbb{H}$. In particular, $\mathbb{H} \subset \mathbb{G}$ is a closed subgroup and M is preserved under the action of unitary antipode $\widehat{R}$ and the scaling group $\widehat{\tau}_t$.*

Remark 1.36. Theorem 1.35 in the above formulation is in fact [45, Theorem 3.1] after one applies the co-duality techniques of [34].

Proof. There are two key steps we need to prove the Theorem. First is that $\mathrm{Rep}(\ell^\infty(\widehat{\mathbb{G}}))$, the category of finite-dimensional unitary representations of the von Neumann algebra $\ell^\infty(\widehat{\mathbb{G}})$, endowed with the following tensor structure: if $\phi, \phi'\colon \ell^\infty(\widehat{\mathbb{G}}) \to \mathsf{B}(\mathcal{H}), \mathsf{B}(\mathcal{H}')$ are *-homomorphisms, then $\phi ⓣ \phi' = \phi' \otimes \phi \circ \widehat{\Delta}$, and with its canonical tensor functor into Hilb_f is naturally unitarily tensor equivalent to the category $\mathrm{Rep}(\mathbb{G})$ with its canonical fiber functor $F_{\mathbb{G}}$. This equivalence is given as follows: to any *-representation in a Hilbert space $\mathcal{H}$ of finite dimension $\phi\colon \ell^\infty(\widehat{\mathbb{G}}) \to \mathsf{B}(\mathcal{H})$ we associate a corepresentation $U_\phi = (\phi \otimes \mathrm{id})\mathrm{W}^{\mathbb{G}} \in \mathsf{M}(\mathsf{K}(\mathcal{H}) \otimes C(\mathbb{G}))$. That this assignment is a natural unitary tensor equivalence follows from Theorem 1.13 (in fact, in case of compact quantum groups this was noted earlier, cf. [68, Section 5]) and (1.12). It is clear from the construction that this equivalence intertwines the tensor functors into $Hilb_f$.

Now observe that any finite dimensional representation of $\ell^\infty(\widehat{\mathbb{G}})$ gives rise to a (still finite dimensional) representation of M by precomposition with the map γ. In this way we obtain a functor $G\colon \mathrm{Rep}(\mathbb{G}) \to \mathrm{Rep}(\mathsf{M})$, where the tensor structure in $\mathrm{Rep}(\mathsf{M})$ is introduced as above: $\phi, \phi'\colon \ell^\infty(\widehat{\mathbb{G}}) \to \mathsf{B}(\mathcal{H}), \mathsf{B}(\mathcal{H}')$ are *-homomorphisms, then $\phi ⦶ \phi' = (\phi' \otimes \phi) \circ (\gamma^{-1} \otimes \gamma^{-1}) \circ \widehat{\Delta} \circ \gamma$ (this is well defined, because $\gamma(\mathsf{M})$ is invariant). To be more precise, let $U, U' \in \mathrm{Rep}(\mathbb{G})$ and let $\phi_U, \phi_{U'}\colon \ell^\infty \to \mathsf{B}(\mathcal{H}), \mathsf{B}(\mathcal{H}')$ be the associated *-homomorphisms (where we already used the existence of the fiber functor $U, U' \mapsto \mathcal{H}, \mathcal{H}'$). The object $\phi_U \circ \gamma$ is well defined, as $F_{\mathbb{G}}(\phi_U \circ \gamma) = \mathcal{H}$ are finite dimensional (likewise for U'). Let then $f \in \mathrm{Mor}(U, U') = \mathsf{B}_{\mathbb{G}}(\mathcal{H}, \mathcal{H}') \subseteq \mathsf{B}(\mathcal{H}, \mathcal{H}')$ be an intertwiner. This means in particular that for all $x \in \ell^\infty(\widehat{\mathbb{G}})$ we have $f\phi_U(x) = \phi_{U'}(x) f \in \mathsf{B}(\mathcal{H}, \mathcal{H}')$. Hence putting $x = \gamma(y)$ for $y \in \mathsf{M}$ we get that $f \in \mathsf{B}(\mathcal{H}, \mathcal{H}')$ is an intertwiner between the *-representations $\phi_U \circ \gamma$ and $\phi_{U'} \circ \gamma$ of M, thus $f \in \mathsf{B}_{\mathsf{M}}(\mathcal{H}, \mathcal{H}')$ is well defined.

It is clear from the construction that the functor G is linear and tensor (for the tensor product $⦶$ defined as above). Moreover, it has the following property: for any $V \in \mathrm{Rep}(\mathsf{M})$ there exists $U \in \mathrm{Rep}(\mathbb{G})$ such that $V \subset G(U)$. This is due to the fact that γ is injective and that M and $\ell^\infty(\widehat{\mathbb{G}})$ are type I atomic von Neumann algebras, hence the inclusion γ is well understood (e.g. by means of the Bratteli diagram): any matrix block of M is sent to a numbers of subblocks of some matrix blocks of $\ell^\infty(\widehat{\mathbb{G}})$ by means of formal identity map and conjugation by an unitary. In particular, as $\mathbb{G}$ has the trivial representation $\mathbb{1}$, there exists a 1×1 matrix block in the decomposition of $\ell^\infty(\widehat{\mathbb{G}})$ into matrix algebras. Because $\gamma\colon \mathsf{M} \to \ell^\infty(\widehat{\mathbb{G}})$ is an inclusion, there is object $\mathbb{1}' \in \mathrm{Rep}(\mathsf{M})$ which correspond to a one-dimensional representation of M (i.e. M admit a character) and this object is obtained by $G(\mathbb{1}) = \mathbb{1}'$.

Observe that $\mathbb{1}'$ is a unit object in $\mathrm{Rep}(\mathsf{M})$. Indeed, $\mathbb{1}$ correspond to the counit $\widehat{\varepsilon}\colon \ell^\infty(\widehat{\mathbb{G}}) \to \mathbb{C}$. Let then $\phi\colon \mathsf{M} \to \mathsf{B}(\mathcal{H})$ be a finite-dimensional representation. Then $\mathbb{1}'$, by construction, correspond to the map $\widehat{\varepsilon} \circ \gamma\colon \mathsf{M} \to \mathbb{C}$. Then for any $x \in \mathsf{M}$ we have:

$$\begin{aligned}\phi ⦶ (\widehat{\varepsilon} \circ \gamma)(x) = (\phi \otimes \widehat{\varepsilon})(\gamma^{-1} \otimes \mathrm{id})\widehat{\Delta}(\gamma(x)) = \phi(\gamma^{-1}\Big((id \otimes \widehat{\varepsilon})\widehat{\Delta}(\gamma(x))\Big)) =\\ = \phi(\gamma^{-1}(\gamma(x))) = \phi(x)\end{aligned}$$

by evoking the defining properties of the counit.

The second step is now easy. To conclude, it is enough to show that $G(U)$ has a conjugate and evoke to Lemma 1.34. Indeed, the existence of a conjugate object to $G(U)$ follows from existence of $\bar{U} \in \mathrm{Rep}(\mathbb{G})$, the object conjugate to U. If then $r \in \mathrm{Mor}(\mathbb{1}, \bar{U} \otimes U)$ and $\bar{r} \in \mathrm{Mor}(\mathbb{1}, U \otimes \bar{U})$ is a solution to the conjugate equations, then by the fact that G is tensor functor and $\mathbb{1}'$ is unit, a simple application of G to the conjugate equations show that $G(\bar{U})$ is conjugate to $G(U)$ (and in particular that $G(r)$ and $G(\bar{r})$ solve the conjugate equations for $G(U)$ and $\overline{G(U)}$).

We now use Theorem 1.32 to see that $\mathrm{Rep}(\mathsf{M})$ with the canonical fiber functor is in fact (equivalent to) $\mathrm{Rep}(\mathbb{H})$ for some compact quantum group $\mathbb{H}$. Repeating the first paragraph we observe that in fact $\ell^\infty(\widehat{\mathbb{H}}) \cong \mathsf{M}$, which finishes the proof. $\square$

1.3.6 Intrinsic subgroup

Any locally compact quantum group $\mathbb{G}$ has its maximal classical subgroup $Gr(\widehat{\mathbb{G}})$, also called the group of characters of $\mathbb{G}$. It is given as follows: consider the universal enveloping C^*-algebra $C_0^u(\mathbb{G})$ and the commutator ideal of it, i.e. the ideal generated by $\{xy - yx : x, y \in C_0^u(\mathbb{G})\}$, call this ideal I. Then the quotient map

$$q_{\mathbb{G}}\colon C_0^u(\mathbb{G}) \to C_0^u(\mathbb{G})/_I =: C_0(Gr(\widehat{\mathbb{G}}))$$

identifies the spectrum of the (commutative) C^*-algebra $C_0^u(\mathbb{G})/_I$, denoted $Gr(\widehat{\mathbb{G}})$, with a closed (quantum) subgroup of $\mathbb{G}$. The commutativity of the C^*-algebra $C_0(Gr(\widehat{\mathbb{G}}))$ ensures us that $Gr(\widehat{\mathbb{G}})$ is a coamenable quantum group, so we drop the $\cdot^u$ decoration.

Alternatively, one can go to the dual quantum group $\widehat{\mathbb{G}}$ and ask for group-like elements in its appropriate C^*- algebras describing, e.g. take

$$Gr(\widehat{\mathbb{G}}) = \{x \in \mathsf{M}(C_0^u(\widehat{\mathbb{G}})) | \Delta^u_{\widehat{\mathbb{G}}}(x) = x \otimes x, x \neq 0\}$$

Instead of $C_0^u(\widehat{\mathbb{G}})$ one can alternatively take $L^\infty(\widehat{\mathbb{G}})$. It follows that elements of $Gr(\widehat{\mathbb{G}})$ are unitary ([32, Theorem 3.9]) and this set with the inherited weak*-topology is a locally compact group (where the multiplication and inverse are Gelfand transforms of restrictions of $\widehat{\Delta}$ and $\widehat{S}$). There are other ways to describe the group $Gr(\widehat{\mathbb{G}})$, see [20, 32].

Chapter 2

Hopf image

2.1 Introduction and results

This chapter is devoted to studying the following concept: let $\mathbb{G}$ be a locally compact quantum group (in the sense of Kustermans-Vaes). Let B be a C^*-algebra and let $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}), \mathsf{B})$ be a Woronowicz morphism. We think of it as Gelfand dual of a map $\widehat{\beta}\colon \mathbb{X} \to \mathbb{G}$ from a quantum space into a quantum group and ask what is the closed quantum subgroup (in the sense of Vaes) of $\mathbb{G}$ generated by $\widehat{\beta}(\mathbb{X}) \subset \mathbb{G}$.

Formally speaking, we consider the following category, denoted by $\mathcal{C}_\beta$. Objects of $\mathcal{C}_\beta$ are triples $(\pi, \mathbb{H}, \tilde{\beta})$ such that $\mathbb{H}$ is a closed quantum subgroup of $\mathbb{G}$ (in the sense of Vaes) such that $\pi \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$ is the associated morphism intertwining the coproducts and $\tilde{\beta} \circ \pi = \beta$ (as Woronowicz morphisms), i.e. the map β factors through the C^*-algebra of the subgroup $C_0^u(\mathbb{H})$ (where $\mathbb{H}$ is embedded in $\mathbb{G}$ using π) and $\tilde{\beta} \circ \pi = \beta$ is the factorization. For two objects $\mathbb{h} = (\pi, \mathbb{H}, \tilde{\beta}), \mathbb{k} = (\pi', \mathbb{K}, \beta') \in Ob(\mathcal{C}_\beta)$, a morphism $\varphi \in \mathrm{Mor}_{\mathcal{C}_\beta}(\mathbb{h}, \mathbb{k})$ is a Woronowicz morphism of the C^*-algebras $\varphi \in \mathrm{Mor}(C_0^u(\mathbb{K}), C_0^u(\mathbb{H}))$ which intertwines the respective coproducts and such that the following diagram commutes:

$$\begin{tikzcd}
C_0^u(\mathbb{G}) \arrow[rr, "\beta"] \arrow[dr, "\pi"] \arrow[dd, "\pi'"'] & & \mathsf{B} \\
& C_0^u(\mathbb{H}) \arrow[ur, "\tilde{\beta}"'] & \\
C_0^u(\mathbb{K}) \arrow[ur, "\varphi"] \arrow[uurr, bend right, "\tilde{\beta}'"'] & &
\end{tikzcd}$$

Diagram 1: Morphisms in $\mathcal{C}_\beta$

The object we are interested in is the initial object of the category $\mathcal{C}_\beta$.

The aim of this chapter is to

1. construct/establish the existence of this initial object,
2. describe it as thoroughly as possible,
3. compare our construction to existing notions of generation in the quantum context.

The following are the main results:

Theorem 2.1. *Given a locally compact quantum group $\mathbb{G}$ and a morphism β as above, there always exists the initial object of the category $\mathcal{C}_\beta$.*

In what follows, this initial object will be called *the Hopf image of the morphism β*. If this initial object happens to be the whole $\mathbb{G}$, then β will be called *generating morphism*. In [6], in the context of compact quantum groups, such morphisms were called *inner faithful* (or, sometimes, *faithful in the discrete dual group sense*). The latter name no longer match the situation of locally compact quantum groups, and the former produces some ambiguity, as some expect such a definition should indicate a relation with the adjoint action.

In Section 2.3 we address the question of comparison our construction of Hopf image to other notions of generation that appeared in literature previously in certain specific cases. The outcome is as expected: the notions of generation in both the discrete quantum group sense and in the compact quantum group sense can be interpreted in the language of Hopf image in locally compact quantum group sense and they coincide (after some minor modifications in certain cases).

2.2 Construction of Hopf image

2.2.1 First steps towards the construction

The goal of this part is to construct a quantum group $\mathbb{H}$ which will later be shown to satisfy the defining properties of Hopf image. So let us fix a morphism $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}), \mathsf{B})$, where B is some C^*-algebra, the universal, reduced and semireduced Kac-Takesaki operators will be denoted $\mathbb{W}, \mathrm{W}, \mathbb{W}, \mathbb{W}$, respectively.

Application of $\Lambda_{\widehat{\mathbb{G}}} \otimes \Lambda_{\widehat{\mathbb{G}}} \otimes \mathrm{id}$ to both sides of (1.12) yields:

$$(\Delta_{\widehat{\mathbb{G}}} \otimes \mathrm{id})(\mathbb{W}) = \mathbb{W}_{23}\mathbb{W}_{13} \tag{2.1}$$

But as the comultiplication on the reduced level is implemented by W, (2.1) together with (1.7) can be rewritten as

$$(\sigma \otimes \mathrm{id})(W_{12}^* \mathbb{W}_{23} W_{12}) = \mathbb{W}_{23}\mathbb{W}_{13}. \tag{2.2}$$

Evaluating $(\sigma \otimes \mathrm{id})$ at both sides of (2.2) yields:

$$\mathrm{W}_{12}^* \mathbb{W}_{23} \mathrm{W}_{12} = \mathbb{W}_{13}\mathbb{W}_{23}, \tag{2.3}$$

hence

$$\mathbb{W}_{23} W_{12} \mathbb{W}_{23}^* = \mathrm{W}_{12}\mathbb{W}_{13}. \tag{2.4}$$

Let us denote $X = (\mathrm{id} \otimes \beta)\mathbb{W} \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes \mathsf{B})$. Computing the value of $(\mathrm{id} \otimes \mathrm{id} \otimes \beta)$ at both sides of the equality (2.4) results in:

$$X_{23}\mathrm{W}_{12}X_{23}^* = \mathrm{W}_{12}X_{13} \tag{2.5}$$

or, equivalently,

$$(\Delta_{\widehat{\mathbb{G}}} \otimes \mathrm{id})X = X_{23}X_{13} \tag{2.6}$$

Applying $(\omega \otimes \mathrm{id} \otimes \mathrm{id})$ to both sides of (2.5) we obtain:

$$X(a \otimes \mathbb{1})X^* = (\omega \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{W}_{12}X_{13}), \tag{2.7}$$

where $a = (\omega \otimes \mathrm{id})\mathrm{W}$. As

$$\begin{aligned} \mathrm{W}_{12} &\in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}) \otimes \mathsf{B}) \subseteq \mathsf{M}(\mathsf{K}(L^2(\mathbb{G})) \otimes C_0(\mathbb{G}) \otimes \mathsf{B}) \\ X_{13} &\in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}) \otimes \mathsf{B}) \subseteq \mathsf{M}(\mathsf{K}(L^2(\mathbb{G})) \otimes C_0(\mathbb{G}) \otimes \mathsf{B}) \end{aligned} \tag{2.8}$$

we get that $X(a \otimes \mathbb{1})X^* \in \mathsf{M}(C_0(\mathbb{G}) \otimes \mathsf{B})$ and hence we can define the map $\theta\colon C_0(\mathbb{G}) \to \mathsf{M}(C_0(\mathbb{G}) \otimes \mathsf{B})$ by

$$C_0(\mathbb{G}) \ni a \overset{\theta}{\longmapsto} X(a \otimes \mathbb{1})X^* \in \mathsf{M}(C_0(\mathbb{G}) \otimes \mathsf{B}). \tag{2.9}$$

Let us assume that $\mathsf{B} \subseteq \mathsf{B}(\mathcal{H})$ is (faithfully) represented on a Hilbert space $\mathcal{H}$. Then we can view θ as a representation $\theta \in \operatorname{Rep}(C_0(\mathbb{G}), L^2(\mathbb{G}) \otimes \mathcal{H})$. One has:

$$(\mathrm{id} \otimes \theta)(\mathrm{W}) = \mathrm{W}_{12} X_{13} \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}) \otimes \mathsf{B}) \subseteq \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes \mathsf{K}(L^2(\mathbb{G}) \otimes \mathcal{H})).$$

As $\mathrm{W} \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}))$ generates $C_0(\mathbb{G})$ in the sense of Section 1.2.3 (cf. Lemma 1.7 and Section 1.3.1), we conclude that:

Proposition 2.2. $\theta \in \operatorname{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes \mathsf{B})$.

Now we are in position which allows us to state the main construction. Let

$$\mathcal{M}_0 = \{(\mathrm{id} \otimes \omega) X | \omega \in B^*\} \subseteq L^\infty(\widehat{\mathbb{G}})$$

As $X \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes \mathsf{B})$, $\mathcal{M}_0 \subseteq \mathsf{M}(C_0(\widehat{\mathbb{G}})) \subseteq L^\infty(\widehat{\mathbb{G}})$. Denote by $\mathcal{M}$ the *-algebra generated by $\mathcal{M}_0$ and by M_0 its norm-closure.

Proposition 2.3. *$\mathcal{M}_0' = M_0'$ is a von Neumann algebra*

Proof. Indeed, as we have

$$T \in \mathcal{M}_0' \iff X(T \otimes \mathbb{1}) = (T \otimes \mathbb{1})X \iff (T \otimes \mathbb{1})X^* = X^*(T \otimes \mathbb{1}) \iff$$
$$\iff X(T^* \otimes \mathbb{1}) = (T^* \otimes \mathbb{1})X \iff T^* \in \mathcal{M}_0'$$

□

Thus also $\mathsf{M}_1 = \mathcal{M}_0''$ is a von Neumann algebra.

Let M_{BV} be the smallest von Neumann algebra containing M_0 (so in particular containing M_1), which is invariant (under $\Delta_{\widehat{\mathbb{G}}}$) and preserved by $\tau^{\widehat{\mathbb{G}}}, R^{\widehat{\mathbb{G}}}$. The existence of such von Neumann algebra follows from standard argument: it is the intersection of all von Neumann subalgebras of $L^\infty(\widehat{\mathbb{G}})$ that are $R^{\mathbb{G}}$-, $\tau^{\mathbb{G}}$- and $\Delta_{\widehat{\mathbb{G}}}$-invariant: this collection is non-empty because $L^\infty(\widehat{\mathbb{G}})$ itself is such an algebra. Later on we will see that it can be constructed more explicitely. Thanks to Theorem 1.25, there exists $\mathbb{H} \subset \mathbb{G}$ such that $L^\infty(\widehat{\mathbb{H}}) = \mathsf{M}$, in particular, we have a map $\pi \in \operatorname{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$ coming from Theorem 1.23, which is linked to the embedding $\mathsf{M}_{BV} \subseteq L^\infty(\widehat{\mathbb{G}})$ via (1.23).

2.2.2 Properties of algebra M_1

Lemma 2.4. *Let $\beta \in \operatorname{Mor}(C_0^u(\mathbb{G}), \mathsf{B})$ be a morphism of C^*-algebras, let $\mathsf{C} = \beta(C_0^u(\mathbb{G})) \subseteq \mathsf{M}(\mathsf{B})$. Let $\mathsf{B} \subseteq \mathsf{B}(\mathcal{H})$ be a non-degenerate representation. Let now:*

- $\mathsf{M}_1 = \{(\mathrm{id} \otimes \omega \circ \beta)(\mathbb{W}) : \omega \in \mathsf{B}^*\}''$
- $\mathsf{M}_2 = \{(\mathrm{id} \otimes \omega \circ \beta)(\mathbb{W}) : \omega \in \mathsf{C}^*\}''$
- $\mathsf{M}_3 = \{(\mathrm{id} \otimes \omega \circ \beta)(\mathbb{W}) : \omega \in \mathsf{B}(\mathcal{H})^*\}''$
- $\mathsf{M}_4 = \{(\mathrm{id} \otimes \omega \circ \beta)(\mathbb{W}) : \omega \in \mathsf{B}(\mathcal{H})_*\}''$
- $\mathsf{M}_5 = \{(\mathrm{id} \otimes \omega_{\xi,\eta} \circ \beta)(\mathbb{W}) : \xi, \eta \in \mathcal{H}\}''$

Then

- $\beta \in \operatorname{Mor}(C_0^u(\mathbb{G}), \mathsf{C})$
- $\mathsf{M}_1 = \mathsf{M}_2 = \mathsf{M}_3 = \mathsf{M}_4 = \mathsf{M}_5$

Remark 2.5. By first part of Lemma 2.4 we see that we can restrict our attention to maps $\beta: C_0^u(\mathbb{G}) \to \mathsf{B}$ that are surjective (philosophically speaking, the maps that are Gelfand duals of an embedding $\widehat{\beta}: \mathbb{X} \hookrightarrow \mathbb{G}$ of the quantum space $\mathbb{X}$ as a closed quantum subset of $\mathbb{G}$, where $\mathsf{B} = C_0(\mathbb{X})$).

Proof. The first part is a standard reasoning, we only indicate its main steps. Firstly one shows that $\mathsf{C} \subseteq \mathsf{B}(\mathcal{H})$ is a non-degenerate representation by using that β is a morphism, $\mathsf{B} \subseteq \mathsf{B}(\mathcal{H})$ is non-degenerate and Cohen factorization theorem. Secondly, one shows that $\mathsf{M}(\mathsf{C}) \subseteq \mathsf{M}(\mathsf{B})$ by using the Hilbert space description of the space of multipliers. The last step is to check that $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}), \mathsf{C})$, which amounts to checking that $\beta(C_0^u(\mathbb{G})) \cdot \mathsf{C} = \mathsf{C}$, which once again follows from Cohen factorization theorem.

It is obvious that $\mathsf{M}_4 \subseteq \mathsf{M}_3$. To show that $\mathsf{M}_3 \subseteq \mathsf{M}_4$, let us fix an element $\omega \in \mathsf{B}(\mathcal{H})^*$ and consider $x = (\mathrm{id} \otimes \omega)X$. Recall that $\mathsf{B}(\mathcal{H})_* \subseteq \mathsf{B}(\mathcal{H})^*$ is weak*-dense, so pick $(\omega_n)_{n\in N} \subseteq \mathsf{B}(\mathcal{H})_*$ such that $\omega_n \to \omega$ in weak*-topology. Then $x_n = (\mathrm{id} \otimes \omega_n)X \in \mathsf{M}_4$ and by Lemma 1.12 we have that $x_n \to x$ in σ-WOT. Hence $x \in \mathsf{M}_3$ by σ-WOT closedness of the latter and we are done.

Now as every functional in $\mathsf{B}(\mathcal{H})^*$ restricts to B and C we have that $\mathsf{M}_3 \subseteq \mathsf{M}_1, \mathsf{M}_2$. But as $\mathsf{B}, \mathsf{C} \subseteq \mathsf{B}(\mathcal{H})$ are closed, any continuous functional from B^* and C^* extends to a continuous functional in $\mathsf{B}(\mathcal{H})^*$ by Hahn-Banach theorem, so $\mathsf{B}(\mathcal{H})^* \twoheadrightarrow \mathsf{C}^*, \mathsf{B}^*$. In particular, this means $\mathsf{M}_1, \mathsf{M}_2 \subseteq \mathsf{M}_3$.

For the equality $\mathsf{M}_4 = \mathsf{M}_5$ recall that the linear span of vector functionals is norm dense in $\mathsf{B}(\mathcal{H})_*$ (see, e.g. [14, III.2.1.4]). Now by standard calculation we show that if $\omega_n \to \omega$ in norm, then $(\mathrm{id} \otimes \omega_n)X \to (\mathrm{id} \otimes \omega)X$ in the weak operator topology. Pick then $\xi \in \mathcal{H}$, we have:

$$
\begin{aligned}
|\langle \xi | (\mathrm{id} \otimes (\omega - \omega_n))X | \xi \rangle| &= |(\omega_\xi \otimes (\omega - \omega_n))X| \\
\leq \|\omega_\xi \otimes (\omega - \omega_n)\| \|X\| &= \|\xi\|^2 \|\omega - \omega_n\| \|X\| \to 0.
\end{aligned}
$$

By WOT-closedness of M_5 any element of the generating set of M_4 is in fact in M_5, so we once again conclude by von Neumann's bicommutant Theorem. □

Proposition 2.6. *The algebra M_1 is invariant, i.e. $\widehat{\Delta}(\mathsf{M}_1) \subseteq \mathsf{M}_1 \overline{\otimes} \mathsf{M}_1$. If moreover $\tau_t^u(\ker(\beta)) \subseteq \ker(\beta)$ (in particular if $\mathbb{G}$ is Kac type), then M_1 is is preserved by $\widehat{\tau}_{-t}$ for each individual $t \in \mathbb{R}$.*

Proof. For the invariance, let us first pick $x = (\mathrm{id} \otimes \langle \xi | \cdot | \eta \rangle)X \in \mathsf{M}_5$ for some $\xi, \eta \in \mathcal{H}$, we will show that $\widehat{\Delta}(x) \in \mathsf{M}_5 \overline{\otimes} \mathsf{M}_5$. Pick an orthonormal basis $(e_j)_{j\in J}$ of $\mathcal{H}$ and recall that $\mathbb{1} = \sum_{j\in J} |e_j\rangle\langle e_j|$ is a WOT-convengent resolution of identity into rank one projections. We compute:

$$
\begin{aligned}
\widehat{\Delta}(x) &= (\widehat{\Delta} \otimes \langle \xi | \cdot | \eta \rangle)(X) = (\mathrm{id} \otimes \mathrm{id} \otimes \langle \xi | \cdot | \eta \rangle)(X_{23} X_{13}) = \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes \langle \xi | \cdot | \eta \rangle)(X_{23}(\mathbb{1} \otimes \mathbb{1} \otimes \sum_{j\in J} |e_j\rangle\langle e_j|) X_{13}) = \\
&= \sum_{j\in J} (\mathrm{id} \otimes \mathrm{id} \otimes \langle \xi | \cdot | \eta \rangle)(X_{23}(\mathbb{1} \otimes \mathbb{1} \otimes |e_j\rangle\langle e_j|) X_{13}) = \\
&= \sum_{j\in J} (\mathrm{id} \otimes \mathrm{id} \otimes \langle \xi | \cdot | e_j \rangle)(X_{23})(\mathrm{id} \otimes \mathrm{id} \otimes \langle e_j | \cdot | \eta \rangle)(X_{13}) = \\
&= \sum_{j\in J} (\mathrm{id} \otimes \langle e_j | \cdot | \eta \rangle)(X) \otimes (\mathrm{id} \otimes \langle \xi | \cdot | e_j \rangle)(X) \in \mathsf{M}_5 \overline{\otimes} \mathsf{M}_5
\end{aligned}
$$

We conclude by normality of $\widehat{\Delta}$ and equality $\mathsf{M}_1 = \mathsf{M}_5$ obtained in Lemma 2.4.

Let $t \in \mathbb{R}$. If $\ker(\beta)$ is τ_t^u invariant, let $\omega \in \mathsf{B}^*$. Then there exists (necessarily unique) functional $\omega_t \in \mathsf{B}^*$ such that $\omega \circ \beta \circ \tau_t^u = \omega_t \circ \beta$. Indeed, using $\mathsf{M}_2 = \mathsf{M}_1$, i.e. assuming that $\beta: C_0^u(\mathbb{G}) \to \mathsf{B}$ is a surjection, given $b \in \mathsf{B}$ we find $a \in C_0^u(\mathbb{G})$ such that $\beta(a) = b$. Define

$$
\omega_t(b) = \omega(\beta(\tau_t^u(a))).
$$

The boundedness of ω_t is obvious, provided that it is well defined. To this end, let us assume $a' \in C_0^u(\mathbb{G})$ is such that $\beta(a') = b$ as well. Then

$$
\omega(\beta(\tau_t^u(a))) = \omega(\beta(\tau_t^u(a'))) \iff \omega(\beta(\tau_t^u(a - a'))) = 0
$$

But as we picked a' so that $a - a' \in \ker \beta$, the above follows from $\tau_t^u(\ker(\beta)) \subseteq \ker(\beta)$.

Now having ω_t we use (the semi-reduced version of) (1.16) to compute:

$$\widehat{\tau}_{-t}((\mathrm{id}\otimes\omega\circ\beta)\mathbb{W}) = \widehat{\tau}_{-t}((\mathrm{id}\otimes\omega\circ\beta)(\widehat{\tau}_t\otimes\tau^u_t)\mathbb{W}) = (\mathrm{id}\otimes\omega_t\circ\beta)\mathbb{W}$$

hence the generating set for the von Neumann algebra M_2 is $\widehat{\tau}_{-t}$-invariant, which is enough to conclude global $\widehat{\tau}_{-t}$-invariance of M_2. □

Lemma 2.7. *Let $\mathsf{M}\subseteq L^\infty(\mathbb{G})$ be a von Neumann algebra that is invariant.*

1. *Let $\mathsf{M}_R\subseteq L^\infty(\mathbb{G})$ be the smallest von Neumann algebra containing M and closed under the unitary antipode R. Then it is invariant.*
2. *Let $\mathsf{M}_\tau\subseteq L^\infty(\mathbb{G})$ be the smallest von Neumann algebra containing M and closed under the scaling group τ_t for all $\in\mathbb{R}$. Then it is invariant.*
3. *Let $\mathsf{M}_{R,\tau}\subseteq L^\infty(\mathbb{G})$ be the smallest von Neumann algebra containing M and closed under the unitary antipode R and the scaling group τ_t for all $t\in\mathbb{R}$. Then it is equal to $(\mathsf{M}_R)_\tau=(\mathsf{M}_\tau)_R$.*

Proof. The first two items follow from the relations $\Delta(R(x))=\sigma\circ R\otimes R\circ\Delta(x)$ and $\Delta(\tau_t(x))=\tau_t\otimes\tau_t\circ\Delta(x)$, so that $\mathsf{M}_R=(\mathsf{M}\cup R(\mathsf{M}))''$ and $\mathsf{M}_\tau=(\bigcup_{t\in\mathbb{R}}\tau_t(\mathsf{M}))''$ are the desired von Neumann algebras. The last item is a combination of the first two together with the relation $\tau_t\circ R=R\circ\tau_t$. □

Remark 2.8. In particular, the algebra M_{BV} described at the end of Section 2.2.1 is obtained by only closing M_1 with respect to the unitary antipode $\widehat{R}$ and scaling group $\widehat{\tau}_t$. In fact, even closing M_1 under $\widehat{\tau}$-invariance is enough, as we will now see.

Theorem 2.9. *The minimal $\widehat{\Delta}$-, $\widehat{\tau}$- and $\widehat{R}$-invariant subalgebra of $L^\infty(\widehat{\mathbb{G}})$ containing M_1 is given by*

$$\mathsf{M}_{BV}=(\bigcup_{t\in\mathbb{R}}\widehat{\tau}_t(\mathsf{M}_1))''.$$

In particular, $\mathsf{M}_{BV}=\mathsf{M}_1$ if $\mathbb{G}$ compact or discrete or if $\tau^u_t(\ker(\beta))\subseteq\ker(\beta)$ for all $t\in\mathbb{R}$.

Proof. For the purpose of the proof, let us call the right hand side von Neumann algebra of the statement of Theorem 2.9 M_{RHS}. This algebra is clearly $\widehat{\tau}_t$ invariant for all $t\in\mathbb{R}$, and as M_1 is invariant, we conclude from Lemma 2.7 that M_{RHS} is again invariant. To see that $\mathsf{M}_{RHS}=\mathsf{M}_{BV}$, we need to show that it is preserved by the unitary antipode $\widehat{R}$.

Firstly, for $t\in\mathbb{R}$ and $\omega\in\mathsf{B}(\mathcal{H})_*$ let us denote $x_{\omega,t}=(\widehat{\tau}_t\otimes\omega)X$ and observe that $\mathsf{M}_1=\mathsf{M}_4$ is generated by $x_{\omega,0}$ for all $\omega\in\mathsf{B}(\mathcal{H})_*$. Furthermore, for $t\in\mathbb{R}$ fixed, $\widehat{\tau}_t(\mathsf{M}_1)$ is generated by $x_{\omega,t}$ for all $\omega\in\mathsf{B}(\mathcal{H})_*$. In turn, M_{RHS} is generated by $x_{\omega,t}$ for all $\omega\in\mathsf{B}(\mathcal{H})$ and all $t\in\mathbb{R}$. Indeed, all elements $x_{\omega,t}\in\mathsf{M}_{RHS}$ from the above description. The converse inclusion follows easily from von Neumann bicommutant theorem: if y commutes with all $x_{\omega,t}$ for all ω and t, we see that $y\in\bigcap_{t\in\mathbb{R}}\widehat{\tau}_t(\mathsf{M}_1)'\subseteq(\bigcup_{t\in\mathbb{R}}\widehat{\tau}_t(\mathsf{M}_1))'$.

Now we use $\mathsf{M}_1=\mathsf{M}_4$ from Lemma 2.4. Let us pick $\omega\in\mathsf{B}(\mathcal{H})$. Then ω^* defined as $\omega^*(a)=\overline{\omega(a^*)}$ is again a normal functional, as the adjoint * is σ-weakly continuous. Then by (1.11) we have that $(\mathrm{id}\otimes\omega)(X^*)=[(\mathrm{id}\otimes\omega^*)(X)]^*=x^*_{\omega^*,0}\in\mathcal{D}(\widehat{S})=\mathcal{D}(\widehat{\tau}_{i/2})$ (recall that in fact X is antirepresentation, so X^* is a representation of $\mathbb{G}$). Hence

$$\widehat{R}((\mathrm{id}\otimes\omega)(X^*))=(\widehat{\tau}_{-i/2}\circ\widehat{S})((\mathrm{id}\otimes\omega)X^*)=\widehat{\tau}_{-i/2}((\mathrm{id}\otimes\omega)X) \tag{2.10}$$

But as M_{RHS} is $\widehat{\tau}$-invariant, it is also preserved by its analytic generator, as it is defined uniquely: the analytic generator of $\widehat{\tau}_{-t}\restriction_{\mathsf{M}_{RHS}}$ is precisely $(\widehat{\tau}_{-i/2})\restriction_{\mathsf{M}_{RHS}}$. Hence $\widehat{R}(x^*_{\omega^*,0})\in\mathsf{M}_{RHS}$ for all $\omega\in\mathsf{B}(\mathcal{H})_*$ and in turn $\widehat{R}(x_{\omega,0})\in\mathsf{M}_{RHS}$ for all $\omega\in\mathsf{B}(\mathcal{H})_*$. Next, thanks to the relation $\widehat{\tau}_t\circ\widehat{R}=\widehat{R}\circ\widehat{\tau}_t$, it follows that $\widehat{R}(x_{\omega,t})=\widehat{\tau}_t(\widehat{R}(x_{\omega,0}))\in\widehat{\tau}_t(\mathsf{M}_{RHS})=\mathsf{M}_{RHS}$. Together with the description of the generating set in first step, this finishes the proof of the main assertion.

The "in particular" part follows from the observation that in all these cases the algebra M_1 is automatically $\widehat{\tau}$-invariant: in case of $\mathbb{G}$ compact this was Theorem 1.35, in case $\mathbb{G}$ discrete this was Proposition 1.27 and the case $\tau^u_t(\ker(\beta))\subseteq\ker(\beta)$ for all $t\in\mathbb{R}$ follows from the discussion in Proposition 2.6.

□

2.2.3 Verification of defining properties

This part is devoted to showing that the quantum subgroup $\mathbb{H}$ constructed in Section 2.2.1 indeed satisfies the defining properties of Hopf image, i.e. firstly, there exists $\tilde{\beta} \in \mathrm{Mor}(C_0^u(\mathbb{H}), \mathsf{B})$ as described in Diagram 1, that is showing that $(\pi, \mathbb{H}, \tilde{\beta}) \in \mathcal{C}_\beta$, and secondly, that it is an initial object of the category $\mathcal{C}_\beta$.

Lemma 2.10. *Let $\mathbb{K}$ be a Vaes-closed quantum subgroup of $\mathbb{G}$ and denote tha associated Hopf morphism by $\pi_\mathbb{K} \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{K}))$ and $\gamma\colon L^\infty(\widehat{\mathbb{K}}) \to L^\infty(\widehat{\mathbb{G}})$. Then $(\pi_\mathbb{K}, \mathbb{K}, \tilde{\beta}) \in \mathcal{C}_\beta$, i.e. there exists $\tilde{\beta} \in \mathrm{Mor}(C_0^u(\mathbb{K}), \mathsf{B})$ such that $\beta = \tilde{\beta} \circ \pi_\mathbb{K}$ if and only if $M_0 \subseteq \gamma(L^\infty(\widehat{\mathbb{K}}))$.*

Proof. Let then $(\pi_\mathbb{K}, \mathbb{K}, \tilde{\beta}) \in \mathcal{C}_\beta$ and pick $\omega \in \mathsf{B}^*$. Using (1.22) and (1.23), we have that

$$(\mathrm{id} \otimes \omega)(\mathrm{id} \otimes \beta)(\mathbb{W}^\mathbb{G}) = (\mathrm{id} \otimes \omega)(\mathrm{id} \otimes \tilde{\beta} \circ \pi_\mathbb{K})(\mathbb{W}^\mathbb{G}) = \gamma\Big((\mathrm{id} \otimes \omega \circ \tilde{\beta})\mathbb{W}^\mathbb{K}\Big)$$

Hence the algebra $\mathcal{M}_0$ constructed for β and the corresponding algebra constructed for $\tilde{\beta}$, seen as subalgebras of $L^\infty(\widehat{\mathbb{G}})$, coincide, hence so do their C^*-envelopes M_0 and the corresponding one for $\tilde{\beta}$. This shows the necessity.

Assume that $M_0 \subseteq L^\infty(\widehat{\mathbb{K}})$. In order to get a morphism $\tilde{\beta} \in \mathrm{Mor}(C_0^u(\mathbb{K}), \mathsf{B})$ it is enough to show that X^* is a representation of $\widehat{\mathbb{K}}$ by Theorem 1.13.

Now observe that Lemma 1.7 shows that $X \in \mathsf{M}(M_0 \otimes \mathsf{B}) \subseteq \mathsf{M}(C_0(\widehat{\mathbb{H}}) \otimes \mathsf{B})$. But from the fact that $\Delta_{\widehat{\mathbb{K}}}$ and $\Delta_{\widehat{\mathbb{G}}} \restriction_{\gamma(L^\infty(\widehat{\mathbb{K}}))}$ coincide and from (2.6), we have that $X_{23}X_{13} = (\Delta_{\widehat{\mathbb{K}}} \otimes \mathrm{id})X$, so X^* satisfies hypothesis of Theorem 1.13. □

Proof of Theorem 2.1. From Lemma 2.10 we get that $\mathbb{H}$ constructed at the end of Section 2.2.1 can be endowed with the morphism $\tilde{\beta}$ completing the desired factorization, i.e. $(\pi, \mathbb{H}, \tilde{\beta}) \in \mathcal{C}_\beta$. Let now $\Bbbk = (\pi_\mathbb{K}, \mathbb{K}, \beta') \in \mathcal{C}_\beta$. From Lemma 2.10 we have that $M_0 \subseteq L^\infty(\widehat{\mathbb{K}})$ and $L^\infty(\widehat{\mathbb{K}})$ is a $R^{\widehat{\mathbb{G}}}$-,$\tau_t^{\widehat{\mathbb{G}}}$- and $\Delta_{\widehat{\mathbb{G}}}$-invariant subalgebra of $L^\infty(\widehat{\mathbb{G}})$. As $L^\infty(\widehat{\mathbb{H}})$ is chosen to be a minimal von Neumann subalgebra with this property, we necessarily have $L^\infty(\widehat{\mathbb{H}}) \subseteq L^\infty(\widehat{\mathbb{K}})$. In particular the inclusion map satisfies the defining property of $\mathbb{H}$ being a Vaes-closed quantum subgroup of $\mathbb{K}$, so we conclude by Theorem 1.23. □

2.2.4 More on $X \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes \mathsf{B})$ and $\theta \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes \mathsf{B})$

In this section we investigate the mutual relation between the objects describing the *embedding* $\mathbb{X} \hookrightarrow \mathbb{G}$ as phrased in Remark 2.5, i.e. the morphism $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}), \mathsf{B})$, the unitary antirepresentation $X \in \mathsf{M}(C_0(\mathbb{G}) \otimes \mathsf{B})$ and the morphism $\theta \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes \mathsf{B})$ in the spirit of [43].

From the discussion in Section 2.2.1 it is clear that out of β one can canonically construct the unitary X, which is an anticorepresentation of $C_0(\widehat{\mathbb{G}})$. But Theorem 1.13 (applied to X^* as in the proof of Lemma 2.10) shows that to a unitary $X \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes \mathsf{B})$ there corresponds a unique morphism $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}), \mathsf{B})$.

Again, from the discussion in Section 2.2.1 it is clear that out of a unitary X, which is a corepresentation of $C_0(\widehat{\mathbb{G}})$ one can uniquely construct the morphism $\theta \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes \mathsf{B})$. Observe that this morphism satisfies the following condition: $(\Delta_\mathbb{G} \otimes \mathrm{id}) \circ \theta = (\mathrm{id} \otimes \theta) \circ \Delta_\mathbb{G}$. Indeed, for $a \in C_0(\mathbb{G})$ we have that

$$(\Delta_\mathbb{G} \otimes \mathrm{id}) \circ \theta(a) = (\Delta_\mathbb{G} \otimes \mathrm{id})(X(a \otimes \mathbb{1})X^*) = \mathrm{W}_{12}X_{13}(a \otimes \mathbb{1} \otimes \mathbb{1})X_{13}^*\mathrm{W}_{12}^* \tag{2.11}$$

Using (2.5) and the fact that $X_{23}(a \otimes \mathbb{1} \otimes \mathbb{1})X_{23}^* = a \otimes \mathbb{1} \otimes \mathbb{1}$ (because X is unitary and X_{23} commutes with $a \otimes \mathbb{1} \otimes \mathbb{1}$), we can continue calculations from (2.11) and get:

$$\begin{aligned} \mathrm{W}_{12}X_{13}(a \otimes \mathbb{1} \otimes \mathbb{1})X_{13}^*\mathrm{W}_{12}^* &= X_{23}\mathrm{W}_{12}X_{23}^*(a \otimes \mathbb{1} \otimes \mathbb{1})X_{23}\mathrm{W}_{12}^*X_{23}^* = \\ &= X_{23}\mathrm{W}_{12}(a \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}_{12}^*X_{23}^* = X_{23}\Big(\mathrm{W}(a \otimes \mathbb{1})\mathrm{W}^*\Big)_{12} X_{23} = \\ &= X_{23}(\Delta_\mathbb{G}(a))X_{23}^* = (\mathrm{id} \otimes \theta) \circ \Delta_\mathbb{G}(a) \end{aligned} \tag{2.12}$$

Proposition 2.11. *Assume $\theta \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes \mathsf{B})$ is such that $(\Delta_{\mathbb{G}} \otimes \mathrm{id}) \circ \theta = (\mathrm{id} \otimes \theta) \circ \Delta_{\mathbb{G}}$. Then there is a unique unitary $X \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes \mathsf{B})$ such that $\theta(a) = X(a \otimes \mathbb{1})X^*$ and X is an antirepresentation of $\widehat{\mathbb{G}}$.*

Proof. The proof is essentially the same as first part of the proof of [43, Theorem 5.3], but we repeat it for sake of completeness.

Denote $\tilde{X} = \mathrm{W}_{12}^* ((\mathrm{id} \otimes \theta)(\mathrm{W})) \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{G}) \otimes \mathsf{B})$. We will show that $\mathrm{W}_{23}\tilde{X}_{124}\mathrm{W}_{23}^* = \tilde{X}_{134}$, then using Theorem 1.14 we conclude that $\tilde{X} \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes \mathbb{C}\mathbb{1} \otimes \mathsf{B})$, so in fact there exists $X \in \mathsf{M}(C_0(\widehat{\mathbb{G}})\mathbb{1} \otimes \mathsf{B})$ with $\tilde{X} = X_{13}$. We compute

$$
\begin{aligned}
\mathrm{W}_{23}\tilde{X}_{124}\mathrm{W}_{23}^* &= \mathrm{W}_{23}\mathrm{W}_{12}^*\mathrm{W}_{23}^*\mathrm{W}_{23}\Big((\mathrm{id} \otimes \theta)\mathrm{W}\Big)_{124}\mathrm{W}_{23}^* = \\
&= \mathrm{W}_{13}^*\mathrm{W}_{12}^*(\mathrm{id} \otimes \Delta_{\mathbb{G}} \otimes \mathrm{id})\Big((\mathrm{id} \otimes \theta)\mathrm{W}\Big) = \\
&= \mathrm{W}_{13}^*\mathrm{W}_{12}^*\Big((\mathrm{id} \otimes \mathrm{id} \otimes \theta)(\mathrm{id} \otimes \Delta_{\mathbb{G}})\mathrm{W}\Big) = \\
&= \mathrm{W}_{13}^*\mathrm{W}_{12}^*\Big((\mathrm{id} \otimes \mathrm{id} \otimes \theta)\mathrm{W}_{12}\mathrm{W}_{23}\Big) = \\
&= \mathrm{W}_{13}^*\Big((\mathrm{id} \otimes \mathrm{id} \otimes \theta)\mathrm{W}_{23}\Big) = \tilde{X}_{134}
\end{aligned}
$$

Let us now check that $\theta(a) = X(a \otimes \mathbb{1})X^*$. In fact we will show that $\theta(a)_{13} = \tilde{X}(a \otimes \mathbb{1} \otimes \mathbb{1})\tilde{X}^*$, together with the fact that $\tilde{X} = X_{13}$ this is enough. We have

$$
\begin{aligned}
\tilde{X}(a \otimes \mathbb{1} \otimes \mathbb{1})\tilde{X}^* &= \mathrm{W}_{12}^*\Big((\mathrm{id} \otimes \theta)\mathrm{W}\Big)(a \otimes \mathbb{1} \otimes \mathbb{1})\Big((\mathrm{id} \otimes \theta)\mathrm{W}^*\Big)\mathrm{W}_{12} = \\
&= \mathrm{W}_{12}^*\Big((\mathrm{id} \otimes \theta)(\mathrm{W}(a \otimes \mathbb{1})\mathrm{W}^*)\Big)\mathrm{W}_{12} = \\
&= \mathrm{W}_{12}^*\Big((\mathrm{id} \otimes \theta)\Delta_{\mathbb{G}}(a)\Big)\mathrm{W}_{12} = \\
&= \mathrm{W}_{12}^*\Big((\Delta_{\mathbb{G}} \otimes \mathrm{id})\theta(a)\Big)\mathrm{W}_{12} = \\
&= \mathrm{W}_{12}^*(\mathrm{W}_{12}\theta(a)_{13}\mathrm{W}_{12}^*)\mathrm{W}_{12} = \theta(a)_{13}
\end{aligned}
$$

Showing that X is a corepresentation amounts to showing that $(\widehat{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})\tilde{X} = \tilde{X}_{234}\tilde{X}_{134}$. Using reduced version of (1.12) we get that

$$
\begin{aligned}
(\widehat{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})\tilde{X} &= \Big(((\widehat{\Delta} \otimes \mathrm{id})\mathrm{W}^*) \otimes \mathbb{1}\Big)(\widehat{\Delta} \otimes \theta)\mathrm{W}) \\
= \mathrm{W}_{13}^*\mathrm{W}_{23}^*((\mathrm{id} \otimes \mathrm{id} \otimes \theta)\mathrm{W}_{23})((\mathrm{id} \otimes \mathrm{id} \otimes \theta)\mathrm{W}_{13}) &= \mathrm{W}_{13}^*\tilde{X}_{234}(\widehat{\Delta} \otimes \theta)\mathrm{W}) = \tilde{X}_{234}\tilde{X}_{134}
\end{aligned}
$$

where in the last equality we used the fact that $\tilde{X}_{234} = X_{24}$ commutes with W_{13}^*. To prove uniqueness, assume $Y \in \mathsf{M}(C_0(\mathbb{G}) \otimes \mathsf{B})$ is another such unitary. Because slices of W are dense in $C_0(\mathbb{G})$, we have that

$$X(a \otimes \mathbb{1})X^* = Y(a \otimes \mathbb{1})Y^*$$

for all $a \in C_0(\mathbb{G})$ is equivalent to saying that

$$X_{23}\mathrm{W}_{12}X_{23}^* = Y_{23}\mathrm{W}_{12}Y_{23}^*$$

Rearranging terms, this is equivalent to

$$\mathrm{W}_{12}^*(Y_{23}^*X_{23})\mathrm{W}_{12} = Y_{23}^*X_{23}$$

which, in turn, is equivalent to

$$(\widehat{\Delta}\otimes\mathrm{id})(Y^*X) = Y_{13}^*X_{13}$$

We conclude from Corollary 1.15 that in fact $X(\mathbb{1}\otimes u) = Y$ for some unitary $u\in\mathsf{M}(\mathsf{B})$. Applying $(\widehat{\Delta}\otimes\mathrm{id})$ to both sides of this equality we get

$$\begin{aligned} X_{23}X_{13}(\mathbb{1}\otimes\mathbb{1}\otimes u) = (\widehat{\Delta}\otimes\mathrm{id})(X(u\otimes\mathbb{1})) &= (\widehat{\Delta}\otimes\mathrm{id})Y \\ &= Y_{23}Y_{13} = X_{23}(\mathbb{1}\otimes\mathbb{1}\otimes u)X_{13}(\mathbb{1}\otimes\mathbb{1}\otimes u) \end{aligned}$$

and hence $u=\mathbb{1}$, which finishes the proof. □

Summarizing, there are three equivalent ways of studying an embedding $\mathbb{X}\hookrightarrow\mathbb{G}$ of a locally compact quantum space into a locally compact quantum group (we recall that $\mathsf{B} = C_0(\mathbb{X})$), these are as follows:

1. the morphism $\beta\in\mathrm{Mor}(C_0^u(\mathbb{G}),\mathsf{B})$;
2. the unitary $X\in\mathsf{M}(C_0(\widehat{\mathbb{G}})\otimes\mathsf{B})$, which is an anticorepresentation of $C_0(\widehat{\mathbb{G}})$ and
3. the morphism $\theta\in\mathrm{Mor}(C_0(\mathbb{G}),C_0(\mathbb{G})\otimes\mathsf{B})$ satisfying

   $$(\Delta_{\mathbb{G}}\otimes\mathrm{id})\circ\theta = (\mathrm{id}\otimes\theta)\circ\Delta_{\mathbb{G}},$$

   which corresponds to the partial action $\mathbb{X}\curvearrowright\mathbb{G}$ by right shifts.

Fixing a non-degenerate representation of $\mathsf{B}\subseteq\mathsf{B}(\mathcal{H})$ and denoting by B'' the WOT-closure of B in the WOT-topology induced by this embedding, we can – similarily as in the case of homomorphisms, study these objects in the von Neumann algebraic context. Indeed, we have $X\in L^\infty(\widehat{\mathbb{G}})\overline{\otimes}\mathsf{B}''\subseteq L^\infty(\widehat{\mathbb{G}})\overline{\otimes}\mathsf{B}(\mathcal{H})$ and $\theta\colon L^\infty(\mathbb{G})\to L^\infty(\mathbb{G})\overline{\otimes}\mathsf{B}''$ satisfying

$$(\Delta_{\mathbb{G}}\otimes\mathrm{id})\circ\theta = (\mathrm{id}\otimes\theta)\circ\Delta_{\mathbb{G}},$$

because θ is obtained by conjugating with a unitary and as such extends to the WOT-closure of $C_0(\mathbb{G})$. We will switch between this viewpoints freely later on.

2.3 Comparison to other notions of generation

2.3.1 Hopf image in the sense of Banica & Bichon

The focus of this section is the study of the relationship between the notion of Hopf image for compact quantum groups in the sense of Theorem 2.1 (i.e. in the revised sense of [49]) and the notion of Hopf image discussed in [6]. Let us recall the main steps of the construction given in [6].

Let us fix a CQG-algebra $\mathrm{Pol}(\mathbb{G})$ and a *-homomorphism $\beta\colon\mathrm{Pol}(\mathbb{G})\to\mathcal{B}$, where $\mathcal{B}$ is some *-algebra (the construction of [6] deals with a more general context of Hopf algebras over a general field). Consider the free monoid F^+ over the alphabet $\mathbb{Z}$, denote the empty word by $\emptyset$ and the length of a word $w\in F^+$ by $\ell(w)\in\mathbb{N}$. To any $g\in F^+$ we associate an algebra $\mathcal{B}^g$ and a morphism $\beta_g\colon\mathrm{Pol}(\mathbb{G})\to\mathcal{B}^g$ in the following manner:

1. $\mathcal{B}^\emptyset = \mathbb{C}$ and $\beta_\emptyset = \varepsilon_{\mathbb{G}}$, the counit of $\mathrm{Pol}(\mathbb{G})$,
2. for $n\in\mathbb{Z}$ we have: $\mathcal{B}^n = \mathcal{B}$ iff n is even and $\mathcal{B}^n = \mathcal{B}^{op}$ iff n is odd; then we define $\beta_n = \beta\circ S_{\mathbb{G}}^n$, where $S_{\mathbb{G}}$ is the antipode of $\mathrm{Pol}(\mathbb{G})$ (recall that for the Hopf *-algebra we have $S^{-1}(x) = S(x^*)^*)$).
3. for $x,y\in F^+$ with $\ell(x),\ell(y)\geq 1$ define $\mathcal{B}^{xy} = \mathcal{B}^x\otimes_{alg}\mathcal{B}^y$ and $\beta_{xy} = \beta_x\otimes\beta_y$.

Then with the notation $I_\beta = \bigcap_{g\in F^+} \ker(\beta_g)$, the CQG-algebra $\mathrm{Pol}(\mathbb{H}) = \mathrm{Pol}(\mathbb{G}) \big/ I_\beta$ is the Hopf image of the *-homomorphism β, i.e. it is the initial object in the category defined analogously to $\mathcal{C}_\beta$ (note that in [6] the terminology was that the minimal factorization was the final object, this is due to sticking to the opposite convention: in our definition the order on $\mathcal{C}_\beta$ is according to quantum groups, whereas in [6] the order is according to Hopf algebras).

In order to compare the notion of Hopf image in the sense of [6] and in our sense, let us pick a *-homomorphism $\overline{\beta}: C^u(\mathbb{G}) \to \mathsf{B}$ and consider the algebraic version of $\overline{\beta}$, which is defined as follows:

$$\beta{:=}\, \overline{\beta}\restriction_{\mathrm{Pol}(\mathbb{G})}: \mathrm{Pol}(\mathbb{G}) \to \overline{\beta}(\mathrm{Pol}(\mathbb{G})) =: \mathcal{B}.$$

Let now $\mathrm{Pol}(\mathbb{H}_1)$ be the Hopf image of β in the sense of [6] (together with $q_1: \mathrm{Pol}(\mathbb{G}) \to \mathrm{Pol}(\mathbb{H}_1)$ and $\tilde{\beta}_1: \mathrm{Pol}(\mathbb{H}_1) \to \mathcal{B}$ such that $\beta = \tilde{\beta}_1 \circ q_1$). Let also $C^u(\mathbb{H}_2)$ be the Hopf image as a locally compact quantum group, i.e. given by Theorem 2.1 (together with $\overline{q}_2: C^u(\mathbb{G}) \to C^u(\mathbb{H}_2)$ and $\overline{\beta}_2: C^u(\mathbb{H}_2) \to \mathsf{B}$ such that $\overline{\beta} = \overline{\beta}_2 \circ \overline{q}_2$). Our aim is to show that $\mathbb{H}_1 = \mathbb{H}_2$, i.e. $C^u(\mathbb{H}_1) = C^u(\mathbb{H}_2)$.

Thanks to Theorem 1.28 we know that the Hopf *-algebraic description and the universal C^*-algebraic description of subgroups coincide, and hence the standard categorical reasoning concerning uniqueness of a minimal object shows that in fact $\mathbb{H}_1 = \mathbb{H}_2$ as compact quantum groups. Moreover, one can see than in fact $\tilde{\beta}_1 = \overline{\beta}_2 \restriction_{\mathrm{Pol}(\mathbb{H}_1)}$. We only sketch the steps of the reasoning.

Consider the map $\tilde{\beta}_1: \mathrm{Pol}(\mathbb{H}_1) \to \mathcal{B} \subseteq \mathsf{B}$. The universal C^*-envelope of $\mathrm{Pol}(\mathbb{H}_1)$ provides an extension of it: $\overline{\beta}_1: C^u(\mathbb{H}_1) \to \mathsf{B}$ such that $\overline{\beta}_1 \restriction_{\mathrm{Pol}(\mathbb{H}_1)} = \tilde{\beta}_1$. Similarily, the map $q_1: Pol(\mathbb{G}) \to Pol(\mathbb{H}_1) \subseteq C^u(\mathbb{H}_1)$ can be extended to the universal C^*-envelope $\overline{q}_1: C^u(\mathbb{G}) \to C^u(\mathbb{H}_1)$ such that $\overline{q}_1 \restriction_{\mathrm{Pol}(\mathbb{G})} = q_1$. Observe that $\overline{\beta} = \overline{\beta}_1 \circ \overline{q}_1$: the equality holds on the dense set $\mathrm{Pol}(\mathbb{G})$ because of the definition of $\overline{q}_1$, the fact that $q_1(\mathrm{Pol}(\mathbb{G})) = \mathrm{Pol}(\mathbb{H})$ and the definition of $\overline{\beta}_1$. Hence the triple $(\overline{q}_1, \mathbb{H}_1, \overline{\beta}_1) \in \mathcal{C}_\beta$, so from the definition of $C^u(\mathbb{H}_2)$ we have that there exists a surjection $\psi: C^u(\mathbb{H}_1) \to C^u(\mathbb{H}_2)$ intertwining the respective coproducts, hence $\mathbb{H}_2 \subset \mathbb{H}_1$

Proceeding similarly on the level of algebraic factorizations, with the role of $\mathbb{H}_1$ and $\mathbb{H}_2$ reversed, uniqueness of minimal factorization established in [6] show that in fact $\mathbb{H}_1 \subset \mathbb{H}_2$. Then we conclude by Theorem 1.28.

2.3.2 Hopf image in the sense of Skalski & Sołtan

This section is devoted to describing the relationship between the Hopf image described in Theorem 2.1 and the notion of Hopf image for compact quantum groups in the sense of [49]. Let us recall the basics of that construction. Let $\beta: C(\mathbb{G}) \to \mathsf{B}$ be a unital *-homomorphism – here $C(\mathbb{G})$ denotes **a** version of the compact quantum group $\mathbb{G}$, non-necessarily universal nor reduced. Denote

$$\beta_n = \underbrace{(\beta \otimes \ldots \otimes \beta)}_{n} \circ \Delta^{(n-1)}: C(\mathbb{G}) \to \mathsf{B}^{\otimes n} \tag{2.13}$$

Sticking to the philosophy saying that β is the Gelfand-dual of an embedding $\mathbb{X} \subset \mathbb{G}$ (as noted earlier, we can replace B with the range of β and it does not affect any step of the construction), we may say that β_n is the Gelfand dual of an embedding $\mathbb{X}^n \subset \mathbb{G}$. Then one defines $I_\beta = \bigcap_{n\geq 1} \ker(\beta_n)$ and shows that I_β is a closed Hopf *-ideal, i.e. $C(\mathbb{G}) \big/ I_\beta$ carries a quantum group structure compatibile with the one of $\mathbb{G}$. By this we mean that, denoting $C(\mathbb{H}) = C(\mathbb{G}) \big/ I_\beta$, the coproduct desceds through the quotient map onto $C(\mathbb{H})$ in such a way that the quotient map intertwines these coproducts. Moreover I_β is a maximal closed Hopf *-ideal contained in $\ker(\beta)$, so $\mathbb{H}$ is a minimal quantum subgroup containing $\mathbb{X}$ (i.e. admitting a factorization as in the definition of the Hopf image): this is simply because $I_\beta \subseteq \ker(\beta)$; call the factorization $\tilde{\beta}_0$. For the details we refer to [49, Theorem 4.1]

The above construction is slightly incompatible with our philosophy. Namely, the philosophy of [49] is that different C^*-algebras correspond to different quantum spaces, so in particular the quantum group corresponding to $C^r(\mathbb{G})$, and the quantum group corresponding to $C^u(\mathbb{G})$, are different, whereas in our approach they are just different C^*-algebras describing the same quantum group.

More specifically, we would like to have a map between the universal C^*-algebras corresponding to $\mathbb{G}$ and $\mathbb{H}$. In the approach of [49] whenever the algebra B is of the form $C(\mathbb{K})$ and the map β intertwines the respective coproducts, the result is just $\mathbb{K}$, without paying attention to the version of the algebra $C(\mathbb{K})$. In particular, the case of the reducing morphism $\Lambda_{\mathbb{G}}: C^u(\mathbb{G}) \to C^r(\mathbb{G})$, in the philosophy of [49] is a good compact quantum groups morphism, whereas in our philosophy its Hopf image should be equal to $(\mathrm{id}, \mathbb{G}, \Lambda_{\mathbb{G}}) \in \mathcal{C}_{\Lambda_{\mathbb{G}}}$.

In order to make the two approaches more compatible, consider the morphism $\beta: C^u(\mathbb{G}) \to \mathsf{B}$ and the algebra $C(\mathbb{H}) = C^u(\mathbb{G}) \big/ I_\beta$ that is the result of the construction of [49]. The quotient map $q: C^u(\mathbb{G}) \to C^u(\mathbb{G}) \big/ I_\beta = C(\mathbb{H})$ can be lifted to a morphism $\varphi: C^u(\mathbb{G}) \to C^u(\mathbb{H})$, that is there exists a map $r: C^u(\mathbb{H}) \to C(\mathbb{H})$ such that $q = r \circ \varphi$, by Proposition 1.20. Observe that the resulting $C^u(\mathbb{H})$ satisfies $\mathbb{h} = (\varphi, \mathbb{H}, \tilde{\beta}_0 \circ r) \in \mathcal{C}_\beta$.

Recall that Theorem 1.24 ensures us that $\mathbb{H}$ is a Vaes-closed quantum subgroup of $\mathbb{G}$ (the map φ shows that $\mathbb{H}$ is Woronowicz-closed quantum subgroup of $\mathbb{G}$). Now that $\mathbb{h} = (\varphi, \mathbb{H}, \tilde{\beta}_0 \circ r) \in \mathcal{C}_\beta$, i.e. $(\tilde{\beta} \circ r) \circ \varphi = \beta$ is obvious, as we have

$$(\tilde{\beta}_0 \circ r) \circ \varphi = \tilde{\beta}_0 \circ (r \circ \varphi) = \tilde{\beta}_0 \circ q = \beta$$

where in the second equality we used that $q = r \circ \varphi$ and in the last equality the defining property of Hopf image in the sense of [49].

We only have to show that the object $\mathbb{h}$ is initial in this category, i.e. coincides with the Hopf image as constructed in Theorem 2.1. To this end, let $\mathbb{h}' = (\varphi', \mathbb{H}', \tilde{\beta}') \in \mathcal{C}_\beta$ be the Hopf image of morphism $\beta: C^u(\mathbb{G}) \to \mathsf{B}$ as a locally compact quantum group (i.e. the one given by Theorem 2.1). We will show that $\mathbb{h} = \mathbb{h}'$. Firstly, from initiality of $\mathbb{h}'$ it follows that there exists a surjection $\psi: C^u(\mathbb{H}) \to C^u(\mathbb{H}')$. On the other hand, the construction of Hopf image in the sense of [49] applied for $\tilde{\beta}': C^u(\mathbb{H}') \to \mathsf{B}$ yields some algebra $C(\mathbb{H}'')$ and a quotient map p; observe that $C(\mathbb{H}'') = C(\mathbb{H})$. Indede, because $p \circ \psi$ enables us to see $C(\mathbb{H}'')$ as a quotient of $C^u(\mathbb{G})$ and we know that the Hopf image in the sense of [49] of the map $\tilde{\beta} \circ r$ is $C(\mathbb{H})$ it follows that there exists a surjection $C(\mathbb{H}'') \to C(\mathbb{H})$ compatible with r and $p \circ \psi$. But using this surjection we get that $C(\mathbb{H})$ is a quotient of $C^u(\mathbb{H}')$, whose minimal factorization was $C(\mathbb{H}'')$, so this surjection is an isomorphism.

Now we see that there exists a surjection $\psi: C^u(\mathbb{H}) \to C^u(\mathbb{H}')$ and a surjection $p: C^u(\mathbb{H}') \to C(\mathbb{H})$, so restricting this maps to $\mathrm{Pol}(\mathbb{H}')$ and $\mathrm{Pol}(\mathbb{H})$ and remembering that $p \circ \psi \restriction_{\mathrm{Pol}(\mathbb{H})} = \mathrm{id}_{\mathrm{Pol}(\mathbb{H})}$ we see that $\mathrm{Pol}(\mathbb{H}') = \mathrm{Pol}(\mathbb{H})$ and hence $C^u(\mathbb{H}') = C^u(\mathbb{H})$ (cf. the proof of Theorem 1.28).

2.3.3 Topological generation á la Brannan, Collins & Vergnioux

The goal of this section is to discuss the notion of *compact quantum group generated by two closed quantum subgroups* in the sense of [15] in the context of Hopf image and to extend it to the non-compact case.

Let then $\mathbb{G}$ be a locally compact quantum group and let $\mathbb{H}_1, \mathbb{H}_2$ be its two Vaes-closed subgroups (denote $\pi_i: C_0^u(\mathbb{G}) \to C_0^u(\mathbb{H}_i)$ for $i = 1, 2$ the corresponding Hopf surjection, by $\gamma_i: L^\infty(\widehat{\mathbb{H}}_i) \to L^\infty(\widehat{\mathbb{G}})$ the corresponding inclusions and by $V^{\mathbb{H}_i} \in L^\infty(\widehat{\mathbb{G}}) \overline{\otimes} L^\infty(\mathbb{H}_i)$ the corresponding bicharacters). Consider the two ideals: $C_0(\mathbb{G} \setminus (\mathbb{H}_1 \cup \mathbb{H}_2)) := \ker(\pi_1) \cap \ker(\pi_2)$ and $C_0(\mathbb{G} \setminus (\mathbb{H}_1 \cdot \mathbb{H}_2)) := \ker((\pi_1 \otimes \pi_2) \circ \Delta^u_{\mathbb{G}})$ and the two quotients

$$q^{\cup}: C_0^u(\mathbb{G}) \to C_0(\mathbb{H}_1 \cup \mathbb{H}_2) = C^u(\mathbb{G}) \big/ C_0(\mathbb{G} \setminus (\mathbb{H}_1 \cup \mathbb{H}_2)) \tag{2.14}$$

and

$$q^{\bullet}: C_0^u(\mathbb{G}) \to C_0(\mathbb{H}_1 \cdot \mathbb{H}_2) = C^u(\mathbb{G}) \big/ C_0(\mathbb{G} \setminus (\mathbb{H}_1 \cdot \mathbb{H}_2)) \tag{2.15}$$

Theorem 2.12. *The following von Neumann subalgebras of $L^\infty(\widehat{\mathbb{G}})$ are equal:*

- $\mathsf{M}_{1,2}$, *the smallest von Neumann algebra containing both $\gamma_1(L^\infty(\widehat{\mathbb{H}}_1))$ and $\gamma_2(L^\infty(\widehat{\mathbb{H}}_2))$;*

- $\mathsf{M}_{\bullet} = \{(\mathrm{id} \otimes \omega \circ q^{\bullet})\mathbb{W} \colon \omega \in C_0(\mathbb{H}_1 \cdot \mathbb{H}_2)^*\}''$;
- $\mathsf{M}_{\cup} = \{(\mathrm{id} \otimes \omega \circ q^{\cup})\mathbb{W} \colon \omega \in C_0(\mathbb{H}_1 \cup \mathbb{H}_2)^*\}''$.
- $\mathsf{M}_{V,1,2} = \{(\mathrm{id} \otimes \omega_1 \otimes \omega_2)V_{12}^{\mathbb{H}_1} V_{13}^{\mathbb{H}_2} : \omega_i \in L^1(\mathbb{H}_i)\}''$

Proof. $\mathsf{M}_{1,2} = \mathsf{M}_{\cup}$. Observe that $\ker(q^{\cup}) = \ker(\pi_1 \oplus \pi_2)$, hence using first part of Lemma 2.4 we may replace $q^{\cup}$ with $\pi_1 \oplus \pi_2$ in the definition of $\mathsf{M}_{\cup}$ (and the functionals are on a different C^*-algebra then). Recall that $(C_0^u(\mathbb{H}_1) \oplus C_0^u(\mathbb{H}_2))^* = (C_0^u(\mathbb{H}_1))^* \oplus (C_0^u(\mathbb{H}_2))^*$, hence every ω appearing in the definition of $\mathsf{M}_{\cup}$ can be written as $\omega = \omega_1 \oplus \omega_2$ for $\omega_i \in (C_0^u(\mathbb{H}_i))^*$. Hence

$$\mathsf{M}_{\cup} = (\gamma_1(L^\infty(\widehat{\mathbb{H}}_1)) + \gamma_2(L^\infty(\widehat{\mathbb{H}}_2)))'' = \mathsf{M}_{1,2}$$

as desired.

$\mathsf{M}_{1,2} = \mathsf{M}_{\bullet}$. Recall that the linear span of the functionals of the form $\omega_1 \otimes \omega_2$ on $\mathsf{A} \otimes \mathsf{B}$ is weak*-dense in $(\mathsf{A} \otimes \mathsf{B})^*$ for any C^*-algebras A, B. Further, as $(\mathrm{id} \otimes q^{\bullet})\mathbb{W} = ((\mathrm{id} \otimes \pi_1)\mathbb{W})_{12}((\mathrm{id} \otimes \pi_2)\mathbb{W})_{13}$, to compute $\mathsf{M}_{\bullet}$ it is enough to elucidate the von Neumann algebra generated by operators $((\mathrm{id} \otimes \omega_1 \circ \pi_1)\mathbb{W})((\mathrm{id} \otimes \omega_2 \circ \pi_2)\mathbb{W})$ thanks to Lemma 1.12. But it is then clear that

$$\mathsf{M}_{\cup} = (\gamma_1(L^\infty(\widehat{\mathbb{H}}_1)) \cdot \gamma_2(L^\infty(\widehat{\mathbb{H}}_2)))'' = \mathsf{M}_{1,2}$$

as desired.

$\mathsf{M}_{V,1,2} = \mathsf{M}_{1,2}$. Similarily as in the previous step, it is immediate to see that

$$\mathsf{M}_{V,1,2} = (\gamma_1(L^\infty(\widehat{\mathbb{H}}_1)) \cdot \gamma_2(L^\infty(\widehat{\mathbb{H}}_2)))'' = \mathsf{M}_{1,2}$$

□

Observe that in the case $\mathbb{G}$ compact, we can apply the procedure of Skalski and Sołtan to $q^{\bullet}$ and $q^{\cup}$. It turns out that the versions of $\mathbb{H}$ given by their procedure for both these maps are the same. Then we have

Proposition 2.13. *The Hopf image of the morphims $q^{\bullet}$ and $q^{\cup}$, in the sense of Skalski and Sołtan, are the same.*

Proof. Recall that one has $\varepsilon_{\mathbb{G}} = \varepsilon_{\mathbb{H}_1} \circ \pi_1 = \varepsilon_{\mathbb{H}_2} \circ \pi_2$. Then it follows that $C_0(\mathbb{G} \setminus (\mathbb{H}_1 \cdot \mathbb{H}_2)) \subset C_0(\mathbb{G} \setminus (\mathbb{H}_1 \cup \mathbb{H}_2))$. Indeed, let $x \in C_0(\mathbb{G} \setminus (\mathbb{H}_1 \cdot \mathbb{H}_2))$, then we compute

$$\begin{aligned}\pi_1(x) = (\mathrm{id} \otimes \varepsilon_{\mathbb{H}_1}) \circ \Delta_{\mathbb{H}_1}(\pi_1(x)) = (\pi_1 \otimes (\varepsilon_{\mathbb{H}_1} \circ \pi_1)) \circ \Delta_{\mathbb{G}}(x) = \\ (\pi_1 \otimes (\varepsilon_{\mathbb{H}_2} \circ \pi_2)) \circ \Delta_{\mathbb{G}}(x) = (\mathrm{id} \otimes \varepsilon_{\mathbb{H}_2})(q^{\bullet}(x)) = 0\end{aligned}$$

And a similar calculation works with π_2 in place of π_1 (then the counit need to appear on the first leg).

On the other hand one has that $\ker(q_2^{\cup}) \subseteq \ker(q^{\bullet})$, where $q_2^{\cup} = (q^{\cup} \otimes q^{\cup}) \circ \Delta_{\mathbb{G}}$ (cf. Section 2.3.2). Indeed, because $\ker(q^{\cup}) \subseteq \ker(\pi_j)$, one can write the factorization $\pi_j = r_j \circ q^{\cup}$, where $r_j \colon C(\mathbb{H}_1 \cup \mathbb{H}_2) \to C^u(\mathbb{H}_j)$. Let $x \in \ker(q_2^{\cup})$, then one has

$$0 = (q^{\cup} \otimes q^{\cup}) \circ \Delta_{\mathbb{G}}(x) \tag{2.16}$$

Applying to both sides of (2.16) the morphism $r_1 \otimes r_2$ we get that

$$0 = (r_1 \circ q^{\cup} \otimes r_2 \circ q^{\cup}) \circ \Delta_{\mathbb{G}}(x) = (\pi_1 \otimes \pi_2) \circ \Delta_{\mathbb{G}}(x)$$

Hence $x \in \ker(q^{\bullet})$, as desired.

Now, proceeding as in Section 2.3.2 we see that the biggest Hopf *-ideal contained in $\ker(q^{\cup})$, call it $I^{\cup}$, and the biggest Hopf *-ideal, contained in $\ker(q^{\bullet})$, coincide: as $I^{\cup} \subseteq \ker(q^{\cup}) \subseteq \ker(q^{\bullet})$, we get that $I^{\cup} \subseteq I^{\bullet}$. On the other hand, as $I^{\bullet} \subseteq \ker(q^{\bullet}) \subseteq \ker(q_2^{\cup})$, we have that $I^{\bullet} \subseteq I^{\cup}$. □

Definition 2.14. We call the Hopf image of either of the maps $q^{\bullet}$ and $q^{\cup}$ the closed quantum subgroup generated by $\mathbb{H}_1$ and $\mathbb{H}_2$ and denote it by $\overline{\langle \mathbb{H}_1, \mathbb{H}_2 \rangle}$.

Now we are ready to discuss the connection of the subgroup generated by two subgroups and the construction of [15] concerning generation. Starting from now, every quantum group appearing in Section 2.3.3 will be compact.

Observe that whenever $U \in \mathsf{M}(\mathsf{K}(\mathcal{H}_U) \otimes C^u(\mathbb{G}))$ is a representation of $\mathbb{G}$, then $U^{\mathbb{H}_i} := (\mathrm{id} \otimes \pi_i)U \in \mathsf{M}(\mathsf{K}(\mathcal{H}_U) \otimes C^u(\mathbb{H}_i))$ is a representation of $\mathbb{H}_i$. Moreover, for the spaces of intertwiners the following inclusion holds:

$$\mathrm{Mor}_{\mathbb{G}}(U, V) \subseteq \mathrm{Mor}_{\mathbb{H}_i}(U^{\mathbb{H}_i}, V^{\mathbb{H}_i}).$$

Let us also denote $\mathrm{Fix}_{\mathbb{G}}(U) = \mathrm{Mor}_{\mathbb{G}}(\mathbb{1}, V)$, the space of fixed points of a $\mathbb{G}$-representation U.

Denote $h_i = h_{\mathbb{H}_i} \circ \pi_i \in C^u(\mathbb{G})^*$ the push-forward of the Haar state and recall that for $\omega, \eta \in C^u(\mathbb{G})^*$ one defines their convolution $\omega \star \eta \in C^u(\mathbb{G})^*$ as $\omega \star \eta = (\omega \otimes \eta) \circ \Delta^u_{\mathbb{G}}$.

As a last ingredient, let us pick some class C of finite-dimensional unitary representations of $\mathbb{G}$ which generate $\mathrm{Rep}(\mathbb{G})$ as tensor category (i.e. every finite dimensional unitary representation of $\mathbb{G}$ is equivalent to a subrepresentation of the tensor product of some members of C).

Definition 2.15 ([15, Definition 4 & Proposition 3.5])**.** We say that $\mathbb{G}$ is topologically generated by $\mathbb{H}_1, \mathbb{H}_2$ if one of the following equivalent conditions holds:

1. $\mathrm{Mor}_{\mathbb{G}}(U, V) = \mathrm{Mor}_{\mathbb{H}_1}(U^{\mathbb{H}_1}, V^{\mathbb{H}_1}) \cap \mathrm{Mor}_{\mathbb{H}_2}(U^{\mathbb{H}_2}, V^{\mathbb{H}_2})$ for every pair of finite dimensional unitary representations U, V of $\mathbb{G}$;
2. $\mathrm{Fix}_{\mathbb{G}}(U) = \mathrm{Fix}_{\mathbb{H}_1}(U^{\mathbb{H}_1}) \cap \mathrm{Fix}_{\mathbb{H}_1}(U^{\mathbb{H}_2})$ for all $U \in C$;
3. For every $a \in \mathrm{Pol}(\mathbb{G})$ we have $h(a) = \lim_{k\to\infty}(h_1 \star h_2)^{\star k}(a)$.

The last condition can be phrased equivalently as follows: let us denote by $p_{\mathbb{G}} = (\mathrm{id} \otimes h_{\mathbb{G}})\mathbb{W}^{\mathbb{G}} \in \ell^\infty(\widehat{\mathbb{G}})$ the Kazhdan projection of $\mathbb{G}$. Then the last condition is precisely

$$p_{\mathbb{G}} = WOT - \lim_{n\to\infty} \left(\gamma_1(p_{\mathbb{H}_1})\gamma_2(p_{\mathbb{H}_2})\right)^n \tag{2.17}$$

where $\gamma_i: \ell^\infty(\widehat{\mathbb{H}}_i) \to \ell^\infty(\widehat{\mathbb{G}})$ are maps coming from Theorem 1.24 identifying $\mathbb{H}_i$ as Vaes-closed quantum subgroups of $\mathbb{G}$. Now let us remark the following:

Theorem 2.16. *The quantum group $\mathbb{G}$ is topologically generated by the quantum subgroups $\mathbb{H}_1$ and $\mathbb{H}_2$ (in the sense of Definition 2.15) if and only if $\mathbb{G} = \overline{\langle \mathbb{H}_1, \mathbb{H}_2 \rangle}$ (in the sense of Definition 2.14).*

Proof. The proof was outlined in [15, Remark 3], we include it for sake of completeness. Recall that Corollary 3.4 (which, in the compact case, is [6, Corollary 8.2]) states that $\beta = (\pi_1 \otimes \pi_2) \circ \Delta$ is generating if and only if $\mathrm{Fix}_{\mathbb{G}}(U) = \mathrm{Fix}(U_\beta)$ for all representations U of $\mathbb{G}$, where $U_\beta = U^{\mathbb{H}_1}_{12} U^{\mathbb{H}_2}_{13}$. We compute

$$\xi \in \mathrm{Fix}(U_\beta) \iff (U^{\mathbb{H}_1}_{12})^*(\xi \otimes \mathbb{1} \otimes \mathbb{1}) = U^{\mathbb{H}_2}_{13}(\xi \otimes \mathbb{1} \otimes \mathbb{1}) \iff U^{\mathbb{H}_1}(\xi \otimes \mathbb{1}) = \xi \otimes \mathbb{1} = U^{\mathbb{H}_2}(\xi \otimes \mathbb{1})$$

where the last step is obtained by observing that as $(\mathrm{id} \otimes \varepsilon_{\mathbb{G}})U = \mathbb{1}$, the belonging $U(\xi \otimes \mathbb{1}) \in \mathcal{H}_U \otimes \mathbb{C}\mathbb{1}$ can happen only if $U(\xi \otimes \mathbb{1}) = \xi \otimes \mathbb{1}$. The above computation is enough to conclude $\mathrm{Fix}(U_\beta) = \mathrm{Fix}(U^{\mathbb{H}_1}) \cap \mathrm{Fix}(U^{\mathbb{H}_2})$. $\square$

2.3.4 Discrete quantum group generation in the sense of Izumi & Vergnioux

In [28, 58, 59] following definition of generation is implicit: let $\mathbb{G}$ be a discrete quantum group and let $\mathcal{D} \subseteq Irr(\widehat{\mathbb{G}})$. We say that $\mathcal{D}$ generates $\mathbb{G}$ if any element $v \in Irr(\widehat{\mathbb{G}})$ is a subrepresentation of a tensor product of some members of $\mathcal{D}$ or their contragredients, i.e. $v \subset d_1^{\epsilon_1} \mathbin{\bigcirc\mkern-15mu\top} \ldots \mathbin{\bigcirc\mkern-15mu\top} d_n^{\epsilon_n}$ for some choice of (possibly repeating) elements $d_1, \ldots, d_n \in \mathcal{D}$ and "signs" $\epsilon_j \in \{\ , -\}$ (i.e. given $d \in \mathcal{D}$ and $\epsilon \in \{\ , -\}$, the representation d^ϵ is one of the following two: $d, \bar{d}$, the latter possibility occurs precisely when $\epsilon = -$). The way to connect it to our notion of Hopf image is the following: for any $d \in \mathcal{D} \subseteq Irr(\widehat{\mathbb{G}})$ let $\pi^d: c_0(\mathbb{G}) \to M_{\dim d}(\mathbb{C})$ be *-homomorphism associated to a representative $u^d \in M_{\dim d}(C(\widehat{\mathbb{G}}))$ by means of Theorem 1.13 (as any irreducible representation of $\widehat{\mathbb{G}}$ is finite dimensional, these expressions makes sense). Consider the C^*-algebra $\mathsf{D} = \oplus_{d\in\mathcal{D}} M_{\dim d}(\mathbb{C})$ (the C_0-direct sum) and the morphism $\pi^{\mathcal{D}} = \oplus_{d\in\mathcal{D}}\pi^d \in \mathrm{Mor}(c_0(\mathbb{G}), \mathsf{D})$.

Similarily, given a morphism $\phi \in \mathrm{Mor}(c_0(\mathbb{G}), \mathsf{B})$, we know from general theory of compact quantum groups that it decomposes into a sum of irreducible components (with some multiplicities). Let us then write $\phi = \oplus_{s\in\mathcal{S}} m_s\phi_s$, where $\mathcal{S} \subseteq Irr(\widehat{\mathbb{G}})$ is the support of ϕ, ϕ_s is the standard realization of $s \in Irr(\mathbb{G})$ and $m_s \in \{1, \dots, \infty\}$ is its multiplicity, Then we have that

Proposition 2.17. *1. The family $\mathcal{D}$ generate $\mathbb{G}$ in the sense of Izumi & Vergnioux if and only if the morphism $\pi^{\mathcal{D}}$ is generating.*

2. The morphism ϕ is generating if and only if its support $\mathcal{S}$ generate $\mathbb{G}$ in the sense of Izumi & Vergnioux.

Proof. (1) It is clear that we can assume $\mathcal{D} = \overline{\mathcal{D}}$, as this amounts to picking $\mathcal{D}_1 = \mathcal{D} \cup \overline{\mathcal{D}}$. Indeed, saying that $v \subset d_1^{\epsilon_1} \mathbin{\text{\textcircled{T}}} \dots \mathbin{\text{\textcircled{T}}} d_n^{\epsilon_n}$ for $d_j \in \mathcal{D}$ with appropriate signs is equivalent to saying that $v \subset d_1' \mathbin{\text{\textcircled{T}}} \dots \mathbin{\text{\textcircled{T}}} d_N'$ for some $d_j' \in \mathcal{D}_1$. Observe that in this situation we have that $X = \oplus_{d\in\mathcal{D}} u^d$ and then $\mathcal{M}_0 = \{(\mathrm{id} \otimes \omega)X : \omega \in \mathsf{D}^*\}$ is a *-closed subset of $L^\infty(\mathbb{G})$. This is because $(u^d)^* = (u^{\overline{d}})^\top$, hence picking $\omega \in \mathsf{D}^*$ and denoting $\omega^*(x) = \overline{\omega(x^*)}$, we can compute:

$$[(\mathrm{id} \otimes \omega)X]^* = (\mathrm{id} \otimes \omega^*)(X^*) = (\mathrm{id} \otimes \omega^*)[(\oplus_{d\in\mathcal{D}} u^d)^*] = (\mathrm{id} \otimes \omega^*)[\oplus_{d\in\mathcal{D}} (u^d)^\top] = (\mathrm{id} \otimes \tilde{\omega})(X)$$

where $(\tilde{\omega}(u^d_{i,j}) = \omega^*(u^d_{j,i})$. Let us now turn to better description of the algebra $\mathsf{M}_1 = (\mathcal{M}_0)''$ in the spirit of Lemma 2.4. To this end, let $\mathcal{H}_d = \mathbb{C}^{\dim d}$ be the carrier Hilbert space of a representation $d \in \mathcal{D}$ and let

$$\widetilde{\mathcal{M}_0} = \{(\mathrm{id} \otimes \omega_{\xi,\eta} \circ \pi^d)X : \xi, \eta \in \mathcal{H}_d \text{ for some } d \in \mathcal{D}\}$$

where $\omega_{\xi,\eta} = \langle \xi | \cdot | \eta \rangle$. Then obviously $\mathrm{span}_{\mathbb{C}} \widetilde{\mathcal{M}_0} \subseteq \mathrm{Pol}(\widehat{\mathbb{G}})$ is a *-closed subcoalgebra. Observe furthermore that $(\widetilde{\mathcal{M}_0})'' = (\mathcal{M}_0)''$ by w^*-density of the span of the vector functionals, precisely as in the proof of Lemma 2.4. As a consequence, $\pi^{\mathcal{D}}$ is generating if and only if $\widetilde{\mathcal{M}} = alg(\widetilde{\mathcal{M}_0})$ is WOT-dense Hopf *-algebra. By uniqueness of a dense Hopf *-algebra in $C(\widehat{\mathbb{G}})$ we conclude that in $\pi^{\mathcal{D}}$ is generating if and only if $\widetilde{\mathcal{M}} = \mathrm{Pol}(\widehat{\mathbb{G}})$.

On the other hand, the way an algebra is generated out of a vector subspace can be described in steps. Namely, $\widetilde{\mathcal{M}} = \mathrm{span}_{\mathbb{C}} \bigcup_{n\geq 1} (\widetilde{\mathcal{M}_0})^{\cdot n}$, (see (1.3)). Thus given any $v \in Irr(\widehat{\mathbb{G}})$ and picking a realization $u^v \in M_{\dim v}(C(\widehat{\mathbb{G}}))$ we get that for a ON basis $(e_j)_{j=1}^{\dim v}$, we know that the coefficients satisfy the belonging $u^v_{i,j} = (\omega_{e_i,e_j} \otimes)u^v \in \mathrm{Pol}(\mathbb{G})$.

Assume that $\pi^{\mathcal{D}}$ is generating. Then $u^v_{i,j} \in (\widetilde{\mathcal{M}_0})^N$ for sufficiently large N, as it has only finitely many coefficients. Therefore denoting $\pi^v: c_0(\mathbb{G}) \to M_{\dim v}(\mathbb{C})$ the realization of u^v by means of Theorem 1.13, we get that

$$\pi^v \prec \bigoplus_{n=1}^{N} \bigoplus_{d_1,\dots,d_n\in\mathcal{D}} \pi^{d_1 \mathbin{\text{\textcircled{T}}} \dots \mathbin{\text{\textcircled{T}}} d_n}$$

But as v is irreducible, by Theorem 1.5 this is equivalent to saying that

$$\pi^v \subset \bigoplus_{n=1}^{N} \bigoplus_{d_1,\dots,d_n\in\mathcal{D}} \pi^{d_1 \mathbin{\text{\textcircled{T}}} \dots \mathbin{\text{\textcircled{T}}} d_n}$$

Once again, as v is irreducible, being contained in a direct sum of a family of representations is equivalent to being contained in one of the summands: for some $1 \leq k \leq N$ and for some $d_1, \dots, d_k \in \mathcal{D}$ we have that

$$\pi^v \subset \pi^{d_1 \mathbin{\text{\textcircled{T}}} \dots \mathbin{\text{\textcircled{T}}} d_k}$$

as desired.

The converse amounts to reading the above reasoning backwards. The step using Theorem 1.5 can be omitted, as $\pi \subset \rho$ implies $\pi \prec \rho$ without any assumptions.
(2) Once noted that the Hopf images of the maps $\phi = \oplus_{s\in\mathcal{S}} m_s\phi_s$ and $\pi^{\mathcal{S}} = \oplus_{s\in\mathcal{S}}\phi_s$ are the same, one concludes by evoking part (1). $\square$

As a corollary, we can rephrase finite generation in the context of discrete quantum groups.

Definition 2.18. Let $\mathbb{G}$ be a discrete quantum group. We say that $\mathbb{G}$ is finitely generated if there exists a finite dimensional C^*-algebra B and a morphism $\beta \in \operatorname{Mor}(c_0(\mathbb{G}), \mathsf{B})$, which is generating.

Recall that a discrete quantum group $\mathbb{G}$ is finitely generated in the sense of Izumi & Vergnioux (see [59], implicit in [28, Section 3]) if there exists a finite subset $\mathcal{D} = \overline{\mathcal{D}} \subseteq Irr(\widehat{\mathbb{G}})$ such that $\mathbb{1}_{Irr(\widehat{\mathbb{G}})} \notin \mathcal{D}$ and any element of $Irr(\widehat{\mathbb{G}})$ is contained in a multiple tensor product of elements of $\mathcal{D}$. We have the following

Theorem 2.19. *Let $\mathbb{G}$ be a discrete quantum group. Then the following are equivalent:*

1. *$\mathbb{G}$ is finitely generated (in the sense of Definition 2.18),*
2. *$\mathbb{G}$ is finitely generated in the sense of Izumi & Vergnioux and*
3. *$\widehat{\mathbb{G}}$ is a compact matrix quantum group.*

Proof. The equivalence of first two items follows from Proposition 2.17: (2) $\implies$ (1) is precisely one of the implications of Proposition 2.17. (1) $\implies$ (2) is the other implication of Proposition 2.17 after we observe that as $\beta\colon c_0(\mathbb{G}) = C^*(\widehat{\mathbb{G}}) \to \mathsf{B}$ is a finite-dimensional representation of $\widehat{\mathbb{G}}$, it decomposes into a sum of irreducibles: $\beta = \oplus_{d\in\mathcal{D}} k_d\pi^d$ (where k_d denotes the multiplicity of $d \in \mathcal{D}$). As B is finite-dimensional, the set $\mathcal{D}$ is finite, hence $\mathcal{D}_1 = (\mathcal{D} \cup \overline{\mathcal{D}}) \setminus \{\mathbb{1}_{Irr(\widehat{\mathbb{G}})}\}$ is a generating set appearing in the definition of finite generation in the sense of Izumi & Vergnioux.

If $\widehat{\mathbb{G}}$ is a compact matrix quantum group and $u \in M_n(C(\widehat{\mathbb{G}}))$ is a fundamental corepresentation, then any element of $Irr(\widehat{\mathbb{G}})$ appears as a subrepresentation of sufficiently large tensor product of u. Hence decomposing u into sum of irreducibles and proceeding as in (1) $\implies$ (2) we obtain a generating set in the sense of Izumi & Vergnioux. Conversly, if $\mathcal{D}$ is a generating set in the sense of Vergnioux, then $u^{\oplus\mathcal{D}} = \oplus_{d\in\mathcal{D}} u^d$ is a fundamental corepresentation. $\square$

Remark 2.20. The equivalence (2) $\iff$ (3) in the above Theorem was a folklore among the quantum groups community. We wanted to state it, so that the connection to our definition of finite generation was made explicit. The finite generation as phrased in Definition 2.18 has a topological flavour, not visible in the condition (3), whereas condition (2) could be thought of as being to strong from the topological perspective.

Chapter 3

Properties and examples

3.1 Introduction

This chapter is devoted to explaining properties of the construction of the Hopf image mimicking the properties of a subgroup generated by a subset in the classical case. We state the formulation of the results in the first part and present the proofs in the second part. In the last part of this chapter we also discuss some examples. They are of three types: firstly, we comment on the case of classical group. Then we summarize examples implicit in Chapter 2. We end with reformulating the key construction of [15] in the language of Hopf image.

Recall that if the Hopf image of a morphism $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}), \mathsf{B})$ is the whole $\mathbb{G}$, then we call β *generating*. We also occasionally write $\mathsf{B} = C_0(\mathbb{X})$.

Theorem 3.1 (Separation of homomorphisms)**.** *Assume β is generating. Consider two quantum group homomorphisms $\mathbb{G} \to \mathbb{K}$ described by $\varphi, \tilde{\varphi} \in \mathrm{Mor}(C_0^u(\mathbb{K}), C_0^u(\mathbb{G}))$. If their restrictions to $\mathbb{X}$ coincides, i.e. $\beta\circ\varphi = \beta\circ\tilde{\varphi}$, then the homomorphisms coincide on the whole of $\mathbb{G}$: $\varphi = \tilde{\varphi}$. If $\mathbb{G}$ is discrete, the converse also holds: if β is not generating, then there exist a quantum group $\mathbb{K}$ and two different quantum group homomorphisms $\varphi_1, \varphi_2 \in \mathrm{Mor}(C_0^u(\mathbb{K}), C_0^u(\mathbb{G}))$ such that $\varphi_1 \restriction_{\mathbb{X}} = \varphi_2 \restriction_{\mathbb{X}}$.*

We have another characterization of generating morphisms in terms of the partial action θ, described in Section 2.2.4, and the map called β-restriction, described in Section 3.2.2.

Theorem 3.2. *If $\mathbb{G}$ is Kac, compact or discrete, then the following statements are equivalent:*

1. $\{(\mathrm{id}\otimes\omega\circ\beta)\mathbb{W} : \omega \in \mathsf{B}^*\}'' = L^\infty(\widehat{\mathbb{G}})$
2. *If $x \in C_0(\mathbb{G})$ satisfies $\theta(x) = x \otimes \mathbb{1}$, then $x \in \mathbb{C}\mathbb{1}$.*
3. *The map $\mathrm{Rep}(\mathbb{G}) \ni U \mapsto U^\beta$ is injective.*
4. *The morphism β is generating.*

The the β-restriction map, informally speaking, is responsible for restricting the family of unitaries indexed by $\mathbb{G}$, describing a representation of $\mathbb{G}$, to the quantum space $\mathbb{X}$. Let us use the following notation: whenever $\pi \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$ is a homomorphism of quantum groups, we denote by $\pi^r = \Lambda_{\mathbb{H}}\circ\pi \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0(\mathbb{H}))$ the reduction of this morphism.

Theorem 3.3. *Let $\mathbb{H}_1, \mathbb{H}_2 \subset \mathbb{G}$ be Vaes-closed quantum subgroups of a locally compact quantum group $\mathbb{G}$ identified via $\pi_i \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$. Denoting by $U^{\pi_i^r} \in \mathsf{B}(\mathcal{H}_U)\overline{\otimes}L^\infty(\mathbb{H}_i)$ the restriction of a corepresentation $U \in \mathsf{B}(\mathcal{H}_U)\overline{\otimes}L^\infty(\mathbb{G})$ to the subgroup $\mathbb{H}_i$ for $i = 1, 2$ we have that the following conditions are equivalent:*

1. $\mathbb{G} = \overline{\langle \mathbb{H}_1, \mathbb{H}_2 \rangle}$ *(in the sense of Section 2.3.3);*

2. *for all corepresentations $U \in \mathsf{B}(\mathcal{H}_U)\overline{\otimes}L^\infty(\mathbb{G})$ of $\mathbb{G}$ we have that*

$$\{(\mathrm{id}\otimes\omega)(U) : \omega \in L^1(\mathbb{G})\}'' = \{(\mathrm{id}\otimes\omega_1\otimes\omega_2)(U_{12}^{\pi_1^r}U_{23}^{\pi_2^r}) : \omega_i \in L^1(\mathbb{H}_i)\}''; \tag{3.1}$$

3. *for the right regular corepresentation $\mathrm{W} \in \mathsf{B}(L^2(\mathbb{G}))\overline{\otimes}L^\infty(\mathbb{G})$ we have that*

$$\{(\mathrm{id}\otimes\omega)(\mathrm{W}) : \omega \in L^1(\mathbb{G})\}'' = \{(\mathrm{id}\otimes\omega_1\otimes\omega_2)(\mathrm{W}_{12}^{\pi_1^r}\mathrm{W}_{23}^{\pi_2^r}) : \omega_i \in L^1(\mathbb{H}_i)\}''. \tag{3.2}$$

Recall that for two corepresentations $U \in \mathsf{B}(\mathcal{H}_U)\overline{\otimes}L^\infty(\mathbb{G})$ and $\tilde{U} \in \mathsf{B}(\mathcal{H}_{\tilde{U}})\overline{\otimes}L^\infty(\mathbb{G})$ of a locally compact quantum group $\mathbb{G}$ we denote by $\mathrm{Mor}_{\mathbb{G}}(U,\tilde{U}) = \{T \in \mathsf{B}(\mathcal{H}_U,\mathcal{H}_{\tilde{U}}) : (T\otimes\mathbb{1})U = \tilde{U}(T\otimes\mathbb{1})\}$ the set of intertwiners between U and $\tilde{U}$. Then one obviously has $\mathrm{Mor}_{\mathbb{G}}(U,\tilde{U}) \subseteq \mathrm{Mor}_{\mathbb{H}}(U^{\pi^r},\tilde{U}^{\pi^r})$ for every closed quantum subgroup $\mathbb{H} \subset \mathbb{G}$.

Corollary 3.4. *We have that $\mathbb{G} = \overline{\langle\mathbb{H}_1,\mathbb{H}_2\rangle}$ (in the sense of Section 2.3.3) if and only if for all corepresentations $U \in \mathsf{B}(\mathcal{H}_U)\overline{\otimes}L^\infty(\mathbb{G})$ and $\tilde{U} \in \mathsf{B}(\mathcal{H}_{\tilde{U}})\overline{\otimes}L^\infty(\mathbb{G})$ we have that*

$$\mathrm{Mor}_{\mathbb{G}}(U,\tilde{U}) = \mathrm{Mor}_{\mathbb{H}_1}(U^{\pi_1^r},\tilde{U}^{\pi_1^r}) \cap \mathrm{Mor}_{\mathbb{H}_2}(U^{\pi_2^r},\tilde{U}^{\pi_2^r}) \tag{3.3}$$

Let us remark that this result is an extension of [6, Corollary 8.2] to the non-compact setting.

Theorem 3.5. *Assume $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}),\mathsf{B})$ is such that B is commutative.*

1. *The Hopf image of β is a classical group. In particular, $\mathbb{G}$ is classical if β is generating.*
2. *Assume B is finite dimensional and $\mathbb{G}$ is discrete. The free group on $\dim\mathsf{B}$ generators has the universal property in the category of discrete quantum groups. In other words, free group is free in the category of discrete quantum groups, not only in the category of discrete groups.*

By the universal property of the free group we mean the fact that any map from a set $\{1,\dots,n\} \to G$ extends uniquely to a homomorphism $\mathbb{F}_n \to G$, where G is a group. Theorem 3.5 says that G can be replaced with a discrete quantum group $\mathbb{G}$.

Now the content of the consecutive Theorem is best seen in the follwing diagram:

$$\begin{tikzcd}
C_0^u(\mathbb{G}) \arrow[rr, two heads, "\beta"] \arrow[dr, "\pi"] \arrow[ddd, two heads, "q_{\mathbb{G}}"'] & & \mathsf{B} \arrow[ddd, two heads, "q_{\mathsf{B}}"] \\
& C_0^u(\mathbb{H}) \arrow[ur, two heads, "\tilde{\beta}"] \arrow[d, two heads, "q_{\mathbb{H}}"] & \\
& C_0(Gr(\widehat{\mathbb{H}})) \arrow[dr, "\tilde{b}"] & \\
C_0(Gr(\widehat{\mathbb{G}})) \arrow[ur, "p"] \arrow[rr, two heads, "b"] & & C_0(\sigma(\mathsf{B}))
\end{tikzcd}$$

Diagram 2: Diagram of Theorem 3.6

Here $q_{\mathbb{G}}\colon C_0^u(\mathbb{G}) \to C_0(Gr(\widehat{\mathbb{G}}))$ is the canonical embedding of the group of characters $Gr(\widehat{\mathbb{G}}) \subset \mathbb{G}$ (likewise for $\mathbb{H}$); $C_0(\sigma(\mathsf{B}))$ is the C^*-algebra obtained by quotiening out the commutator ideal and $\sigma(\mathsf{B})$ denotes the spectrum of this commutative C^*-algebra, q_{B} denotes this particular quotient map. Now p is obtained as follows: as $q_{\mathbb{H}}\circ\pi$ has commutative target, it factors through $C(Gr(\widehat{\mathbb{G}}))$ and $p\circ q_{\mathbb{G}} = q_{\mathbb{H}}\circ\pi$. Similarily, we obtain b as the map completing the factorization of $q_{\mathsf{B}}\circ\beta$ through $q_{\mathbb{G}}$ and $\tilde{b}$ completes the factorization of $q_{\mathsf{B}}\circ\tilde{\beta}$ through $q_{\mathbb{H}}$.

Theorem 3.6. *If $\mathbb{H}$ is the Hopf image of the map β, then the Hopf image of b contans $Gr(\widehat{\mathbb{H}})$. In other words, the Gelfand dual $\hat{b}\colon\sigma(\mathsf{B}) \to Gr(\widehat{\mathbb{G}})$ satisfies $\overline{\langle\hat{b}[\sigma(\mathsf{B})]\rangle} \supseteq Gr(\widehat{\mathbb{H}})$.*

The converse of this Theorem amounts to showing that if the map β is generating, then its abelianized version is generating as well. However, it is not valid in full generality, as it is easily seen in the following example. Putting to Diagram 2 $C_0^u(\mathbb{G}) = C^*_{max}(\mathbb{F}_2)$, the full group C^*-algebra of the free group $\mathbb{F}_2$, $\mathsf{B} = C^*_r(\mathbb{F}_2)$, the reduced group C^*-algebra of this group and $\beta = \Lambda$, the reduction morphism, we obtain that $C_0(Gr(\mathbb{F}_2)) = C^*(\mathbb{Z}^2) = C(\mathbb{T}^2)$, whereas $\sigma(\mathsf{B}) = \varnothing$ (that is, $\mathbb{F}_2$ is C^*-simple, this was first observed by R. Powers in [46]). In fact, even more can be shown. Let say that the quantum group has property (FAG), standing for *faithful abelianized generation*, if the following hold:

(FAG). Assume that $\mathbb{G}$ is a compact quantum group that is generated by its two subgroups $\mathbb{H}_1, \mathbb{H}_2$. Then its group of character $Gr(\widehat{\mathbb{G}})$ is generated by the respective groups of characters $Gr(\widehat{\mathbb{H}}_1)$ and $Gr(\widehat{\mathbb{H}}_2)$.

Our motivation for introducting Property (FAG) were Theorem 3.17 and Theorem 3.18. These conclusions were a tool that we wanted to use to give a more thorough description of the lattice of subgroups of some known examples of quantum groups. However, it seems that it is very rare to have this property and in particular, we show in Chapter 4, among others, that S_4^+ fail to posses it. Unfortunately, we were unable to find any simpler example of quantum group without these property. It is also hard to provide a non-trivial example of a quantum group with these property. Often duals of finite groups satisfy (FAG) due to the following phenomenon: its quantum subgroups are either to small to generate it, or already contain all of the characters of the original group (this is the case e.g. if $\mathbb{G} = \widehat{S_n}$, because then $Gr(S_4) = \mathbb{Z}_2$, or for $\mathbb{G} = \widehat{SL_2(\mathbb{F}_3)}$, when $Gr(\widehat{\mathbb{G}}) = \mathbb{Z}_3$ is contained in both nontrivial proper subgroups of $\widehat{SL_2(\mathbb{F}_3)}$).

3.2 The results

3.2.1 Separation of homomorphisms

Let now $\mathbb{G}, \mathbb{K}$ be locally compact quantum groups and let $\varphi \in \mathrm{Mor}(C_0^u(\mathbb{K}), C_0^u(\mathbb{G}))$ be a quantum group homomorphism $\mathbb{G} \to \mathbb{K}$ with corresponding bicharacter denoted by $V \in \mathsf{M}(C_0(\widehat{\mathbb{K}}) \otimes C_0(\mathbb{G}))$, it is given by $V = (\mathrm{id} \otimes \Lambda_{\mathbb{G}} \circ \varphi) \mathcal{W}^{\mathbb{K}}$. Let B be a C^*-algebra, $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}), \mathsf{B})$, the corresponding unitary $X \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes \mathsf{B})$ is given by $V = (\mathrm{id} \otimes \beta) \mathcal{W}^{\mathbb{G}}$. Let us describe in detail the unitary corresponding to $\beta \circ \varphi$. It is given by $Y = (\mathrm{id} \otimes \beta \circ \varphi) \mathcal{W}^{\mathbb{K}}$.

Lemma 3.7. *The unitaries X, Y, V obey the following equation:*

$$Y_{13} = V_{12}^* X_{23} V_{12} X_{23}^* \in \mathsf{M}(C_0(\widehat{\mathbb{K}}) \otimes \mathsf{K}(L^2(\mathbb{G})) \otimes \mathsf{B})$$

Proof. Let $\widehat{\varphi} \in \mathrm{Mor}(C_0^u(\widehat{\mathbb{G}}), C_0^u(\widehat{\mathbb{K}}))$ be the Hopf morphism describing the homomorphism dual to $\mathbb{G} \to \mathbb{K}$ (described by φ), recall they are linked by (1.21). Application of $\Lambda_{\widehat{\mathbb{K}}} \otimes \Lambda_{\mathbb{G}}$ to both sides of (1.21) yields

$$V = (\mathrm{id} \otimes \Lambda_{\mathbb{G}} \circ \varphi) \mathcal{W}^{\mathbb{K}} = (\Lambda_{\widehat{\mathbb{K}}} \circ \widehat{\varphi} \otimes id) \mathbb{W}^{\mathbb{G}}$$

Application of $\Lambda_{\widehat{\mathbb{K}}} \circ \widehat{\varphi} \otimes \mathrm{id} \otimes \beta$ to both sides of (1.14) gives:

$$(LHS) = (\Lambda_{\widehat{\mathbb{K}}} \circ \widehat{\varphi} \otimes \mathrm{id} \otimes \beta) \mathbb{W}_{13}^{\mathbb{G}} = (\mathrm{id} \otimes \mathrm{id} \otimes \beta) \Big(\Lambda_{\widehat{\mathbb{K}}} \circ \widehat{\varphi} \otimes \mathrm{id} \otimes \mathrm{id}) \mathbb{W}_{13}^{\mathbb{G}} \Big) =$$
$$= (\mathrm{id} \otimes \mathrm{id} \otimes \beta) \Big(\Lambda_{\widehat{\mathbb{K}}} \otimes \mathrm{id} \otimes \varphi) \mathbb{W}_{13}^{\mathbb{G}} \Big) = \Big((\mathrm{id} \otimes \beta \circ \varphi) \mathcal{W}^{\mathbb{G}} \Big)_{13} = Y_{13}$$

and

$$(RHS) = (\Lambda_{\widehat{\mathbb{K}}} \circ \widehat{\varphi} \otimes \mathrm{id} \otimes \beta)\Big((\mathbb{W}^{\mathbb{G}}_{12})^* \mathcal{W}^{\mathbb{G}}_{23} \mathbb{W}^{\mathbb{G}}_{12} (\mathcal{W}^{\mathbb{G}}_{23})^* \Big) =$$

$$= \Big((\Lambda_{\widehat{\mathbb{K}}} \circ \widehat{\varphi} \otimes \mathrm{id}) \mathbb{W}^{\mathbb{G}} \Big)^*_{12} \Big((\mathrm{id} \otimes \beta) \mathcal{W}^{\mathbb{G}} \Big)_{23} \Big((\Lambda_{\widehat{\mathbb{K}}} \circ \widehat{\varphi} \otimes \mathrm{id}) \mathbb{W}^{\mathbb{G}} \Big)_{12} \Big((\mathrm{id} \otimes \beta) \mathcal{W}^{\mathbb{G}} \Big)^*_{23} =$$

$$= \Big((\mathrm{id} \otimes \Lambda_{\mathbb{G}} \circ \varphi)(\mathcal{W}^{\mathbb{K}})^* \Big)_{12} X_{23} \Big((\mathrm{id} \otimes \Lambda_{\mathbb{G}} \circ \varphi)(\mathcal{W}^{\mathbb{K}}) \Big)_{12} X^*_{23} = V^*_{12} X_{23} V_{12} X^*_{23}$$

□

We are now ready to give the proof of one of the implications of Theorem 3.1. Let us settle the notation: we consider two quantum group homomorphisms $\mathbb{G} \to \mathbb{K}$. These are described by: Hopf morphisms $\varphi, \tilde{\varphi} \in \mathrm{Mor}(C^u_0(\mathbb{K}), C^u_0(\mathbb{G}))$, bicharacters $V, \tilde{V} \in \mathsf{M}(C_0(\widehat{\mathbb{K}}) \otimes C_0(\mathbb{G}))$ and right quantum group homomorphisms $\rho, \tilde{\rho} \in \mathrm{Mor}(C_0(\mathbb{K}), C_0(\mathbb{G}) \otimes C_0(\mathbb{K}))$ (the objects without tilde describe one quantum group homomorphism and the objects with tilde describe the second quantum group homomorphism).

Theorem 3.8. *With the above notation, if β is generating and $\beta \circ \varphi = \beta \circ \tilde{\varphi}$, then $\varphi = \tilde{\varphi}$. In other words, a homomorphism $\mathbb{G} \to \mathbb{K}$ is determined uniquely by its values on the generating set.*

Proof. If $\beta \circ \varphi = \beta \circ \tilde{\varphi}$, then the corresponding unitaries coincide: $Y = \tilde{Y} \in \mathsf{M}(C_0(\widehat{\mathbb{K}}) \otimes \mathsf{B})$. Using Lemma 3.7 one may rewrite this as

$$\tilde{V}^*_{12} X_{23} \tilde{V}_{12} X^*_{23} = V^*_{12} X_{23} V_{12} X^*_{23}$$

or equivalently

$$\tilde{V}^*_{12} X_{23} \tilde{V}_{12} = V^*_{12} X_{23} V_{12}.$$

By slicing with $\omega \in \mathsf{B}^*$ on the third leg we see that the right quantum group homomorphisms ρ and $\tilde{\rho}$ agree on the M_1 (they are normal *-homomorphisms). But applying $(\tau^{\widehat{\mathbb{K}}}_t \otimes \tau^{\mathbb{G}}_t \otimes \mathrm{id})$ and remembering that $(\tau^{\widehat{\mathbb{K}}}_t \otimes \tau^{\mathbb{G}})(V) = V$ (and likewise for $\tilde{V}$) we see that ρ and $\tilde{\rho}$ have the same values on $\widehat{\tau}_t(\mathsf{M}_1)$ and once again, by normality of $\rho, \tilde{\rho}$, they coincide on $(\bigcup_{t \in \mathbb{R}} \widehat{\tau}_t(\mathsf{M}_1))'' = \mathsf{M}_{BV}$ (by Theorem 2.9). By assumption, $\mathsf{M}_{BV} = L^\infty(\widehat{\mathbb{G}})$, hence $\rho = \tilde{\rho}$. This ends the proof, as ρ and $\tilde{\rho}$ determines the homomorphisms $\mathbb{G} \to \mathbb{K}$ uniquely (see Section 1.3.3).

□

In order to give the converse implication of Theorem 3.1 in case $\mathbb{G}$ is a discrete quantum group, let us fix some notation. Firstly, let $\beta \in \mathrm{Mor}(c_0(\mathbb{G}), \mathsf{B})$ be a morphism with Hopf image $(\pi, \mathbb{H}, \tilde{\beta})$ such that $\mathbb{H} \subsetneq \mathbb{G}$. Denote by $\mathbb{K} = \mathbb{G} *_{\mathbb{H}} \mathbb{G}$. Using [60, Theorem 3.4 & Corollary 3.5], one can describe $\mathbb{K}$ as follows.

Consider first the free product $\mathbb{K}' = \mathbb{G} * \mathbb{G}$, it is given by the C^*-algebra $C(\widehat{\mathbb{K}'}) = C(\widehat{\mathbb{G}}) * C(\widehat{\mathbb{G}})$ (amalgamated over $\mathbb{C}\mathbb{1}$). Denote by i_1, i_2 the maps $C(\widehat{\mathbb{G}}) \to C(\widehat{\mathbb{G}}) * C(\widehat{\mathbb{G}}) = C(\widehat{\mathbb{K}'})$ putting the copy of $C(\widehat{\mathbb{G}})$ in the first and second spot, respectively, these maps are Hopf morphisms. Denote by $\widehat{\pi}\colon C(\widehat{\mathbb{H}}) \to C(\widehat{\mathbb{G}})$ the homomorphism dual to π.

Lemma 3.9. *$\widehat{\pi}$ is not surjective.*

Proof. Assume it is. Using (1.22) we then have that $\gamma\colon L^\infty(\widehat{\mathbb{H}}) \to L^\infty(\widehat{\mathbb{G}})$ is surjective, which contradicts our assumption $\mathbb{H} \neq \mathbb{G}$. □

Denote also $I \subseteq C(\widehat{\mathbb{K}'})$ the closed ideal generated by $\{i_1 \circ \widehat{\pi}(x) - i_2 \circ \widehat{\pi}(x) : x \in C(\widehat{\mathbb{H}})\}$.

Lemma 3.10. *There exists $y \in C^u(\mathbb{G})$ such that $i_1(y) - i_2(y) \notin I$.*

Proof. Consider the smallest C^*-subalgebra of $C^u(\widehat{\mathbb{K}})$ generated by $i_1(C^u(\widehat{\mathbb{G}}))$ and $i_2(\widehat{\pi}(C^u(\widehat{\mathbb{H}}))$ inside $C^u(\widehat{\mathbb{K}})$: let it be called A. From the concrete description of the free product (it is usually given by a Fock-type construction) it is clear that $\mathsf{A} \neq C^u(\widehat{\mathbb{K}})$ (this of course needs Lemma 3.9). Let then $\omega \in C^u(\widehat{\mathbb{K}})^*$ be a non-zero functional such that $\omega\restriction_{\mathsf{A}} = 0$. Then in particular $\omega\restriction_I = 0$. Let $y \in C^u(\mathbb{G}) \setminus \widehat{\pi}(C^u(\widehat{\mathbb{H}}))$. Then

$$\omega(i_1(y) - i_2(y)) = -\omega(i_2(y))$$

It is now clear that for a given y as above one can manufacture ω such that $\omega(i_2(y)) \neq 0$ and $\omega\restriction_{\mathsf{A}} = 0$. Indeed, first pick non-zero $\tilde{\omega} \in C^u(\widehat{\mathbb{G}})$ such that $\tilde{\omega}\restriction_{\widehat{\pi}(C^u(\widehat{\mathbb{H}}))} = 0$ and then take $\tilde{\omega}\circ i_2 \in C^u(\widehat{\mathbb{K}})^*$. The assertion now follows. $\square$

Then one has $q\colon C(\widehat{\mathbb{K}'}) \to C(\widehat{\mathbb{K}}) = {C(\widehat{\mathbb{K}'})}/{I}$. Consider the morphisms $\varphi_j = \widehat{q\circ i_j}\colon c_0(\mathbb{K}) \to c_0(\mathbb{G})$ for $j = 1, 2$. Then one has

Proposition 3.11. *The homomorphisms φ_j coincide on B, i.e. $\beta\circ\varphi_1 = \beta\circ\varphi_2$ and do not coincide on the whole of $\mathbb{G}$.*

Proof. Observe that it is enough to show that φ_j coincide on $\mathbb{H}$, i.e.

$$\pi\circ\varphi_1 = \pi\circ\varphi_2 \tag{3.4}$$

because $\beta = \tilde{\beta}\circ\pi$. The equality (3.4) is equivalent to the equality

$$\widehat{\pi\circ\varphi_1} = \widehat{\pi\circ\varphi_2} \tag{3.5}$$

But composition of morphisms satisfies $\widehat{\varphi\circ\phi} = \widehat{\phi}\circ\widehat{\varphi}$, hence (3.5) is equivalent to

$$q\circ i_1\circ\widehat{\pi} = \widehat{\varphi_1}\circ\widehat{\pi} = \widehat{\varphi_2}\circ\widehat{\pi} = q\circ i_2\circ\widehat{\pi} \tag{3.6}$$

which holds in the quotient $C(\widehat{\mathbb{K}}) = {C(\widehat{\mathbb{K}'})}/{I}$, as desired.

Now using $y \in C^u(\widehat{\mathbb{G}})$ from Lemma 3.10 we can see that $\varphi_1 \neq \varphi_2$. Indeed,

$$\widehat{\varphi}_1(y) - \widehat{\varphi}_2(y) = q(i_1 - i_2(y)) \neq 0$$

from the definition of y. Hence $\widehat{\varphi}_1 \neq \widehat{\varphi}_2$ and consequently $\varphi_1 \neq \varphi_2$. $\square$

Now Theorem 3.8 and Proposition 3.11 constitute the full proof of Theorem 3.1. Let us remark that the converse to Theorem 3.8 without assuming discreteness is not valid even in the group case: the categorical perspective hints that the canonical choice is the coproduct in the category of groups with a fixed common subgroup (or a pushout of the obvious diagram). The latter need not exist in full generality, and in the case of discrete groups is precisely the amalgamated free product.

3.2.2 β-restriction and generating morphisms

Let A be a C^*-algebra and let $U \in \mathsf{M}(\mathsf{A} \otimes C_0(\mathbb{G}))$ be a representation of $\mathbb{G}$ on A. Reasoning similarly as in the proof of Proposition 2.11, one can prove the following proposition:

Proposition 3.12. *Let $\theta \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes \mathsf{B})$ be a morphism satisfying*

$$(\Delta_{\mathbb{G}} \otimes \mathrm{id})\circ\theta = (\mathrm{id} \otimes\theta)\circ\Delta_{\mathbb{G}}.$$

There exists a unique unitary element $Y \in \mathsf{M}(\mathsf{A} \otimes \mathsf{B})$ such that $(\mathrm{id} \otimes\theta)U = U_{12}Y_{13}$.

Definition 3.13. Let $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}), \mathsf{B})$ be a morphism, $\theta \in \mathrm{Mor}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes \mathsf{B})$ be the morphism assigned to β via (2.9), let $U \in \mathsf{M}(\mathsf{A} \otimes C_0(\mathbb{G}))$ be a representation of $\mathbb{G}$ in A. Then $Y \in \mathsf{M}(\mathsf{A} \otimes \mathsf{B})$ obtained by Proposition 3.12 will be denoted U^β and called the β-restriction of U.

Remark 3.14. Observe that (2.5) may be interpreted as $\mathrm{W}^\beta = X = (\mathrm{id}\otimes\beta)\mathbb{W}$.

Theorem 3.15. *With the notations as above, consider the following four statements.*

1. $\{(\mathrm{id}\otimes\omega)X : \omega \in \mathsf{B}^*\}'' = L^\infty(\widehat{\mathbb{G}})$
2. *If* $x \in L^\infty(\mathbb{G})$ *satisfies* $\theta(x) = x \otimes \mathbb{1}$, *then* $x \in \mathbb{C}\mathbb{1}$.
3. *The map* $\mathrm{Rep}(\mathbb{G}) \ni U \mapsto U^\beta$ *is injective.*
4. *The morphism* β *is generating.*

We have (1) $\implies$ (2) $\implies$ (3) $\implies$ (4)*. Moreover,* (4) $\implies$ (1) *provided that* $\mathbb{G}$ *is compact or discrete or* $\tau^u_t(\ker(\beta)) \subseteq \ker(\beta)$ *for all* $t \in \mathbb{R}$ *(in particular, if* $\mathbb{G}$ *is Kac).*

Proof. (1) $\implies$ (2). This follows from general co-duality theory developed in [34, Section 3] (cf. also [36, 35]), but in this particular case the reasoning is easy. Let $a \in L^\infty(\mathbb{G})$ be such that $\theta(a) = a \otimes \mathbb{1}$. Using (2.9) we can rewrite this as:

$$X(a \otimes \mathbb{1}) = (a \otimes \mathbb{1})X \tag{3.7}$$

Applying $\mathrm{id}\otimes\omega$ to both sides of (3.7), where $\omega \in \mathsf{B}^*$, and using weak-*-continuity of multiplication on bounded sets, we get that $a \in \{(\mathrm{id}\otimes\omega)X : \omega \in \mathsf{B}^*\}''' = L^\infty(\widehat{\mathbb{G}})'$. Thus $a \in L^\infty(\mathbb{G}) \cap L^\infty(\widehat{\mathbb{G}})' = \mathbb{C}\mathbb{1}$ by Proposition 1.16.

(2) $\implies$ (3). Let $U, V \in \mathsf{M}(\mathsf{K}(\mathcal{H} \otimes C_0(\mathbb{G}))$ be two representations of $\mathbb{G}$ in the same Hilbert space $\mathcal{H}$. Assume that $U^\beta = V^\beta \in \mathsf{M}(\mathsf{K}(\mathcal{H}) \otimes \mathsf{B})$. We have:

$$(\mathrm{id}\otimes\theta)(UV^*) = U_{12}U^\beta_{13}(V^\beta_{13})^*V^*_{12} = U_{12}V^*_{12}.$$

Thus condition (2) ensures us that there exista a unitary element $u \in \mathsf{B}(\mathcal{H})$ such that $U = (u\otimes\mathbb{1})V$. Applying $(\mathrm{id}\otimes\Delta)$ to both sides of this equality we get that $u = \mathbb{1}$ as in the last step of the proof of Proposition 2.11.

(3) $\implies$ (4). Assume that β is not generating, i.e. its Hopf image $\mathbb{H}$ satisfies $L^\infty(\widehat{\mathbb{H}}) \subsetneq L^\infty(\widehat{\mathbb{G}})$. Then we have $L^\infty(\widehat{\mathbb{G}})' \subsetneq L^\infty(\widehat{\mathbb{H}})' \subset B(L^2(\mathbb{G}))$, so pick a unitary $u \in L^\infty(\widehat{\mathbb{H}})' \setminus L^\infty(\widehat{\mathbb{G}})'$ (it exists, as von Neumann algebras are spanned by its unitary elements, so if the two von Neumann algebras had the same set of unitaries, they would necessarily coincide). Consider $U = (u \otimes \mathbb{1})\mathrm{W}(u^* \otimes \mathbb{1})$ (it is obvious that $U \in \mathrm{Rep}(\mathbb{G})$). From the definition of u it is clear that $U \neq \mathrm{W}$. But on the other hand we have that

$$U^\beta_{13} = U^*_{12}((\mathrm{id}\otimes\theta)U) = U^*_{12}\Big((\mathrm{id}\otimes\theta)\Big((u \otimes \mathbb{1})W(u^* \otimes \mathbb{1})\Big)\Big) =$$
$$= (u \otimes \mathbb{1} \otimes \mathbb{1})W^*_{12}(u^* \otimes \mathbb{1} \otimes \mathbb{1})(u \otimes \mathbb{1} \otimes \mathbb{1})\Big((\mathrm{id}\otimes\theta)W\Big)(u^* \otimes \mathbb{1} \otimes \mathbb{1}) =$$
$$= (u \otimes \mathbb{1} \otimes \mathbb{1})\mathrm{W}^\beta_{13}(u^* \otimes \mathbb{1} \otimes \mathbb{1}) = (u \otimes \mathbb{1} \otimes \mathbb{1})X_{13}(u^* \otimes \mathbb{1} \otimes \mathbb{1}) = X = \mathrm{W}^\beta_{13}$$

Where the equalities in the last line follow from the fact that $\mathrm{W}^\beta = X$ (cf. Remark 3.14) and the fact that $X \in L^\infty(\widehat{\mathbb{H}})\bar{\otimes}\mathsf{B}(\mathcal{H})$ (for a fixed non-degenerate representation $\pi\colon \mathsf{B} \to \mathsf{B}(\mathcal{H})$) and hence the first leg of X commutes with u.

(4) $\implies$ (1) This was discussed in Proposition 2.6 and Theorem 2.9. $\square$

Remark 3.16. In fact Theorem 3.1 can be deduced from Theorem 3.15 in the case of compact, discrete and Kac case: it relies on the implication (4) $\implies$ (3) for representations coming from bicharacters describing homomorphisms. The converse part of Theorem 3.1, valid for discrete quantum groups, for which (4) $\implies$ (3) holds automatically, is stronger than just this implication: one can detect such injectivity not only in the class of all representations, but also in the class of bicharacters coming from homomorphisms.

3.2.3 Promotion of intertwiners

Let $U \in \mathsf{B}(\mathcal{H})\bar{\otimes}L^\infty(\mathbb{G})$ be a representation of $\mathbb{G}$. Let us call B the C^*-algebra generated by slices of U, then $U \in \mathsf{M}(\mathsf{B} \otimes C_0(\mathbb{G})) \subseteq \mathsf{B}''\bar{\otimes}L^\infty(\mathbb{G})$, where the bicommutant is taken inside $\mathsf{B}(\mathcal{H})$. Let $\varphi \in \mathrm{Mor}(C_0^u(\widehat{\mathbb{G}}), \mathsf{B})$ be the unique morphism such that $U = (\varphi \otimes \mathrm{id})\mathbb{W}$ (given by Theorem 1.13).

We would like to interpret $X = \sigma(U) \in \mathsf{M}(C_0(\mathbb{G}) \otimes \mathsf{B})$ (which is now an anticorepresentation) as a quantum subset $\mathbb{X} \subset \widehat{\mathbb{G}}$. Let then $\theta\colon L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\widehat{\mathbb{G}})\bar{\otimes}\mathsf{B}''$ be the map given by (2.9).

Also, for any corepresentation $\tilde{U} \in \mathsf{M}(\mathsf{A} \otimes C_0(\widehat{\mathbb{G}}))$ there exists a unique $\tilde{U}^\varphi \in \mathsf{M}(\mathsf{A} \otimes \mathsf{B})$ – the restriction of the family of unitaries $\tilde{U}$ to the quantum subset $\mathbb{X} \subset \widehat{\mathbb{G}}$ (given by Proposition 3.12). The precise formula is

$$\tilde{U}^\varphi_{13} = \tilde{U}^*_{12}(\mathrm{id} \otimes \theta)(\tilde{U}).$$

First, let us compute the restriction of $U \in \mathsf{M}(\mathsf{B} \otimes C_0(\mathbb{G}))$ to the subset $\mathbb{H} \subset \mathbb{G}$, where $\mathbb{H}$ is a closed quantum subgroup (via $\pi\colon C_0^u(\mathbb{G}) \to C_0^u(\mathbb{H})$ and with the aid of the bicharacter $V \in \mathsf{M}(C_0(\widehat{\mathbb{H}}) \otimes C_0(\mathbb{G}))$).

$$\begin{aligned} U^{\pi^r}_{13} &= U^*_{12}V_{23}U^*_{12}V^*_{23} \\ &= (\varphi \otimes \mathrm{id} \otimes \Lambda\circ\pi)(\mathbb{W}^*_{12}\mathcal{W}_{23}\mathbb{W}_{12}\mathcal{W}^*_{23}) \\ &= ((\varphi \otimes \Lambda\circ\pi)\mathcal{W})_{13} \end{aligned} \tag{3.8}$$

In fact the above computation works also for Woronowicz-closed quantum subgroups. For later use, let us compute $(\theta \otimes \mathrm{id})V$.

$$\begin{aligned} (\theta \otimes \mathrm{id})V = X_{12}V_{13}X^*_{12} &= (\sigma \otimes \mathrm{id})(U^*_{12}V_{23}U_{12}) \\ &= (\sigma \otimes \mathrm{id})((\varphi \otimes \mathrm{id} \otimes \Lambda\circ\pi)\mathbb{W}^*_{12}\mathcal{W}_{23}\mathbb{W}_{12}) \\ &= (\sigma \otimes \mathrm{id})((\varphi \otimes \mathrm{id} \otimes \Lambda\circ\pi)(\mathbb{W}_{13}\mathcal{W}_{23}) \\ &= ((\varphi \otimes \Lambda\circ\pi)\mathcal{W})_{23}V_{13} = U^{\pi^r}_{23}V_{13} \end{aligned} \tag{3.9}$$

In particular if $\mathbb{H} = \mathbb{G}$ and $V = \mathrm{W}$, then (3.9) simply says that $(\theta \otimes \mathrm{id})\mathrm{W} = U_{23}\mathrm{W}_{13}$. We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. ($1 \implies 2$). We assume that $\mathbb{G} = \overline{\langle \mathbb{H}_1, \mathbb{H}_2 \rangle}$, where the embedding $\mathbb{H}_i \subset \mathbb{G}$ is described by means of a bicharacter $V^{\mathbb{H}_i} \in L^\infty(\widehat{\mathbb{G}})\bar{\otimes}L^\infty(\mathbb{H}_i)$ and morphism $\pi_i \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}_i))$. From Theorem 2.12 this is to say that

$$L^\infty(\widehat{\mathbb{G}}) = \{(\mathrm{id} \otimes \omega)\mathrm{W} : \omega \in L^1(\mathbb{G})\}'' = \{(\mathrm{id} \otimes \omega_1 \otimes \omega_2)V^{\mathbb{H}_1}_{12}V^{\mathbb{H}_2}_{13} : \omega_i \in L^1(\mathbb{H}_i)\}'' \tag{3.10}$$

Let us fix a corepresentation $U \in \mathsf{B}''\bar{\otimes}L^\infty(\mathbb{G})$ and interpret it as a quantum subset $\mathbb{X} \subset \widehat{\mathbb{G}}$ as in the introduction. Let then $\theta\colon L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\widehat{\mathbb{G}})\bar{\otimes}\mathsf{B}''$ be the corresponding morphism. Let us apply the map θ to middle and right hand side of (3.10). The right hand side is then

$$\begin{aligned} \{(\theta \otimes \omega_1 \otimes \omega_2)(V^{\mathbb{H}_1}_{12}V^{\mathbb{H}_2}_{13}) : \omega_i \in L^1(\mathbb{H}_i)\}'' &= \{(\mathrm{id} \otimes \mathrm{id} \otimes \omega_1 \otimes \omega_2)(U^{\pi_1^r}_{23}V^{\mathbb{H}_1}_{13}U^{\pi_2^r}_{24}V^{\mathbb{H}_2}_{14}) : \omega_i \in L^1(\mathbb{H}_i)\}'' \\ &= \{(\mathrm{id} \otimes \mathrm{id} \otimes \omega_1 \otimes \omega_2)(U^{\pi_1^r}_{23}U^{\pi_2^r}_{24}V^{\mathbb{H}_1}_{13}V^{\mathbb{H}_2}_{14}) : \omega_i \in L^1(\mathbb{H}_i)\}'' \end{aligned} \tag{3.11}$$

whereas the left (middle) hand side is

$$\{(\theta \otimes \omega)(\mathrm{W}) : \omega \in L^1(\mathbb{G})\}'' = \{(\mathrm{id} \otimes \mathrm{id} \otimes \omega)(U_{23}\mathrm{W}_{13}) : \omega \in L^1(\mathbb{G})\}'' \tag{3.12}$$

Applying $(\eta \otimes \mathrm{id})$ for $\eta \in L^1(\widehat{\mathbb{G}})$ to all elements appearing in (3.12) and (3.11), by normality, yields:

$$\begin{aligned} &\{(\eta \otimes \mathrm{id} \otimes \omega_1 \otimes \omega_2)(U^{\pi_1^r}_{23}U^{\pi_2^r}_{24}V^{\mathbb{H}_1}_{13}V^{\mathbb{H}_2}_{14}) : \omega_i \in L^1(\mathbb{H}_i)\}'' \\ &\qquad = \{(\eta \otimes \mathrm{id} \otimes \omega)(U_{23}\mathrm{W}_{13}) : \omega \in L^1(\mathbb{G})\}'' \end{aligned}$$

Now letting η run through the whole set $L^1(\widehat{\mathbb{G}})$, we obtain

$$\begin{aligned}\{(\eta\otimes\mathrm{id}\otimes\omega_1\otimes\omega_2)(U_{23}^{\pi_1^r}U_{24}^{\pi_2^r}V_{13}^{\mathbb{H}_1}V_{14}^{\mathbb{H}_2}):\omega_i\in L^1(\mathbb{H}_i),\eta\in L^1(\widehat{\mathbb{G}})\}''\\ =\{(\eta\otimes\mathrm{id}\otimes\omega)(U_{23}\mathrm{W}_{13}):\omega\in L^1(\mathbb{G}),\eta\in L^1(\widehat{\mathbb{G}})\}''\end{aligned}\tag{3.13}$$

Remembering that $(\eta\otimes\mathrm{id})\mathrm{W}$ generate $C_0(\mathbb{G})$ and that $C_0(\mathbb{G})\subseteq \mathsf{B}(L^2(\mathbb{G}))$ is nondegenerate, observe that the natural action of $C_0(\mathbb{G})$ on $\mathsf{B}(L^2(\mathbb{G}))_*$ is non-degenerate (cf. [22, eq. (1.2)]), hence the right-hand side of (3.13) reads as:

$$\begin{aligned}\{(\eta\otimes\mathrm{id}\otimes\omega)(U_{23}\mathrm{W}_{13}):\omega\in L^1(\mathbb{G}),\eta\in L^1(\widehat{\mathbb{G}})\}''\\ =\{(\eta\otimes\mathrm{id}\otimes\omega)(U_{23}):\omega\in L^1(\mathbb{G}),\eta\in L^1(\widehat{\mathbb{G}})\}''\\ =\{(\mathrm{id}\otimes\omega)(U):\omega\in L^1(\mathbb{G})\}''\end{aligned}\tag{3.14}$$

and similarily the left hand side of (3.13) is nothing but:

$$\begin{aligned}\{(\eta\otimes\mathrm{id}\otimes\omega_1\otimes\omega_2)(U_{23}^{\pi_1^r}U_{24}^{\pi_2^r}V_{13}^{\mathbb{H}_1}V_{14}^{\mathbb{H}_2}):\omega_i\in L^1(\mathbb{H}_i),\eta\in L^1(\widehat{\mathbb{G}})\}''\\ =\{(\eta\otimes\mathrm{id}\otimes\omega_1\otimes\omega_2)(U_{23}^{\pi_1^r}U_{24}^{\pi_2^r}):\omega_i\in L^1(\mathbb{H}_i),\eta\in L^1(\widehat{\mathbb{G}})\}''\\ =\{(\mathrm{id}\otimes\omega_1\otimes\omega_2)(U_{12}^{\pi_1^r}U_{23}^{\pi_2^r}):\omega_i\in L^1(\mathbb{H}_i)\}''\end{aligned}\tag{3.15}$$

since $V^{\mathbb{H}_i}$ generates $C_0(\mathbb{H})$ (see Section 1.3.4 and in particular point 3. of Definition 1.21 and point 2. of Definition 1.22 together with Theorem 1.23).

Combining (3.13) with (3.14) and (3.15), we obtain

$$\{(\mathrm{id}\otimes\omega)(U):\omega\in L^1(\mathbb{G})\}''=\{(\mathrm{id}\otimes\omega_1\otimes\omega_2)(U_{12}^{\pi_1^r}U_{23}^{\pi_2^r}):\omega_i\in L^1(\mathbb{H}_i)\}''\tag{3.16}$$

which is precisely condition (2).

($2\implies 3$) is obvious, as one specializes $U=\mathrm{W}$, whereas (3) $\implies$ (1) was already shown in Theorem 2.12 (and used as the starting point of the implication ($1\implies 2$)). ☐

Proof of Corollary 3.4. It is clear (and noted in the introduction to Corollary 3.4) that

$$\mathrm{Mor}_{\mathbb{G}}(U,\tilde U)\subseteq \mathrm{Mor}_{\mathbb{H}_1}(U^{\pi_1^r},\tilde U^{\pi_1^r})\cap \mathrm{Mor}_{\mathbb{H}_2}(U^{\pi_2^r},\tilde U^{\pi_2^r})$$

hence the genuine statement is to obtain the converse containment, under assumption that $\mathbb{G}=\overline{\langle\mathbb{H}_1,\mathbb{H}_2\rangle}$.

Assume first that $U=\tilde U$. Observe that $T\in\mathrm{Mor}_{\mathbb{G}}(U,U)$ is equivalent to $(T\otimes\mathbb{1})U=U(T\otimes\mathbb{1})$, which is equivalent to $T((\mathrm{id}\otimes\omega)U)=((\mathrm{id}\otimes\omega)U)T$ for all $\omega\in L^1(\mathbb{G})$, which is equivalent to $T\in\{(\mathrm{id}\otimes\omega)U:\omega\in L^1(\mathbb{G})\}'$.

Applying von Neumann bicommutant theorem to (3.1) one obtains

$$\{(\mathrm{id}\otimes\omega)(U):\omega\in L^1(\mathbb{G})\}'=\{(\mathrm{id}\otimes\omega_1\otimes\omega_2)(U_{12}^{\pi_1^r}U_{23}^{\pi_2^r}):\omega_i\in L^1(\mathbb{H}_i)\}'\tag{3.17}$$

That we need only one prime follows from the same argument as in Proposition 2.3. If then $T\in\mathrm{Mor}_{\mathbb{H}_i}(U^{\pi_i^r},U^{\pi_i^r})$, then in particular

$$(T\otimes\mathbb{1})U^{\pi_i^r}=U^{\pi_i^r}(T\otimes\mathbb{1})\tag{3.18}$$

or, equivalently, that $T\in\{(\mathrm{id}\otimes\omega_i)U^{\pi_i^r}:\omega_i\in L^1(\mathbb{H}_i)\}'$. Using (3.17) we have now that

$$\begin{aligned}T\in\{(\mathrm{id}\otimes\omega)U:\omega\in L^1(\mathbb{G})\}'&\iff T\in\{(\mathrm{id}\otimes\omega_1\otimes\omega_2)(U_{12}^{\pi_1^r}U_{23}^{\pi_2^r}):\omega_i\in L^1(\mathbb{G}_i)\}'\\ &\iff (T\otimes\mathbb{1}\otimes\mathbb{1})U_{12}^{\pi_1^r}U_{13}^{\pi_2^r}=U_{12}^{\pi_1^r}U_{13}^{\pi_2^r}(T\otimes\mathbb{1}\otimes\mathbb{1})\end{aligned}$$

and the last statement follows obviously from the assumption (3.18).

If now U and $\tilde{U}$ are general, one can consider $U \oplus \tilde{U}$. Observe that then $T \in \mathrm{Mor}(U, \tilde{U})$ if and only if

$$\tilde{T} = \begin{pmatrix} 0 & 0 \\ T & 0 \end{pmatrix} \in \mathrm{Mor}(U \oplus \tilde{U}, U \oplus \tilde{U})$$

To conclude (3.3), we apply the first part of the proof to $U \oplus \tilde{U}$ and $\tilde{T}$. The only things that one needs to verify is that $(U \oplus \tilde{U})^{\pi_i^r} = U^{\pi_i^r} \oplus \tilde{U}^{\pi_i^r}$ (which is clear in view of (1.9)) and that the block form of $\tilde{T}$ remains after we restrict $U \oplus \tilde{U}$ to both $\mathbb{H}_1$ and $\mathbb{H}_2$, which is again obvious.

The other implication is then obtained by taking (3.3) with $U = \tilde{U} = \mathrm{W}$, then one arrives at (the commutant of) (3.2). We are done thanks to Theorem 3.3. $\square$

3.2.4 Commutative target

Fix a morphism $\beta \in \mathrm{Mor}(C_0^u(\mathbb{G}), \mathsf{B})$ and assume that the C^*-algebra B is commutative . Then from general theory of C^*-algebras it follows that β factors through the abelianization of $C_0^u(\mathbb{G})$, we call it $C_0^u(\mathbb{G})_{ab}$. Now it is elementary to check that for any C^*-algebra A, denoting by $\sigma(\mathsf{A})$ the spectrum of this C^*-algebra, i.e. the subset of unit ball of A^* consisting of characters, one has one can identify the abelianization $\mathsf{A}_{ab} = C_0(\sigma(\mathsf{A}))$. Now as $C_0^u(\mathbb{G})_{ab} = C_0(Gr(\widehat{\mathbb{G}}))$ is the C^*-algebra of vanishing at infinity continuous functions on locally compact group (see Section 1.3.6), and that the abelianization, as a quotient map, identifies $Gr(\widehat{\mathbb{G}})$ with a Vaes-closed subgroup of $\mathbb{G}$, the first part of Theorem 3.5 is concluded.

$$\begin{tikzcd}
C_0^u(\mathbb{G}) \arrow[rr, "\beta"] \arrow[dr, "ab"'] & & \mathbb{C}^n \\
& c_0(Gr(\widehat{\mathbb{G}})) \arrow[ur, "\tilde{\beta}"] \arrow[dr, "\beta'"'] & \\
& & c_0(\mathbb{F}_n) \arrow[uu, "I"']
\end{tikzcd}$$

Diagram 3: Diagram of Theorem 3.5

Assume now $\mathsf{B} \cong \mathbb{C}^n$ and $Gr(\widehat{\mathbb{G}})$ discrete. It follows from the preceeding paragraph that there is a factorization $(ab, Gr(\widehat{\mathbb{G}}), \tilde{\beta})$ through the abenialization of $C_0^u(\mathbb{G})$. Gelfand-Naimark theory tells us that $\tilde{\beta}$ comes from a map $b\colon \{1, \ldots, n\} \to Gr(\widehat{\mathbb{G}})$. From the universal property there is $b'\colon \mathbb{F}_n \to Gr(\widehat{\mathbb{G}})$ extending b, and taking its Gelfand transform we have a map $\beta'\colon c_0(Gr(\widehat{\mathbb{G}})) \to c_0(\mathbb{F}_n)$. Now the composition $\beta' \circ ab$ describes a morphism $\mathbb{F}_n \to \mathbb{G}$ extending the Gelfand transform of the map β, the Diagram 3 commutes. We denoted by $\iota\colon \{1, \ldots, n\} \hookrightarrow \mathbb{F}_n$ the canonical (up to permutation) choice of free generators and $I\colon c_0(\mathbb{F}_n) \to \mathbb{C}^n$ is its Gelfand transform. This concludes the second part of Theorem 3.5.

3.2.5 Diagram Theorem and its consequences

Proof of Theorem 3.6. That $\sigma(\mathsf{B}) \subseteq H$ follows from the commutativity of Diagram 2, which is explained in the paragraphs preceeding formulation of Theorem 3.6. Hence $\overline{\langle \sigma(\mathsf{B}) \rangle} \subseteq H$, as H is a locally compact group. $\square$

Theorem 3.17. *Assume $\mathbb{K}, \mathbb{H} \subseteq \mathbb{G}$ are compact quantum groups, where $\mathbb{H}, \mathbb{K} \subset \mathbb{G}$ are closed quantum subgroups, and assume $\mathbb{G}$ has Property (FAG). Let us denote $H = Gr(\widehat{\mathbb{H}}), K = Gr(\widehat{\mathbb{K}})$ and $G = Gr(\widehat{\mathbb{G}})$. Then $Gr(\widehat{\overline{\langle \mathbb{H}, \mathbb{K} \rangle}}) = \overline{\langle H, K \rangle}$. In particular, if $\overline{\langle H, K \rangle} \subseteq G$ is a proper subgroup, then $\mathbb{H}, \mathbb{K}$ do not generate $\mathbb{G}$.*

Proof. As in Theorem 2.12, consider the morphism $q^{\cup}\colon C^u(\mathbb{G}) \to C(\mathbb{H}\cup\mathbb{K})$. By passing to its Hopf image, we may assume it is a generating morphism. Moreover,

$$G = \overline{\langle H \cup K\rangle} \supseteq \sigma(C(\mathbb{H}\cup\mathbb{K})) \supseteq H \cup K$$

where the first inclusion follows from Property (FAG) and the other is obvious. As $Gr(\widehat{\overline{\langle \mathbb{H},\mathbb{K}\rangle}})$ is in particular a closed subgroup of G, the conclusion follows. $\square$

Theorem 3.18. *Let $\mathbb{G}$ be a compact quantum group with property (FAG) and assume $\mathbb{H} \subset \mathbb{G}$ is a maximal proper subgroup, i.e. $\mathbb{H} \neq \mathbb{G}$ and if $\mathbb{H} \subset \mathbb{G}_1 \subset \mathbb{G}$, then either $\mathbb{H} = \mathbb{G}_1$ or $\mathbb{G}_1 = \mathbb{G}$. Assume moreover that $H = Gr(\widehat{\mathbb{H}}) \subsetneq Gr(\widehat{\mathbb{G}}) = G$. Let $\mathbb{K} \subseteq \mathbb{G}$ be a closed quantum subgroup. If $Gr(\widehat{\mathbb{K}}) = K \subset H$ as subgroups of G, then $\mathbb{K} \subset \mathbb{H}$.*

Proof. Consider $\mathbb{K}_1 = \overline{\langle \mathbb{K}, H\rangle}$ and let ut denote $K_1 = Gr(\widehat{\mathbb{K}_1})$. Thanks to Theorem 3.17, we have that $K_1 = \overline{\langle K, H\rangle} = H$, as $K \subseteq H$ as subgroups of G.

Observe that the statement $\mathbb{K} \subset \mathbb{H}$ is equivalent to showing that $\overline{\langle \mathbb{H},\mathbb{K}\rangle} = \mathbb{H}$ (cf. Section 2.3.3). It is also clear that $\mathbb{K} \subset \mathbb{K}_1$.

Now relying on Theorem 3.17 once again, we have that $Gr(\widehat{\overline{\langle \mathbb{K}_1,\mathbb{H}\rangle}}) = \overline{\langle K_1, H\rangle} = H$ and that $\mathbb{H} \subset \overline{\langle \mathbb{H},\mathbb{K}\rangle}$. By maximal properness of $\mathbb{H}$ and because $Gr(\widehat{\overline{\langle \mathbb{K}_1,\mathbb{H}\rangle}}) = H \subsetneq G$ it follows that $\overline{\langle \mathbb{H},\mathbb{K}_1\rangle} = \mathbb{H}$, as desired. Hence $\mathbb{H} \supset \mathbb{K}_1 \supset \mathbb{K}$. $\square$

3.3 Examples

3.3.1 Commutative examples

Let G be a locally compact quantum group, let B be a C^*-algebra and let $\beta \in \mathrm{Mor}(C_0(G), \mathsf{B})$. Thanks to Lemma 2.4 (see also Remark 2.5), we can replace B with a quotient of $C_0(G)$. But quotients of abelian C^*-algebras are always abelian, hence we may assume $\mathsf{B} = C_0(X)$ for some locally compact space X. The Gelfand-Naimark Theory (see Theorem 1.2) enables us to see β as a Gelfand-dual of a continuous embedding $b\colon X \hookrightarrow G$. The Hopf image of the morphism β (even if the target was *a priori* non-commutative) is nothing but $\overline{\langle b(X)\rangle} \subseteq G$.

3.3.2 Woronowicz-closed quantum subgroups

Consider a homomorphism $\mathbb{H} \to \mathbb{G}$ described by a $\varphi \in \mathrm{Mor}(C_0^u(\mathbb{G}), C_0^u(\mathbb{H}))$ and a bicharacter $V \in \mathsf{M}(C_0(\widehat{\mathbb{G}}) \otimes C_0(\mathbb{H}))$. The following lemma is well-known to experts, but we were unable to find appropriate reference for this particular formulation.

Lemma 3.19. *Fix $t \in \mathbb{R}$. Then $\tau_t^{u,\mathbb{H}}\circ\varphi\circ\tau_{-t}^{u,\mathbb{G}} = \varphi$.*

Proof. Recall from Section 1.3.3 that the bicharacter $V \in \mathsf{M}(C_0(\widehat{\mathbb{G}})\otimes C_0(\mathbb{H}))$ satisfies $(\tau_t^{\widehat{\mathbb{G}}}\otimes\tau_t^{\mathbb{H}})V = V$. Let us denote the bicharacter corresponding to φ by V_φ and let us compute the bicharacter corresponding to $\tau_t^{u,\mathbb{H}}\circ\varphi\circ\tau_{-t}^{u,\mathbb{G}}$.

$$\begin{aligned}
&(\mathrm{id}\otimes(\Lambda_{\mathbb{H}}\circ\tau_t^{u,\mathbb{H}}\circ\varphi\circ\tau_{-t}^{u,\mathbb{G}}))\mathbb{W}^{\mathbb{G}} = \\
&= (\mathrm{id}\otimes(\Lambda_{\mathbb{H}}\circ\tau_t^{u,\mathbb{H}}\circ\varphi\circ\tau_{-t}^{u,\mathbb{G}}))(\tau_t^{\widehat{\mathbb{G}}}\otimes\tau_t^{u,\mathbb{G}})\mathbb{W}^{\mathbb{G}} = \\
&= (\tau_t^{\widehat{\mathbb{G}}}\otimes\tau_t^{\mathbb{H}})(\mathrm{id}\otimes(\Lambda_{\mathbb{H}}\circ\varphi))\mathbb{W}^{\mathbb{G}} = \\
&= (\tau_t^{\widehat{\mathbb{G}}}\otimes\tau_t^{\mathbb{H}})V_\varphi = V_\varphi
\end{aligned}$$

The first equality is justified with (1.16), the second with (1.15) and the third by properties of bicharacters. Now, as V_φ is a bicharacter corresponding to quantum group homomorphism φ and as bicharacters are in one-to-one correspondence with quantum group homomorphisms, the proof is finished. $\square$

Let us now assume that a homomorphism $\mathbb{H} \to \mathbb{G}$ identifies $\mathbb{H}$ as a Woronowicz-closed quantum subgroup of $\mathbb{G}$. Then $\tau_t^{u,\mathbb{G}}(\ker(\varphi)) = \ker(\varphi)$ and thus we are in the setting of "In particular" part of Theorem 2.9. Thus the von Neumann algebra generated by slices of the bicharacter V on the right leg is invariant and preserved under $\widehat{\tau}_\bullet$ and $\widehat{R}$: it corresponds to a minimal Vaes-closed quantum subgroup of $\mathbb{G}$ containing $\mathbb{H}$: Vaes-closure of a Woronowicz-closed quantum subgroup.

This construction was independently given in [33] and in [23], although the question of minimality (in the sense of Section 2.1) was not discussed in these papers.

3.3.3 Finitely generated discrete quantum groups

Assume $\mathbb{G}$ is a discrete quantum group. We have seen in Theorem 2.19 that one can manufacture a generating morphism $\beta \in \mathrm{Mor}(c_0(\mathbb{G}), \mathsf{B})$ with finite dimensional B if and only if $\widehat{\mathbb{G}}$ is a compact matrix quantum group (see Section 1.3.2). In the literature there are plenty of examples of such a groups, let us evoke some of them.

1. The quantum $S_qU(2)$ of Woronowicz [64] and its higher-dimensional relatives $S_qU(n)$ [65], for the deformation parameter $q \in \mathbb{R}^\times$. The quantum group $S_qU(2)$ has particularly simple form: $C(S_q(2))$ is generated by two elements α, γ such that γ is normal, $\alpha\gamma = q\gamma\alpha$, $\alpha\gamma^* = q\gamma^*\alpha$ and $\alpha\alpha^* + q\gamma^*\gamma = \alpha^*\alpha + q\gamma^*\gamma = \mathbb{1}$. Then the matrix
$$\begin{pmatrix} \alpha & \gamma \\ -\gamma^* & \alpha^* \end{pmatrix}$$
is a fundamental corepresentation of $S_qU(2)$. Note that for $q = 1$ one obtains the classical group $SU(2)$, and deformation parameter q and q^{-1} leads to the same quantum group. Moreover, for $q \neq 0, 1$ the C^*-algebras underlying $C(S_qU(2))$ are all isomorphic as C^*-algebras, but as quantum groups they are different (apart from the q–q^{-1}-symmetry stated above). Let us also note that $S_qU(2)$ is coamenable, hence the ^{u} decoration is redundant.

2. The class of liberalized quantum groups. Let $G \subseteq O_n$ be a (Lie) group of orthogonal transformations. Then one can write $C(G) = \langle x_{i,j} | x_{i,j}x_{k,l} = x_{k,l}x_{i,j}, \mathcal{R}\rangle$, where $x_{i,j}$ are the coordinate functions and $\mathcal{R}$ denote relations on coordinate functions coming from relations satisfied by elements of the group G (apart from the relation making coordinate functions commute, which we listed separately on purpose). $X = [x_{i,j}]_{1\le i,j\le n}$ is a fundamental corepresentation. One can then form $C^u(G^+) = \langle x_{i,j} | \mathcal{R}\rangle$ and endow this C^*-algebra with a fundamental corepresentation X as previously. In particular, one obtains in this way the universal orthogonal quantum groups O_n^+ previously introduced by Wang [60], and quantum permutation groups S_n^+, constructed previously in [61]. In this generality, these groups were introduced in [8] by Banica and Speicher.

3. Prior to the construction of [8] a similar, but an alternative approach was used in [55] to construct a larger class of quantum groups: now called universal orthogonal quantum groups O_F^+ and universal unitary quantum groups U_F^+, where $F \in M_n(\mathbb{C})$ is a parameter matrix. Consider the matrix $U = [u_{i,j}]_{1\le i,j\le n}$ and the C^*-algebra $C^u(U_F^+)$ generated by n^2-elements $u_{i,j}$, $1 \le i, j \le n$ subject to the following relations:

   (a) The matrix $U \in M_n(C^u(U_F^+))$ is unitary and

   (b) $U^\top F\bar{U}F^{-1} = \mathbb{1} = F\bar{U}F^{-1}U^\top$, where $\bar{U} = [u_{i,j}^*]_{1\le i,j\le n}$ is the entrywise conjugation of U and $U^\top = [u_{j,i}]_{1\le i,j\le n}$ is the transpose of U.

   The matrix U is a fundamental corepresentation of $C^u(U_F^+)$, this determines the quantum groups U_F^+ uniquely. The universal orthogonal quantum groups are obtained by imposing an additional relation $U = \bar{U}$. The previously introduced universal orthogonal quantum groups O_n^+ and universal unitary quantum groups U_n^+ correspond to picking $F = \mathbb{1}$.

4. Surprisingly, it took some years to carry a similar construction to obtain the aforementioned quantum permutation groups S_n^+ in [61] (cf. [49, Section 3]). As we will discuss these groups in Chapter 4, let us recall their definition entirely. Consider the universal C^*-algebra generated by n^2-elements $u_{i,j}$, $1 \leq i, j \leq n$ subject to the following relations:

   (a) the generators $u_{i,j}$ are all projections.

   (b) $\sum_{i=1}^n u_{i,j} = \mathbb{1} = \sum_{j=1}^n u_{i,j}$.

   This C^*-algebra will be denoted $C^u(S_n^+)$. The matrix $U = [u_{i,j}]_{1\leq i,j\leq n}$ is a fundamental corepresentation of $C^u(S_n^+)$, this gives all the quantum group-theoretic data. Moreover, $S_n^+ = S_n$ for $n \leq 3$ and $S_n^+ \supsetneq S_n$ for $n \geq 4$ and S_n^+ is coamenable only if $n \leq 4$ ([3]).

3.3.4 Examples of Brannan, Collins and Vergnioux

In this part we summarize the crucial result of [15, Section 4] in the language of Hopf image. We list some of the subgroups of the quantum group O_n^+:

1. The group of characters $O_n \subset O_n^+$ (see Section 1.3.6).

2. The subgroup of classical permutations $S_n \subset O_n \subset O_n^+$ obtained by canonical *permutation representation* of S_n on $\mathbb{C}^n$.

3. Given natural numbers a, b such that $a + b = n$, consider the morphism $\pi_{a,b}\colon C^u(O_n^+) \to C^u(O_a^+ \hat{*} O_b^+) = C^u(O_a^+) * C^u(O_b^+)$ obtained by sending the upper $a \times a$ corner of the fundamental corepresentation of $C^u(O_n^+)$ to the fundamental corepresentation of O_a^+ entrywise, the lower $b \times b$ corner of the fundamental corepresentation of $C^u(O_n^+)$ to the fundamental corepresentation of O_b^+ entrywise and all other entries to 0.

4. Consider a unit vector $\xi \in \mathbb{R}^n \subseteq \mathbb{C}^n$. Then one has the following subgroup of O_n^+: completing ξ to an orthonormal basis, we can write $\pi_\xi\colon C^u(O_n^+) \to C^u(O_{n-1}^{+,\xi})$ by setting $\pi_\xi(u_{1,1}) = \mathbb{1}$. Abstractly, $C(O_{n-1}^{+,\xi}) \cong C(O_{n-1}^+)$, and $O_{n-1}^{+,\xi}$ correspond to stabilizer subgroup of $\xi \in S^{n-1}$.

Then

1. $O_n^+ = \overline{\langle O_n \cup O_{n-1}^{+,\xi} \rangle}$ for any $\xi \in \mathbb{R}^n \subseteq \mathbb{C}^n$ of norm one and $n \geq 4$.

2. $O_n^+ = \overline{\langle O_{n-1}^{+,\xi_1} \cup O_{n-1}^{+,\xi_2} \rangle}$ for any $\xi_1 \neq \xi_2 \in \mathbb{R}^n \subseteq \mathbb{C}^n$ of norm one and $n \geq 4$.

3. $O_{2n}^+ = \overline{\langle O_n^+ \hat{*} O_n^+ \cup S_{2n} \rangle}$ for $n \geq 2$.

In the above, $\cup$ has to be understood as in Section 2.3.3.

Chapter 4

Applications

4.1 Introduction

This chapter has essentially three parts. In the first part, we provide a first genuinly quantum example of a generating set (i.e. of a generating morphism). This answers partially [49, Question 7.3]: we show that the morphism $\mathbb{I}_{2,4} \to S_4^+$ from quantum increasing sequences into quantum permutations on 4 points, introduced by Curran in [19], is generating. Then we turn into studying some group-theoretic properties of the quantum group S_4^+. In the last part, we use them to show that S_4^+ fail to have Property (FAG). We also draw some other consequences of it, that is, we only use the "easy directions" of the results from Section 4.3 to show in an elementary way that $\widehat{S_4^+}$ is hyperlinear.

4.2 The quantum permutation group S_n^+ and quantum increasing sequences

The algebra of continuous functions on the set of quantum increasing sequences was defined by Curran in [19, Definition 2.1]. Let $k \leq n \in \mathbb{N}$ and let $C(\mathbb{I}_{k,n})$ be the universal C^*-algebra generated by $p_{i,j}$, $1 \leq i \leq n$, $1 \leq j \leq k$ subject to the following relations:

1. the generators $p_{i,j}$ are all projections.
2. each column of the rectangular matrix $P = [p_{i,j}]$ forms a partition of unity: $\sum_{i=1}^n p_{i,j} = \mathbb{1}$ for each $1 \leq j \leq k$.
3. increasing sequence condition: $p_{i,j}p_{i'j'} = 0$ whenever $j < j'$ and $i \geq i'$.

This definition is obtained by the liberalization philosophy: if one denotes by $I_{k,n}$ the set of increasing sequences of length k and values in $\{1, \ldots, n\}$, then it is possible to write a matrix representation: to an increasing sequence $\underline{i} = (i_1 < \ldots < i_k)$ one associates its matrix representation $A(\underline{i}) \in M_{n\times k}(\{0,1\})$ as follows: $A(\underline{i})_{i_l,l} = 1$ and all other entries are set to be 0. One can check that the space of continuous functions on these matrices $C(\{A(\underline{i}) : \underline{i} \in I_{k,n}\})$ is generated by the coordinate functions $x_{i,j}$ subject to the relations introduced above **and** the commutation relation (cf. the discussion after [19, Remark 2.2]).

Curran defined also a *-homomorphism $\beta_{k,n}\colon C(S_n^+) \to C(\mathbb{I}_{k,n})$ ([19, Proposition 2.5]) by:

- $u_{i,j} \mapsto p_{i,j}$ for $1 \leq i \leq n$, $1 \leq j \leq k$,
- $u_{i,k+m} \mapsto 0$ for $1 \leq m \leq n-k$ and $i < m$ or $i > m+k$,

- for $1 \leq m \leq n-k$ and $0 \leq p \leq k$,

$$u_{m+p,k+m} \mapsto \sum_{i=0}^{m+p-1} p_{i,p} - p_{i+1,p+1},$$

where we set $p_{0,0} = \mathbb{1}$, $p_{0,i} = p_{0,i} = p_{i,k+1} = 0$ for $i \geq 1$.

The fact that this *-homomorphism is well defined follows from [19, Proposition 2.4], where some additional relations were identified, and the universal property of $C(S_n^+)$ (see Equation 4.3.1). These maps are defined in such a way that when applied to the commutative C^*-algebras $C(S_n) \to C(I_{k,n})$ (which satisfy the same relations plus commutativity), it is precisely the map of 'completing the increasing sequence to a permutation map'. More precisely, one can draw the diagram of an increasing sequence $\underline{i} = (i_1 < \ldots < i_k)$ in the following manner: drawing k dots in one row and additional n dots in the row below, one connects l-th dot in the upper row to the i_l-th dot in the lower row. Then one draws additional $n-k$ dots in the upper row next to previously drawn k dots and connects them as follows: $(k+1)$-th dot is connected to the leftmost non-connected dot in the bottom row, then $(k+2)$-th dot is connected to the leftmost non-connected dot in the bottom row etc. Finally, one obtains the diagram of a permutation on n letters, which is then called $\beta_{k,n}(\underline{i})$ (for the version of $\beta_{k,n}$ as a map between appropriate commutative C^*-algebras).

Fact 4.1. *$\langle I_{k,n} \rangle = S_n$ for all n and all $k \neq 0, n$, where $I_{k,n} \subseteq S_n$ is seen via the above map.*

Proposition 4.2. *Let $\mathbb{H} \subseteq S_n^+$ be the Hopf image of the map $\beta_{k,n}\colon C(S_4^+) \to C(\mathbb{I}_{k,n})$ for $k \neq 0, n$. Then $S_n \subseteq \mathbb{H} \subseteq S_n^+$*

Proof. The abelianization of $C(\mathbb{I}_{k,n})$ is the commutative C^*-algebra whose spectrum is the space of increasing sequences $I_{k,n}$ and the map $\beta_{k,n}$ on the abelianizations is precisely the canonical 'completing the increasing sequence to a permutation' map, whose image generate the whole of S_n if $k \neq 0, n$, so we conclude by Theorem 3.6. □

In what follows, we restrict our attention to the case $n = 4$, $k = 2$.

Theorem 4.3. *The Hopf image of the map $\beta_{2,4}\colon C(S_4^+) \to C(\mathbb{I}_{2,4})$ is the whole S_4^+.*

Proof. Form Proposition 4.2 we see that the group of characters of $\mathbb{H}$, the Hopf image of β, is the permutation group $Gr(\widehat{\mathbb{H}}) = S_4$. In particular, $\mathbb{H}$ contains the diagonal Klein subgroup, so is one of the groups listed in [5, Theorem 6.1] (see Theorem 4.7). It is easy to check that the group of characters of subgroups contained in [5, Theorem 6.1] (see Theorem 4.7) are equal to S_4 only for the following two groups: S_4 and S_4^+. On the other hand, in [49, Proposition 7.4] it was shown that $C(\mathbb{I}_{2,4}) \cong (\mathbb{C}^2 * \mathbb{C}^2) \oplus \mathbb{C}^2$ (the free product is amalgamated over $\mathbb{C}\mathbb{1}$) is infinite dimensional, hence $\mathbb{H} \neq S_4$. Consequently, $\mathbb{H} = S_4^+$ is the only possibility left. □

4.3 Group-theoretic properties of S_4^+

In order to prove that S_4^+ does not enjoy Property (FAG), we turn to studying some group-theoretic properties of this quantum group. Namely, we show that the automorphisms group $Aut(S_4^+) = S_4$, classify the embeddings of the maximal proper subgroups of S_4^+: the twisted version of $O(2)$ and twisted version of A_5. There are precisely three copies of $O_{-1}(2) \subset S_4^+$, all of them conjugate by an automorphism and there is a unique copy of $A_5^\tau \subset S_4^+$. The key techniques are taken from [5], let us recall them.

4.3.1 Another presentation of S_4^+

Let us recall that the C^*-algebra of continuous functions on $SO(3)$ is the universal C^*-algebra generated by $x_{i,j}$, $1 \leq i, j \leq 3$ subject to the following relations:

1. The matrix $X = (x_{i,j})_{1\leq i,j\leq 3} \in M_3(C^u(SO(3))$ is orthogonal, i.e. $AA^\top = A^\top A = \mathbb{1} \in M_3(C^u(SO(3)))$ and the generators $x_{i,j}$ are self-adjoint.
2. $x_{i,j}x_{k,l} = x_{k,l}x_{i,j}$ for all $1 \leq i,j,k,l \leq 3$.
3. $\sum_{\tau\in S_3} \operatorname{sgn}(\tau)a_{1,\tau(1)}a_{2,\tau(2)}a_{3,\tau(3)} = \mathbb{1}$

It is a routine check (relying on the Stone-Weierstrass theorem) that the above C^*-algebra is indeed the C^*-algebra of continuous functions on the compact group $SO(3)$ and that X being a fundamental corepresentation encodes the matrix multiplication in $SO(3)$.

Definition 4.4. The C^*-algebra of continuous functions on a compact quantum group $SO_{-1}(3)$ is the universal C^*-algebra generated by $a_{i,j}$, $1 \leq i,j \leq 3$ subject to the following relations:

1. The matrix $A = (a_{i,j})_{1\leq i,j\leq 3} \in M_3(C^u(SO_{-1}(3))$ is orthogonal, i.e. $AA^\top = A^\top A = \mathbb{1} \in M_3(C^u(SO_{-1}(3)))$. In particular, the generators $a_{i,j}$ are self-adjoint.
2. $a_{i,j}a_{i,k} = -a_{i,k}a_{i,j}$ for $k \neq j$.
3. $a_{i,j}a_{k,j} = -a_{k,j}a_{i,j}$ for $k \neq i$.
4. $a_{i,j}a_{k,l} = a_{k,l}a_{i,j}$ for $i \neq k$, $j \neq l$.
5. $\sum_{\tau\in S_3} a_{1,\tau(1)}a_{2,\tau(2)}a_{3,\tau(3)} = \mathbb{1}$

The matrix A is a fundamental corepresentation of $C^u(SO_{-1}(3))$: this gives all the quantum group-theoretic data.

By [5, Theorem 3.1], the map $C^u(SO_{-1}(3)) \to C(S_4^+)$ seen as

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \mapsto \frac{1}{4}MUM \tag{4.1}$$

where U is as in and

$$M = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \tag{4.2}$$

is an isomorphism of quantum groups. In particular, $SO_{-1}(3)$ is coamenable, as it has the same corepresentation category as $SO(3)$ (or because it is isomorphic to S_4^+) and we will drop the ^{u} decoration.

4.3.2 Cocycle twists. General Theory.

In what follows, we briefly discuss the twisting procedure and introduce the notation. We stick to the theory of Hopf *-algebras, altough the general procedure works well for general Hopf algebras over any field.

Let H be a Hopf *-algebra with coproduct Δ. Recall that the algebra $H\otimes_{alg} H$ can be given the Hopf *-algebra structure: the coproduct is $\Delta_2 = (\mathrm{id}\otimes\sigma\otimes\mathrm{id})\circ(\Delta\otimes\Delta)$. We will use the Sweedler-Heyneman notation: $\Delta(x) = x_{(1)} \otimes x_{(2)}$. A linear map $\Sigma\colon H\otimes H \to \mathbb{C}$ is called a 2-cocycle if:

1. it is convolution invertible: the neutral element of convolution is $m_{\mathbb{C}}\circ(\varepsilon\otimes\varepsilon)$, the convolution of $\Sigma, \Sigma'\colon H\otimes H \to \mathbb{C}$ is given by $\Sigma * \Sigma' = m_{\mathbb{C}}\circ(\Sigma\otimes\Sigma')\circ\Delta_2$,
2. it satifies the cocycle identity:
$$\Sigma(x_{(1)}, x_{(2)})\Sigma(x_{(2)}y_{(2)}, z) = \Sigma(y_{(1)}, z_{(1)})\Sigma(x, y_{(2)}z_{(2)}) \tag{4.3}$$
and $\Sigma(x,1) = \varepsilon(x) = \Sigma(1,x)$ for $x,y,z \in H$.

Here and in what follows, $m_W: W \otimes W \to W$, for a given algebra W, is the multiplication map $W \otimes_{alg} W \ni x \otimes y \xmapsto{m_W} x \cdot_W y \in W$.

Following [25, 48, 5], a 2-cocycle Σ provides a new Hopf *-algebra H^Σ. As a coalgebra, $H^\Sigma = H$, whereas the product of H^Σ is defined as

$$[x][y] = \Sigma(x_1, y_1)\Sigma^{-1}(x_3, y_3)[x_2 y_2],$$

where an element $x \in H$ is denoted $[x]$ when viewed as an element of H^Σ. In other words, $m_{H^\Sigma} = (\Sigma \otimes m_H \otimes \Sigma^{-1}) \circ \Delta_2^2$.

The antipode of H^Σ can be expressed via the following formula:

$$S^\Sigma([x]) = \Sigma(x_1, S(x_2))\Sigma^{-1}(S(x_4), x_5)[S(x_3)].$$

The Hopf algebras H and H^Σ have equivalent tensor categories of comodules [48].

In our considerations we are interested in the case when the 2-cocycle is induced from Hopf *-algebra quotient (quantum subgroup). Let $\pi: H \to K$ be a Hopf surjection and let $\Sigma: K \otimes K \to \mathbb{C}$ be a 2-cocycle on K. Then $\Sigma_\pi = \Sigma \circ (\pi \otimes \pi): H \otimes H \to \mathbb{C}$ is a 2-cocycle.

Proposition 4.5 ([5, Lemma 4.3])**.** *Let $\pi: H \to K$ be a Hopf surjection and let $\Sigma: K \otimes K \to \mathbb{C}$ be a 2-cocycle. Then there is a bijection between:*

1. *Hopf surjections $f: H \to L$ such that there exists a Hopf surjection $g: L \to K$ satisfying $g \circ f = \pi$, and*
2. *Hopf surjections $\tilde{f}: H^{\Sigma_\pi} \to \tilde{L}$ such that there exists a Hopf surjection $\tilde{g}: \tilde{L} \to K^\Sigma$ satisfying $\tilde{g} \circ \tilde{f} = [\pi(\cdot)]$.*

The bijection is given by $\tilde{f}(\cdot) = [f(\cdot)]$.

4.3.3 Twistings applied to $SO(3)$

Let $H = \mathrm{Pol}(SO(3))$ denote the Hopf *-algebra of representative functions on the group $SO(3)$, let $H' = \mathrm{Pol}(SO_{-1}(3)) \subseteq C(SO_{-1}(3))$ be the unique dense Hopf *-algebra of the quantum group $SO_{-1}(3)$ and let $K = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ denote the group algebra of the Klein group. We denote by t_1, t_2 the canonical generators of the $\mathbb{Z}_2$ factors of the Klein group, let also denote $t_0 = e, t_3 = t_1 t_2$. The restriction of functions on $SO(3)$ to its diagonal subgroup gives a Hopf surjection

$$H \ni x_{i,j} \xmapsto{\pi_d} \delta_{i,j} t_i \in K. \tag{4.4}$$

Let $\Sigma: K \otimes K \to \mathbb{C}$ be the unique linear extention of the mapping

$$\Sigma(t_i, t_j) = \begin{cases} -1 & \text{for } (i,j) \in \{(1,1),(1,3),(2,1),(2,2),(3,2),(3,3)\} \\ 1 & \text{otherwise} \end{cases} \tag{4.5}$$

In other words, for $1 \le i, j \le 2$ we have that $\Sigma(t_i, t_j) = -1$ if and only if $i \le j$ and we extend this definition by bimultiplicativity. Then Σ is the 2-cocycle in the sense of (4.3). We will work with the cocycle $\Sigma_d = \Sigma \circ (\pi_d \otimes \pi_d)$ on H. Note that $\Sigma_d^{-1} = \Sigma_d$.

Theorem 4.6 ([5, Theorem 5.1])**.** *The Hopf *-algebras H^{Σ_d} and H' are isomorphic. The isomorphism is given by $[x_{i,j}] \mapsto a_{i,j}$.*

As a consequence of Theorem 4.6 and Proposition 4.5, the authors obtained the list of subgroups $\widehat{\mathbb{Z}_2 \times \mathbb{Z}_2} \subset \mathbb{G} \subset S_4^+$:

Theorem 4.7 ([5, Theorem 6.1])**.** *The compact quantum groups satysfying $\widehat{\mathbb{Z}_2 \times \mathbb{Z}_2} \subset \mathbb{G} \subset S_4^+$ are precisely:*

1. *S_4^+;*

2. $O_{-1}(2)$*, the cocycle twist of the group* $O(2)$*;*

3. D_n^τ*, for* $n > 4$ *even: the unique non-trivial twist of the dihedral group* D_n*;*

4. A_5^τ*, the unique non-trivial twist of the alternating group* A_5^τ

5. S_4^τ*, the twist of* S_4 *by a cocycle induced from the non-normal Klein subgroup of* S_4 *and*

6. $S_4, A_4, D_4, \mathbb{Z}_2 \times \mathbb{Z}_2$*.*

Moreover, $D_n^\tau \subset O_{-1}(2)$, hence the maximal proper subgroups of S_4^+ containing the diagonal Klein subgroup are precisely $O_{-1}(2)$, A_5^τ, S_4^τ and S_4.

Similarily to (4.4), the Klein group can be embedded into $SO_{-1}(3)$ as follows:

$$H' \ni a_{i,j} \xmapsto{\pi'_d} \delta_{i,j} t_i \in K \tag{4.6}$$

Thus one can define the 2-cocycle $\Sigma'_d\colon H' \otimes H' \to \mathbb{C}$ via $\pi'_d\colon H' \to K$ and in this way obtain another realization of the isomorphism from Theorem 4.6: $(H')^{\Sigma'_d} \cong H$.

4.3.4 Characteristic subgroups.

Let $\mathbb{G}$ be a compact quantum group and let $\mathbb{H}$ be its subgroup: let $\pi\colon C^u(\mathbb{G}) \to C^u(\mathbb{H})$ be a quotient map intertwining the respective coproducts.

Definition 4.8. We will say that $\mathbb{H}$ is a *characteristic subgroup* of $\mathbb{G}$ if for any automorphism of $\mathbb{G}$ (i.e. a Hopf *-homomorphism $\vartheta\colon C^u(\mathbb{G}) \to C^u(\mathbb{G})$), $\mathbb{H}$ is mapped onto $\mathbb{H}$ (i.e. $\pi \circ \vartheta = \chi \circ \pi$ for some automorphism $\chi\colon C^u(\mathbb{H}) \to C^u(\mathbb{H})$, or in other words, $\vartheta(\ker(\pi)) = \ker(\pi)$).

It is clear that this notion can be described equivalently in terms of the underlying Hopf *-algebra and we will use this further without mentioning. There is a canonical example of a characteristic subgroup.

Proposition 4.9. *The group of characters* $Gr(\widehat{\mathbb{G}})$ *of* $\mathbb{G}$ *is characteristic.*

Proof. Let $\vartheta\colon C^u(\mathbb{G}) \to C^u(\mathbb{G})$ be an automorphism of $\mathbb{G}$. As the kernel of the quotient map $q\colon C^u(\mathbb{G}) \to C(Gr(\widehat{\mathbb{G}}))$ is an ideal generated by commutators, and as $\theta([x,y]) = [\theta(x), \theta(y)]$, $\theta(\ker(q)) \subseteq \ker(q)$. The other inclusion follows by applying θ^{-1}. ☐

There is another, more concrete, example of a characteristic subgroup, and we will use it in consecutive sections.

Recall that $H' = \mathrm{Pol}(SO_{-1}(3))$ is the unique dense Hopf *-algebra of the quantum group $SO_{-1}(3)$ and $K = \mathbb{C}[\mathbb{Z}_2 \times \mathbb{Z}_2]$ is the group algebra of the Klein group. The Klein group can be embedded into $SO_{-1}(3)$ via (4.6). But there are other occurences of the Klein group as a subgroup of $SO_{-1}(3)$: this particular one will be called diagonal. Let $\pi\colon H' \to K$ be a Klein subgroup in $SO_{-1}(3)$ and consider the following factorization:

$$\begin{tikzcd} H' \arrow[r, two heads, "q"] \arrow[rr, bend left, "\pi"] & H'_{ab} \arrow[r, two heads, "\theta"] & K \end{tikzcd}$$

In the above diagram, H'_{ab} denotes the the abelianization of H': the Hopf *-algebra quotient of H' by the commutator ideal, q denotes this quotient map.

It is clear that all quotients π onto the group algebra of the Klein group enjoy the above factorization. Let us describe it more explicitely.

Lemma 4.10. H'_{ab} *is precisely the Hopf* **-algebra* $C(S_4)$*, and the map* q *is given as follows: consider the canonical representation* $\rho\colon S^4 \to O(4)$ *and consider the restriction to the subspace* $(1,1,1,1)^\perp$*: this gives an embedding* $\rho\colon S_4 \to O(3)$*,* $q\colon H' \to C(S_4)$ *acts as* $a_{i,j} \xmapsto{q} x_{i,j} \circ \rho$*.*

Proof. Straightforward computation. □

Thus any Klein subgroup of $SO_{-1}(3)$ is a Klein subgroup in S_4; there are two types of Klein groups embedded into S_4: the easy ones, of the form: $\{\mathrm{id}, (1i), (kl), (1i)(kl)\}$ and the diagonal one, of the form $\{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$.

Remark 4.11. The diagonal Klein subgroup, in the above map, consists of the matrices

$$\left\{ I, \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}.$$

Lemma 4.12. *The diagonal Klein subgroup of $SO_{-1}(3)$ is a characteristic subgroup.*

Proof. Any occurence of a Klein subgroup in $SO_{-1}(3)$ appears as a Klein subgroup of S_4. Furthermore, any automorphism of $SO_{-1}(3)$ restricts to an automorphism of $Gr(\widehat{SO_{-1}(3)}) = S_4$. It is then enough to check that the diagonal (in $SO_{-1}(3)$) Klein subgroup of $SO_{-1}(3)$ is precisely the diagonal (in S_4) Klein subgroup of S_4 and that the latter is characteristic in S_4. Both the assertions are easy to check, as $Aut(S_4) = S_4$. □

Just to complete the picture, let us elucidate the easy Klein subgroups, providing a non-example of a characteristic subgroup.

Lemma 4.13. *All the easy Klein subgroups of S_4 are conjugate; the corresponding automorphism of S_4 extends to $SO_{-1}(3)$.*

Proof. Let $\{\mathrm{id}, (12), (34), (12)(34)\}$ and $\{\mathrm{id}, (1i), (2j), (1i)(2j)\}$ be two different Klein subgroups of S_4. It is easy to check that conjugation by $(2i)$ gives the first part of the lemma. In order to get the automorphism $u\colon H \to H$ extending it, simply consider the map $A \mapsto \rho(2i)A\rho(2i)$, where ρ is the map from Lemma 4.10. □

4.3.5 Automorphisms of S_4^+

With Theorem 4.6 and the results of Section 4.3.2 & Section 4.3.4 in hand, we are able to classify all the automorphisms of $SO_{-1}(3)$. Consider an automorphism $\vartheta\colon C(SO_{-1}(3)) \to C(SO_{-1}(3))$ and the following diagram:

$$\begin{tikzcd} H' \arrow[r, two heads, "\theta"] \arrow[rr, bend left, "\pi'_d"] & H' \arrow[r, two heads, "\chi \circ \pi'_d"] & K \end{tikzcd}$$

Thanks to Lemma 4.12, the above diagram is well defined: the Klein subgroup is characteristic, hence $\chi \circ \pi'_d \circ \theta = \pi'_d$ for some automorphism χ of the Klein group. We can then use Proposition 4.5 to 'untwist' this diagram and obtain an automorphism of $SO(3)$ (which should be easier to classify). Apply the cocycle Σ'_d (recall that the Klein group have no nontrivial twist, cf. [5, Lemma 6.2]) and Proposition 4.5 gives us the following diagram:

$$\begin{tikzcd} H \arrow[r, two heads, "\theta^{\Sigma'_d}"] \arrow[rr, bend left, "\pi_d"] & H \arrow[r, two heads, "\chi \circ \pi_d"] & K \end{tikzcd}$$

and $\theta^{\Sigma'_d} = [\theta]$, where $[\cdot]$ is understood as in Section 4.3.2. As any automorphism of $SO(3)$ is inner, it is enough to check which of them preserve the diagonal Klein subgroup. It is clear that conjugation by $\rho(x)$, $x \in S_4$ (where $\rho\colon S_4 \to O(3)$ is introduced in Lemma 4.10), is such an automorphism (as the diagonal Klein subgroup is characteristic in S_4). Using Remark 4.11 one can write the formula for $F \in SO(3)$ that conjugates the diagonal Klein subgroup in $SO(3)$ and arrive at a system of constraints saying that the matrix F has to have two zero entries in each row and column (the remaining entry, because of norm 1 condition in each row and column, has to be ± 1). There are precisely $24 = 4! = |S_4|$ of such matrices, hence we arrive at the following

Theorem 4.14. *Every automorphism of $SO_{-1}(3)$ is given by $A \mapsto \rho(x)^\top A \rho(x)$ for some $x \in S_4$. In other words, $Aut(SO_{-1}(3)) \cong S_4$.*

4.3.6 On the embeddings $O_{-1}(2) \subset SO_{-1}(3)$

Definition 4.15. The C^*-algebra of continuous functions on a compact quantum group $O_{-1}(2)$ is the universal C^*-algebra generated by $\tilde{a}_{i,j}$, $1 \le i,j \le 2$ subject to the relations (1-4) of Definition 4.4, *mutati mutandis*. As previously, the matrix $\tilde{A} = [\tilde{a}_{i,j}]_{1\le i,j\le 2} \in M_2(C(O_{-1}(2)))$ is a fundamental corepresentation of $C(O_{-1}(2)))$.

The following map yields a surjective *-homomorphism interpreted as $O_{-1}(2) \subset SO_{-1}(3)$:

$$a_{i,j} \mapsto \begin{cases} \tilde{a}_{i,j} & \text{for } 1 \le i,j \le 2 \\ \tilde{a}_{1,1}\tilde{a}_{2,2} + \tilde{a}_{1,2}\tilde{a}_{2,1} & \text{for } i=j=3 \\ 0 & \text{otherwise} \end{cases} \quad : C(SO_{-1}(3)) \twoheadrightarrow C(O_{-1}(2)) \tag{4.7}$$

But there are more embeddings $O_{-1}(2) \subset SO_{-1}(3)$. Their classification is contained in the following

Theorem 4.16. *There are three copies of $O_{-1}(2) \subset SO_{-1}(3)$. The three copies are conjugate (via an automorphism descibed in Theorem 4.14).*

Lemma 4.17. *The group of characters of $O_{-1}(2)$ is the dihedral group D_4, equal to*

$$\left\{\mathbb{1}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}, \begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}\right\}.$$

Proof. Let us evoke the conditions (2-3) of definition of $C(O_{-1}(2))$ (Definition 4.15):

2. $\tilde{a}_{i,j}\tilde{a}_{i,k} = -\tilde{a}_{i,k}\tilde{a}_{i,j}$ for $k \neq j$.

3. $\tilde{a}_{i,j}\tilde{a}_{k,j} = -\tilde{a}_{k,j}\tilde{a}_{i,j}$ for $k \neq i$.

After abelianization, these conditions can be written as: $\tilde{a}_{i,j}\tilde{a}_{k,l} = 0$ whenever (i,j) and (k,l) correspond to different entries in the same row or column. In short: in every column and in every row there is (at least) one zero entry. Together with the remaining relations from the definition of $C(O_{-1}(2))$, we see that $C(O_{-1}(2))_{ab}$ is the C^*-algebra of continuous functions on such 2×2 orthogonal matrices, which have one zero entry in each row and each column: this produces the above list of 8 matrices. As the list of groups of order 8 is known, it is enough to observe that this group is nonabelian and contains an order 4 element to conclude that it is isomorphic to D_4. ☐

Proof of the Theorem 4.16. Let $\Phi\colon H' = \mathrm{Pol}(SO_{-1}(3)) \to Pol(O_{-1}(2))$ be a Hopf *-algebra quotient. Consider the following diagram:

$$\begin{tikzcd} H' \arrow[rr, two heads, "\Phi"] \arrow[dr, "q_{SO_{-1}(3)}"'] & & \mathrm{Pol}(O_{-1}(2)) \arrow[dr, "q_{O_{-1}(2)}"] & \\ & C(S_4) \arrow[rr, two heads, "\varphi"] & & C(D_4) \end{tikzcd}$$

The existence of the map φ as in the diagram above follows from the universal property of abelianization. Because all the involved morphisms are Hopf *-algebra morphisms, φ is. Similarily, because all the involved morphisms are surjections, φ is. Thus $\hat{\varphi}$, the Gelfand transform of φ, is a monomorphism $\hat{\varphi}\colon D_4 \hookrightarrow S_4$. Let us take for granted that the image of $\hat{\varphi}$ contains the diagonal Klein subgroup of S_4 (the proof of this statement is postponed to Lemma 4.18 just below the end of the proof of Theorem 4.16).

As the diagonal Klein subgroup is characteristic in $SO_{-1}(3)$, this gives us the following diagram of morphisms:

$$\begin{tikzcd} H' \arrow[r, two heads, "\Phi"] \arrow[rr, bend right, two heads, "\pi'_d"'] & \mathrm{Pol}(O_{-1}(2)) \arrow[r, two heads, "\chi \circ \tilde{\pi}"] & K \end{tikzcd}$$

where $\tilde{\pi}$ is obtained by composing $q_{O_{-1}(2)}$ with Hopf *-algebra quotient map corresponding to restriction to the diagonal Klein subgroup in $\hat{\varphi}(D_4)$ and χ is the automorphism of the Klein group. Using Proposition 4.5, we untwist this diagram and arrive at

$$\begin{tikzcd} H \arrow[r, two heads, "{[\Phi]}"] \arrow[rr, bend right, two heads, "\pi_d"'] & \mathrm{Pol}(O(2)) \arrow[r, two heads, "\Pi"] & K \end{tikzcd}$$

The closed subgroups of $SO(3)$ isomorphic to $O(2)$ are all of the form

$$\left\{ F \begin{pmatrix} A & 0 \\ 0 & \det(A) \end{pmatrix} F^\top : A \in O(2) \right\}$$

for some matrix $F \in SO(3)$ (see, e.g. [26, Theorem 6.1]). The occurence of $O(2)$ in $SO(3)$ coming from the above diagram contains the diagonal Klein subgroup. Because $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq D_4$ is characteristic, we know from (the proof of) Theorem 4.14 that the matrices F are necessirely of the form $\rho(x)$ for some $x \in S_4$. To verify (4.7) it is then enough to check that

$$\begin{aligned} [\det(\tilde{X})] &= [\tilde{x}_{1,1}\tilde{x}_{2,2}] - [\tilde{x}_{1,2}\tilde{x}_{2,1}] = \\ &= \sigma(t_1,t_2)\sigma(t_1,t_2)[\tilde{x}_{1,1}][\tilde{x}_{2,2}] + \sigma(t_1,t_2)\sigma(t_2,t_1)[\tilde{x}_{1,2}][\tilde{x}_{2,1}] = \\ &= \tilde{a}_{1,1}\tilde{a}_{2,2} + \tilde{a}_{1,2}\tilde{a}_{2,1} = \mathrm{perm}(\tilde{A}) \end{aligned}$$

□

Lemma 4.18. *Image of $\hat{\varphi}$ contains the diagonal Klein subgroup of S_4.*

Proof. Up to an inner automorphism, the only way to embed the dihedral group into the symmetric group is via

$$\hat{\varphi}(D_4) = \{\mathrm{id}, (12), (34), (12)(34), (13)(24), (14)(23), (1234), (1432)\}$$

and the diagonal Klein subgroup of S_4 is precisely $\{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$, which is characteristic (hence it appears as a subgroup of any possible occurences of D_4 in S_4). □

In summary, the above Theorem says that any embedding $O_{-1}(2) \subset SO_{-1}(3)$ is given by the following map:

$$A \overset{\Phi_x}{\longmapsto} \rho(x) \begin{pmatrix} \tilde{A} & 0 \\ 0 & \mathrm{perm}(\tilde{A}) \end{pmatrix} \rho(x)^\top \tag{4.8}$$

where $x \in S_4$ and ρ is as in Lemma 4.10 and $\mathrm{perm}(\tilde{A}) = \tilde{a}_{1,1}\tilde{a}_{2,2} + \tilde{a}_{1,2}\tilde{a}_{2,1}$ is the permanent function.

4.3.7 On the embeddings $A_5^\tau \subset SO_{-1}(3)$

Proceeding similarily as in Section 4.3.6, we will now turn to studying embeddings of the other maximal proper subgroup of S_4^+, namely: A_5^τ. Recall that this group is obtained by cocycle twist as described in Section 4.3.2 by the following data: $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq A_4 \subseteq A_5$, where the second inclusion is obtained by making A_4 act on first four letters (out of five on which A_5 act).

Theorem 4.19. *There is a unique copy $A_5^\tau \subset SO_{-1}(3)$.*

Lemma 4.20 ([5, Lemma 6.8])**.** *The group of characters of A_5^τ is the alternating group A_4, equal to $\{\rho(x) : x \in A_4\}$ when viewed as a subgroup of $SO_{-1}(3)$.*

Proof of the Theorem 4.19. Let $\Phi\colon H' = \mathrm{Pol}(SO_{-1}(3)) \to C(A_5^\tau)$ be a Hopf *-algebra quotient. Consider the following diagram:

$$\begin{tikzcd}
H' \arrow[rr, two heads, "\Phi"] \arrow[dr, "q_{SO_{-1}(3)}"'] & & C(A_5^\tau) \arrow[dr, "q_{A_5^\tau}"] & \\
& C(S_4) \arrow[rr, two heads, "\varphi"] & & C(A_4)
\end{tikzcd}$$

The existence of the map φ is due to the same argument as in the proof of Theorem 4.16. Similarily, $\hat{\varphi}$, the Gelfand transform of φ, is a monomorphism $\hat{\varphi}\colon D_4 \hookrightarrow S_4$.

Lemma 4.21. *Image of $\hat{\varphi}$ contains the diagonal Klein subgroup of S_4.*

Proof. There is a unique monomorphism $A_4 \to S_4$ and the conclusion follows. $\square$

As previously, we obtain the following diagram of morphisms:

$$\begin{tikzcd}
H' \arrow[r, two heads, "\Phi"] \arrow[rr, bend right, "\pi'_d"'] & C(A_5^\tau) \arrow[r, two heads, "\tilde{\pi}"] & K
\end{tikzcd}$$

where $\tilde{\pi}$ is obtained by composing $q_{A_5^\tau}$ with Hopf *-algebra quotient map corresponding to restriction to the diagonal Klein subgroup in $\hat{\varphi}(A_4)$. Using Proposition 4.5, we untwist this diagram and arrive at

$$\begin{tikzcd}
H \arrow[r, two heads, "{[\Phi]}"] \arrow[rr, bend right, "\pi_d"'] & A_5 \arrow[r, two heads, "\Pi"] & K
\end{tikzcd}$$

Recall that there are five possible embeddings $A_4 \hookrightarrow A_5$ coming from letting A_4 fix one of the letters $\{1, \dots, 5\}$ on which A_5 acts (and acting on the remaining four in any way, as renaming the four non-fixed among $\{1, \dots, 5\}$ amounts to renaming the four on which A_4 acts canonically). Let us remark that [5, Lemma 4.2] shows that the quantum group A_5^τ does not depend on the choice of embedding $A_4 \subset A_5$, here we show a stronger statement: that $A_5^\tau \subset S_4^+$ does not depend on the choice of embedding $A_4 \subset A_5$.

All the occurences of A_5 as a subgroup of $SO(3)$ correspond to fixing a dodecahedron (or its dual graph, icosahedron), then A_5 is the group of rotations that preserve this fixed dodecahedron, call it $\mathfrak{D}$. But observe that the copy of A_5 we got by the untwisting procedure has additional feature: it contains the canonical copy of $A_4 \subseteq S_4$. This particular occurence of S_4 is the group of rotations preserving the octahedron spanned by $\{\pm e_1, \pm e_2, \pm e_3\}$, call its dual graph – which is a cube – $\mathfrak{C}$. Consider the tetrahedron, whose four vertices are the four non-adjecent vertices of $\mathfrak{C}$ – there are two tetrahedrons like that, call them $\mathfrak{T}_1, \mathfrak{T}_2$. Then $A_4 \subseteq S_4$ consists of those rotations that preserve one (equivalently, both) of tetrahedrons $\mathfrak{T}_1, \mathfrak{T}_2$. In other words, the elements of S_4

are precisely rotations generated by rotations by a multiple of $\frac{\pi}{2}$ around one of the coordinate axis and elements of A_4 are rotations generated by rotations by multiple of π around one of coordinate axis.

Regarding the A_5 group, the dodecahedron $\mathfrak{D}$ has the following feature: as $A_4 \subseteq A_5$, the rotations of $\mathfrak{D}$ that come from A_4 must preserve the system of tetrahedrons $\mathfrak{T}_1, \mathfrak{T}_2$. Observe (see Figure) that there are precisely five ways of incribing a cube into a dodecahedron, or in other words: five ways of circumscribing a dodecahedron about a cube. Graphically speaking, this amounts to fixing an edge of the cube $\mathfrak{C}$ and a base of dodecahedron $\mathfrak{b} \subseteq \mathfrak{D}$. It has five edges, and the five dodecahedrons are obtained by attaching the cube $\mathfrak{C}$ to five different edges of the base $\mathfrak{b}$, see Figure. But these five dodecahedrons differ only by a rotation of this fixed base $\mathfrak{b}$ and hence are the same polyhedra. It follows that this unique polyhedron correspond to a unique $A_5 \subseteq SO(3)$ containing this distinguished copy of A_4. Now as there are 5 possible embeddings $A_4 \subset A_5$ (which are conjugate) and we realized these five embeddings as five ways of circumcribing a dodecahedron about a cube, we conclude that there are no other intermediate A_5 group satisfying $A_4 \subset A_5 \subset SO(3)$.

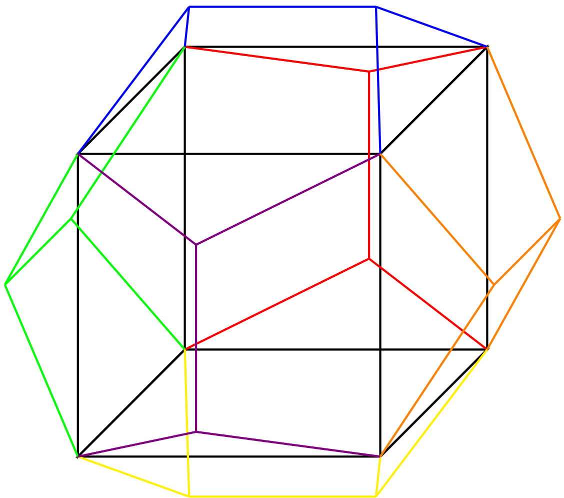

Figure: Cube inscribed into dodecahedron

□

Remark 4.22. The proof of Theorem 4.19 was more complicated than the proof of Theorem 4.16 because $\mathbb{Z}_2 \times \mathbb{Z}_2 \subseteq A_5$ is not characteristic.

4.4 Lack of Property (FAG) for S_4^+ and other consequences

The main motivation for the Property (FAG) was an error in the proof [5, Lemma 6.6]. The strategy of Banica and Bichon in [5] for establishing the list of all subgroups of S_4^+ was, after establishing the list of subgroups containing the diagonal Klein subgroup in Theorem 4.7, to show that every other subgroup of S_4^+ is already a subgroup of one of the smaller subgroup (thus reducing the complexity of the task). It was further completed in [12]: the only subgroups of S_4^+ that are not present in the list of Theorem 4.7 are proper subgroups of $O_{-1}(2)$ (whose subgroups were worked out in [5, Section 7]). We started investigations concerning Property (FAG) because Theorem 3.18 might serve as an alternative, and easy, way to do this. However, we found that it is impossible to use this strategy, as

Proposition 4.23. *S_4^+ does not enjoy Property (FAG).*

Proof. From Theorem 4.7 we know that $S_4^\tau \subseteq S_4^+$ and that $Gr(\widehat{S_4^\tau}) = D_4$ by [5, Lemma 6.7]. Let then $O_{-1}(2) \subseteq S_4^+$ is embedded in such a way that $Gr(\widehat{O_{-1}(2)}) = Gr(\widehat{S_4^\tau})$ as subgroups of

$S_4 = Gr(\widehat{S_4^+})$ (we know from Theorem 4.16 that it is possible to find such a copy of $O_{-1}(2)$). But [5, Theorem 7.1], establishing the full list of subgroups of $O_{-1}(2)$, ensures us that $S_4^\tau \not\subset O_{-1}(2)$ and thus $\mathbb{G} = \overline{\langle O_{-1}(2), S_4^\tau \rangle} = S_4^+$, as this group as strictly bigger than $O_{-1}(2)$. But at the same time, if S_4^+ had (FAG), then

$$Gr(\widehat{\mathbb{G}}) = \langle D_4, D_4 \rangle = D_4 \neq S_4$$

$\square$

An additional consequence of our considerations is the following

Proposition 4.24. $S_4^+ = \langle A_5^\tau \cup S_4 \rangle$

Proof. Let $\mathbb{G} = \langle A_5^\tau \cup S_4 \rangle$. As $Gr(\widehat{\mathbb{G}}) = S_4$ and $\mathbb{G} \neq S_4$ (because $A_5^\tau \not\subset S_4$), we can check on the list of Theorem 4.7 that the only remaining quantum subgroup of S_4^+ with $Gr(\mathbb{G}) = S_4$ is S_4^+ itself. $\square$

Corollary 4.25. $\widehat{S_4^+}$ *is hyperlinear.*

Remark 4.26. Recall $\widehat{\mathbb{G}}$ is hyperlinear if and only if $L^\infty(\mathbb{G}, h_\mathbb{G}) \hookrightarrow \mathsf{R}^\omega$, where R is the hyperfinite II_1 factor and ω is a principal ultrafilter.

Proof. This follows immediatly from the fact that S_4 and A_5^τ are finite (and hence their duals are hyperlinear) and [15, Theorem 3.6]. $\square$

Let us mention here that this result can be also proven by employing the fact that $C(S_4^+)$ is nuclear ($\widehat{S_4^+}$ is amenable). However, the above proof, as the proof of [15, Theorem 3.6], are much more elementary than the proof of nuclearity of $C(S_4^+)$. Furthermore, it is not known whether $\widehat{S_n^+}$ are hyperlinear for $n \geq 5$. A resent anouncement of Alex Chirvasitu shows that $S_n^+ = \overline{\langle S_n, S_{n-1}^+ \rangle}$ for $n \geq 6$, hence if one shows hyperlinearity of S_5^+, the whole family will turn out to be hyperlinear by [15, Theorem 3.6].

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