Two Dimensional YD-modules over $U_q(sl_2)$ are trivial

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Introduction

Assume q is not a root of unity, X. W. Chen and P. Zhang embed $U_q(sl_2)$ into the path coalgebra of the Gabriel quiver D of the coalgebra of $U_q(sl_2)$.

They also describe the category of $U_q(sl_2)$-comodules in terms of representations of the quiver D.

I will present examples of comodules over $U_q(sl_2)$, and show that all YD-modules over $U_q(sl_2)$ are trivial.

Throughout this presentation, k denotes a field of characteristic zero.

The Algebra $U_q(sl_2)$

We define $U_q(sl_2)$ as the algebra generated by the four variables E, F, K, K^{-1} with the relations;

$KK^{-1} = K^{-1}K = 1$

$KEK^{-1} = q^2E$, $KFK^{-1} = q^{-2}F$, and

$[E,F] = \frac{K-K^{-1}}{q-q^{-1}}$

Note that the algebra U_q is Noetherian and has no zero divisors. The set $\{E^iF^jK^l\}_{i,j\in N;l\in Z}$ is a basis of U_q.

The Hopf Algebra Structure on $U_q(sl_2)$

$U_q(sl_2)$ has a Hopf structure with

$$\Delta(E) = 1 \otimes E + E \otimes K, \qquad \Delta(K) = K \otimes K$$
$$\Delta(F) = K^{-1} \otimes F + F \otimes 1, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1}$$

$$\varepsilon(E) = \varepsilon(F) = 0, \qquad \varepsilon(K) = \varepsilon(K^{-1}) = 1$$

$$S(E) = -EK^{-1}, \qquad S(F) = -KF, \qquad S(K) = K^{-1}, \qquad S(K^{-1}) = K.$$

The Path Coalgebra kQ^c

A quiver $Q = (Q_0, Q_1, s, t)$ is a datum, where Q is an oriented graph with
Q_0 the set of vertices and
Q_1 the set of arrows, s and t are two maps from Q_1 to Q_0, such that
$s(a)$ and $t(a)$ are respectively the starting vertex and terminating vertex of
$a \in Q_1$.

A path p of length l in Q is a sequence $p = a_l...a_2a_1$ of arrows a_i, $1 \leq i \leq l$.

A vertex is regarded as a path of length 0.

Given a quiver Q, one defines the path coalgebra kQ^c as follows:

- the underlying space has as basis the set of all paths in Q,
- the coalgebra structure is given by

$$\Delta(p) = \sum_{\beta\alpha = p} \beta \otimes \alpha,$$

$\varepsilon(p) = 0$ if $l \geq 1$,

$\varepsilon(p) = 1$ if $l = 0$ for each path p of length l.

By a graded coalgebra one means a coalgebra C with decomposition $C = \bigoplus_{n \geq 0} C(n)$ of k-space such that

$$\Delta(C(n)) \subseteq \sum_{i+j=n} C(i) \otimes C(j)$$

$$\varepsilon(C(n)) = 0$$

for all $n \geq 1$. Note that a path coalgebra kQ^c is graded with length grading, and it is coradically graded, and

$$kQ^c \simeq Cot_{kQ_0}(kQ_1)$$

Proposition :

Let $C = \bigoplus_{n \geq 0} C(n)$ be a graded coalgebra. Then

(i) There is a unique graded coalgebra map $\theta : C \to Cot_{C(0)}C(1)$ such that $\theta|_{C(i)} = id$ for $i = 0, 1$.

(ii) $\theta(x) = \pi^{\otimes n+1} \circ \Delta^n(x)$ for all $x \in C(n+1)$ and $n \geq 1$, where $\pi : C \to C(1)$ is the projection, and $\Delta^n = (Id \otimes \Delta^{n-1}) \circ \Delta$ for all $n \geq 1$, with $\Delta^0 = id$.

$U_q(sl_2)$ as a Subcoalgebra of a Path Coalgebra

$U_q(sl_2) = \bigoplus_{n\geq 0} C(n)$ is a graded coalgebra with

$$C(0) = \bigoplus_{l\in\mathbb{Z}} kK^l$$

and $C(1)$ has a basis

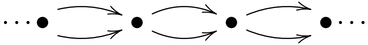
$$\{K^l E, K^l F | l \in \mathbb{Z}\}$$

One has in $C(1)$

$$\Delta(K^{l-1}E) = K^{l-1} \otimes K^{l-1}E + K^{l-1}E \otimes K^l$$

$$\Delta(K^l F) = K^{l-1} \otimes K^l F + K^l F \otimes K^l$$

The quiver D of $U_q(sl_2)$ is of the form



Vertices are labelled by integers, i.e., $D_0 = \{e_l | l \in \mathbb{Z}\}$. Write v as $v = (v_1, ..., v_n)$, where $v_j = 1$ or -1 for each j. Define

$$P_l^{(v)} = a_{|v|} ... a_2 a_1$$

to be the concatenated path in D starting at e_l of lenght $|v|$.

For instance,

$$P_l^{(0)} = e_l,$$

$$P_l^{(1)} = \bullet \longrightarrow \bullet\ ,$$

$P_l^{(1,-1)} = \bullet \longrightarrow \bullet \longrightarrow \bullet$ with starting at the vertex e_l in D.

One can write the set of all paths in D as follows:

$$\{P_l^{(v)} = P_{l-|v|+1}^{(v_{|v|})} \dots P_{l-1}^{(v_2)} P_l^{(v_1)} | l \in \mathbb{Z}, v \in I\}$$

Lemma :

There is a unique graded coalgebra map $\theta : U_q(sl_2) \to kD^c$ such that

$$\theta(K^l) = e_l$$

$$\theta(K^{l-1}E) = P_l^{(1)}$$

$$\theta(K^l F) = P_l^{(-1)}$$

for each integer l.

Theorem :

Assume that q is not a root of unity. Then as a coalgebra $U_q(sl_2)$ is isomorphic to the subcoalgebra of kD^c with the basis

$$\{b(l,n,i) | 0 \leq i \leq n, n \in \mathbb{N}_0, l \in \mathbb{Z}\}$$

$$b(l,n,i) := \sum_{v\in\{\pm 1\}^n, |T_v|=i} \chi(v) P_l^{(v)} \in kD^c$$

where $T_v := \{t | 1 \le t \le n, v_t = 1\}$, and

$\chi(v) := q^{2\sum_{t\in T_v} t}$, if $n \ge 1$, $T_v \ne \emptyset$;

$\chi(v) := 1$, otherwise.

For instance,

$$\begin{aligned} b(l,0,0) &= e_l & b(l,1,0) &= P_l^{(-1)} \\ b(l,1,1) &= q^2 P_l^{(1)} & b(l,2,0) &= P_l^{(-1,-1)} \\ b(l,2,2) &= q^6 P_l^{(1,1)} & b(l,2,1) &= q^2 P_l^{(1,-1)} + q^4 P_l^{(-1,1)} \end{aligned}$$

Comodules of $U_q(sl_2)$

Representations of Quivers

A k-representation of Q is a datum $V = (V_e,\ f_a;\ e \in Q_0,\ a \in Q_1)$,

- V_e is a k-space for each vertex $e \in Q_0$,
- $f_a : V_{s(a)} \to V_{t(a)}$ is a k-linear map for each arrow $a \in Q_1$.

Set $f_p := f_{a_l} \circ \cdots \circ f_{a_1}$ for each path $p = a_l \ldots a_1$, where each a_i is an arrow, $1 \le i \le l$, and $f_e := id$ for $e \in Q_0$

The standard comodule structure on a quiver representation

Let $V = (V_e,\ f_a;\ e \in Q_0,\ a \in Q_1)$ be a representation of a quiver Q, one defines a kQ^c-comodule structure $\rho : V \to V \otimes kQ^c$ as follows;

$$\rho(m) = \sum_{s(p)=e} f_p(m) \otimes p \quad \text{for every } m \in V_e$$

Theorem :

Assume that q is not a root of unity. Then there is an equivalence between the category of the right $U_q(sl_2)$-comodules and the full subcategory of representation of D with the standard comodule structures that satisfy the following conditions:

(i) $f_{l-1}^{(1)} \circ f_l^{(-1)} = q^2 f_{l-1}^{(-1)} \circ f_l^{(1)}$

(ii) For any $m \in V_l$, $f_l^{(m)} = 0$ for all but finitely many paths.

Example

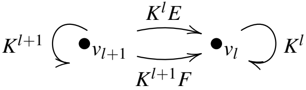
Let l be an integer and n a non-negative integer. For each $\lambda \in k$, one can define a representation V of quiver D as follows:

$$
\begin{aligned}
V_j &:= k, && \text{if} \quad l \leqslant j \leqslant l+n \\
V_j &:= 0, && otherwise; \\
f_j^{(1)} &:= 1, && \text{if} \quad l+1 \leqslant j \leqslant l+n \\
f_j^{(1)} &:= 0, && otherwise; \\
f_j^{(-1)} &:= \lambda q^{-2(l+n-j)}, && \text{if} \quad l+1 \leqslant j \leqslant l+n \\
f_j^{(-1)} &:= 0, && otherwise.
\end{aligned}
$$

And V has an induced right $U_q(sl_2)$ comodule structure

$$\rho(m) = \sum_{s(p)=e} f_p(m) \otimes p$$

which is denoted by $M_{(l,n,\lambda)}$. Let's write these explicitly for $n = 1$;



$$\rho(v_l) = v_l \otimes K^l$$

$$\rho(v_{l+1}) = v_{l+1} \otimes K^{l+1} + v_l \otimes K^l E + \lambda v_l \otimes K^{l+1} F$$

Theorem :

The comodules $M_{(l,n,\lambda)}$ give a complete list of all non-isomorphic, indecomposable Schurian right $U_q(sl_2)$ comodules, where $l \in \mathbb{Z}$, $n \in \mathbb{N}_0$, $\lambda \in (k \cup \{\infty\})$.

A finite-dimensional right $U_q(sl_2)$ comodule (M, ρ) is said to be Schurian, if $dim_k M_j = 1$ or 0 for each integer j, where $M_j := \{m \in M | (Id \otimes \pi_0)\rho(m) = m \otimes e_j\}$ and π_0 is the projection from kD^c to kD_0.

An Example of Two Dimensional YD module over $U_q(sl_2)$

Let $V := M_{(l,1,\lambda)}$ be a two dimensional comodule over $U_q(sl_2)$ and also take a two dimensional module V is generated by w_1 and w_{-1} with the following structure:

$$K^{\pm 1} w_1 = q^{\pm 1} w_1, \qquad K^{\pm 1} w_{-1} = q^{\mp 1} w_{-1}$$

$$E w_1 = 0, \qquad E w_{-1} = w_1$$

$$F w_1 = w_{-1}, \qquad F w_{-1} = 0$$

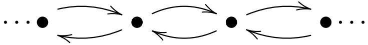
Now let us try to match the module and comodule structures...

Conjecture

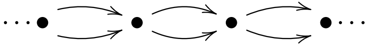
$U_q(sl_2)$ has no irreducible module-comodules of dimension 2 or greater.

Idea of a proof

One knows that representations of sl_2 and $U_q(sl_2)$ are in one-to-one correspondence [Kassel, 1995]. Moreover, irreducible representations come from a specific quiver [Mazorchuk, 2010];



One also knows that the irreducible comodules of $U_q(sl_2)$ come from the following quiver;



I suspect that one cannot make these structures compatible on the same vector space.

In Closing

THANK YOU FOR LISTENING :)