BRAIDED JOIN COMODULE ALGEBRAS OF GALOIS OBJECTS

Piotr M. Hajac

(IMPAN / Warsaw University / University of New Brunswick)

Joint work with L. Dąbrowski, T. Hadfield, E. Wagner

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Goal and plan

Motivation: Go beyond the noncommutative join construction for Hopf algebras (compact quantum groups) to include Galois objects (quantum torsors).

Applications:

1. Quantum coverings from anti-Drinfeld doubles that are used in Hopf-cyclic theory with coefficients.
2. Quantum-torus bundles with potential use in the new machinery of Dąbrowski, Sitarz and Zucca for constructing Dirac operators.

Plan:

1. Recall the basics: classical joins, braidings, Galois objects.
2. Show that the diagonal coaction of noncommutative Hopf algebras on the braided tensor product of Galois objects is a homomorphism of algebras.
3. Construct a braided join comodule algebra of a bi-Galois object, and show that it is a principal comodule algebra for the diagonal coaction. Apply to anti-Drinfeld doubles.
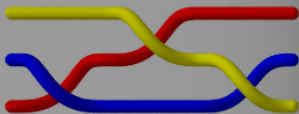
The join $X * Y$ of compact Cartan principal G-bundles X and Y (local triviality not assumed) is again a compact Cartan principal G-bundle for the diagonal G-action on $X * Y$:
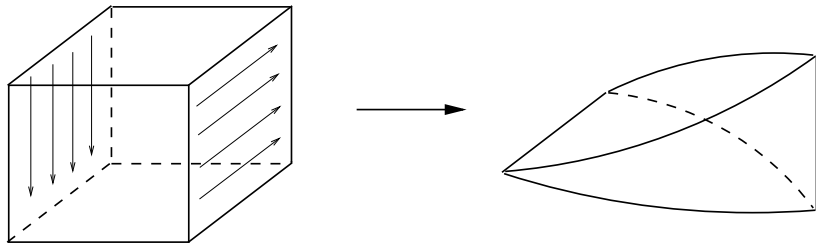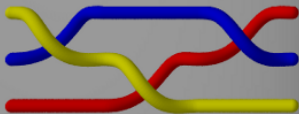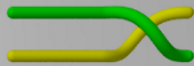
Braidings of algebras

A factorization of two algebras A and A' is a linear map
$\sigma : A' \otimes A \longrightarrow A \otimes A'$ such that

1. $\forall\, a \in A,\ a' \in A':\ \sigma(1 \otimes a) = a \otimes 1$ and $\sigma(a' \otimes 1) = 1 \otimes a'$,

2. $\sigma \circ (m' \otimes \mathsf{id}) = (\mathsf{id} \otimes m') \circ \sigma_{12} \circ \sigma_{23}$,
   $\sigma \circ (\mathsf{id} \otimes m) = (m \otimes \mathsf{id}) \circ \sigma_{23} \circ \sigma_{12}$.

Here m and m' are multiplications in A and A' respectively. If in addition $A' = A$ and the braid equation

$$\sigma_{12} \circ \sigma_{23} \circ \sigma_{12} = \sigma_{23} \circ \sigma_{12} \circ \sigma_{23}$$

is satified, we call σ a braiding.

Factorizations classify all possible associative algebra structures on the vector space $A \otimes A'$ such that A and A' are included in $A \otimes A'$ as unital subalgebras:

$$m_{A \otimes A'} = (m \otimes m') \circ (\mathsf{id} \otimes \sigma \otimes \mathsf{id})\,.$$

Left and right Hopf-Galois extensions

Let H be a Hopf algebra, and P a left (right) H-comodule algebra with coaction

$$_P\Delta(x) = x_{(-1)} \otimes x_{(0)} \quad \text{(left)},$$
$$\Delta_P(x) = x_{(0)} \otimes x_{(1)} \quad \text{(right)}.$$

Define the left (right) coaction-invariant subalgebra as

$$B := {}^{co\,H}P := \{x \in P \mid {}_P\Delta(x) = 1 \otimes x\} \quad \text{(left)},$$
$$B := P^{co\,H} := \{x \in P \mid \Delta_P(x) = x \otimes 1\} \quad \text{(right)}.$$

Definition

We call P a left (right) H-Galois extension of B iff the left (right) canonical map

$$can_L : P \underset{B}{\otimes} P \ni x \otimes y \longmapsto x_{(-1)} \otimes x_{(0)}y \in H \otimes P \quad \text{(left)},$$

$$can_R : P \underset{B}{\otimes} P \ni x \otimes y \longmapsto xy_{(0)} \otimes y_{(1)} \in P \otimes H \quad \text{(right)},$$

is a bijection.

Durdevic braiding

> **Theorem (M. Durdevic)**
>
> *Let P be a left H-Galois extension of B. Then the linear map*
> $$\sigma \colon P \underset{B}{\otimes} P \ni x \otimes y \longmapsto y_{(-1)}{}^{[1]} \otimes y_{(-1)}{}^{[2]} x \, y_{(0)} \in P \underset{B}{\otimes} P$$
> *satisfies the braid equation and factorization identities. Here*
> $h^{[1]} \otimes h^{[2]} := can_L^{-1}(h \otimes 1)$.

Note that σ becomes a flip when P is commutative.

Special cases:

❶ Comodule algebra P is a left H-Galois extension of the ground field (i.e. left Galois object). This is the case we are to explore.

❷ Comodule algebra P is a Hopf algebra H. Then the Durdevic braiding coincides with the Yetter-Drinfeld braiding:
$$\sigma(a \otimes b) = b_{(1)} \otimes S(b_{(2)})ab_{(3)},$$
where S is the antipode of H.

Braiding left Galois objects

Let $\sigma \colon A \otimes A \to A \otimes A$ be a braiding. We denote the vector space $A \otimes A$ with the algebra structure determined by the braiding as $A \underline{\otimes} A$, and call it a braided tensor product algebra.

> **Lemma (Key lemma)**
>
> *Let H be a Hopf algebra and A a bicomodule algebra over H (left and right coactions commute). Assume that A is a left Galois object over H, and that $A \underline{\otimes} A$ is the tensor product algebra braided by the Durdevic braiding. Then the right diagonal coaction*
>
> $$\Delta_{A \underline{\otimes} A} \colon A \underline{\otimes} A \ni a \underline{\otimes} a' \longmapsto a_{(0)} \underline{\otimes} a'_{(0)} \otimes a_{(1)}\, a'_{(1)} \in A \underline{\otimes} A \otimes H$$
>
> *is an algebra homomorphism.*

Braided noncommutative join construction

Definition

Let H be a Hopf algebra and A a bicomodule algebra over H. Assume that A is a left Galois object over H. We call the unital $\mathbb{C}$-algebra

$$A \underset{H}{*} A := \left\{ x \in C([0,1]) \otimes A \underline{\otimes} A \;\middle|\; \begin{array}{l} (\mathrm{ev}_0 \otimes \mathsf{id})(x) \in \mathbb{C} \otimes A, \\ (\mathrm{ev}_1 \otimes \mathsf{id})(x) \in A \otimes \mathbb{C} \end{array} \right\}$$

the H-braided noncommutative join algebra of A. Here ev_r is the evaluation map at $r \in [0,1]$, i.e. $\mathrm{ev}_r(f) = f(r)$.

Main theorem

Lemma

Let $A *_H A$ be the H-braided join algebra of A. Then the formula

$$C([0,1]) \otimes A\underline{\otimes}A \longrightarrow C([0,1]) \otimes A\underline{\otimes}A \otimes H \,,$$
$$f \otimes a \otimes b \longmapsto f \otimes a_{(0)} \otimes b_{(0)} \otimes a_{(1)}b_{(1)} \,,$$

restricts and corestricts to $\delta \colon A *_H A \to (A *_H A) \otimes H$ making it a right H-comodule algebra.

Theorem

Let $A *_H A$ be the H-braided join comodule algebra of A. Assume that the antipode of H is bijective and that A is also a right Galois object. Then the coaction

$$\delta \colon A *_H A \longrightarrow (A *_H A) \otimes H$$

is principal, i.e. the canonical map it induces is bijective and P is H-equivariantly projective as a left B-module. Furthermore, the coaction-invariant subalgebra B is the unreduced suspension of H.

Anti-Drinfeld doubles

Let H be a finite-dimensional Hopf algebra. The multiplication of the anti-Drinfeld double algebra $A(H) := H^* \otimes H$ is defined by

$$(\varphi \otimes h)(\varphi' \otimes h') = \varphi'_{(1)}(S^{-1}(h_{(3)}))\varphi'_{(3)}(S^2(h_{(1)})) \; \varphi\varphi'_{(2)} \otimes h_{(2)}h' \, .$$

Theorem

Let H be a finite-dimensional Hopf algebra. Then the anti-Drinfeld double $A(H)$ is a bicomodule algebra and a left and right Galois object over the Drinfeld double $D(H)$ for coactions given respectively by the formulas:

$$_{A(H)}\Delta(\psi \otimes k) = \psi_{(2)} \otimes S^2(k_{(1)}) \otimes \psi_{(1)} \otimes k_{(2)} \, ,$$
$$\Delta_{A(H)}(\varphi \otimes h) = \varphi_{(2)} \otimes h_{(1)} \otimes \varphi_{(1)} \otimes h_{(2)} \, .$$

IMPAN International Fellowship Programme

Impan International Fellowship Programme (IMPACT)
Coordinator: Piotr M. Hajac

e-mail: P.M.Hajac@impan.pl

IMPACT is a 5-year long International Postdoctoral Fellowship Programme co-financed by the EU through the Marie Curie Action "Co-funding of regional, national and international programmes (COFUND)". It is also partially supported by the Polish Ministry of Science and Higher Education. IMPACT is currently the only COFUND-project run in Poland.

The programme offers the total of 10 years of fellowships, which average to 2 positions each year. All researchers not residing in Poland and working in any area of mathematics are eligible to apply. IMPACT is directed and coordinated by Piotr M. Hajac, Feliks Przytycki, and Andrzej Sitarz. The contact person for practical details is Monika Wysocka. Current IMPACT postdoctoral fellows are Michal Doucha and Liviana Palmisano. Here are their profiles:

POSTDOCTORAL FELLOWS IN IMPACT

I studied at the Institute of Mathematics of the Czech Academy of Sciences and the Faculty of Mathematics and Physics of the Charles University in Prague under the supervision of *Jindřich Zapletal*. I am interested in applications of logic to other areas of mathematics. In particular, I specialize in descriptive set theory and countable and metric model theory. I try to follow the recent program of applying techniques from these disciplines to areas such as functional analysis, topological group theory, topological dynamics, etc., especially in cases where classical methods were not successful.

My project in IM PAN consists mainly of extending the countable model theoretic methods (Fraïssé theory, Hrushovski construction, etc.), that were originally used to produce some countable discrete structures with certain properties, to a non-discrete metric (topological) context. Thus far, this led to constructions of certain universal Polish groups that are of interest to topological group theorists.

Another (closely related) activity is my search for a connection between model-theoretic properties of homogeneous structures and topological properties of their automorphism groups. A lot of research has been recently done on non-Archimedean Polish groups (closed subgroups of the permutation group of integers) as they precisely correspond to automorphism groups of ultrahomogeneous discrete countable structures. This correspondence turned out to be very useful. Similarly, general Polish groups correspond to automorphism groups of "almost ultrahomogeneous" metric structures, and this is a connection I plan to explore.

If one has a clear plan what to work on, and an idea how to approach problems he/she wants to solve (including optimism that it will work out), then working as an IMPACT-fellow enables him/her to fully focus on that project in a great working environment in IM PAN. There are no other duties than working on the project, and possibilities to travel and present results at conferences all year round.

Michal Doucha

I am an IMPACT postdoctoral fellow at Institute of Mathematics of PAN in Warsaw. I defended my Ph.D.-thesis "On the circle functions with a flat interval and Cherry flows" on 12 December 2013 at the Mathematical Department of the University of Orsay (Paris XI).

My research work is essentially concentrated on dynamical systems in low dimensions. The centerpiece of my interest is a class of circle endomorphisms with a flat interval and its applications. The most interesting application is the study of a flow on the bi-dimensional torus the Cherry flow. I study its topological, metrical and ergodic properties. This type of problems originate in theoretical physics, in particular in mechanics.

My research can be considered to be at the interface between the more classical techniques in real dynamical systems theory and the more modern techniques in ergodic dynamics and theoretical physics.

At IMPAN there is an active research group in low dimensional dynamical systems. The research themes of this group corresponds very well to my current scientific interests concentrated on dynamics of the Cherry flows. My stay in Warsaw is provides me with a great opportunity to discuss and broaden my current research interests.

Liviana Palmisano