Quantum permutations of two elements

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1. Frobenius algebras

Let k be an arbitrary base field.

Theorem (Nakayama)

The following are equivalent:

1. *A is a Frobenius algebra, i.e., $A \hookrightarrow A^* = \mathrm{Hom}(A, k)$ as left A-modules.*
2. *There exists an algebra automorphism $\delta : A \to A$ and a linear functional $\tau : A \to k$ such that*
$$\tau(aa') = \tau(a'\delta(a))$$
whose kernel contains no nonzero ideals.
3. *There exists a nondegenerate bilinear form $B : A \times A \to k$ such that*
$$B(aa', a'') = B(a, a'a'').$$

Nakayama automorphism and twisted trace

- The automorphism δ of a Frobenius algebra as above is uniquely determined by τ and is called the Nakayama automorphism.The class of δ up to inner automorphisms of A is independent of the choice of τ.
- A pair (δ, τ) consisting of an automorphism δ and a functional τ such that

$$\tau(aa') = \tau(a'\delta(a))$$

is called a twisted trace.

Examples of Frobenius algebras

- Every finite dimensional semisimple algebra A admits a functional τ coming from traces on simple factors and $\delta = \mathrm{id}$ (by Wedderburn theory).
- The cohomology algebra A of a smooth closed oriented n-fold X admits a functional τ coming from the cap-product with the fundamental class $[X]$ and an automorphism δ coming from the grading, *i.e.*

$$\tau(a) := [X] \frown a = \int_{[X]} a, \quad \delta(a) := (-1)^{p(n-1)} a$$

  if a is homogeneous of degree p (by Poincaré duality).
- Every 2-dimensional topological quantum field theory is equivalent to a commutative Frobenius algebra with trivial Nakayama automorphism.
- Every finite dimensional Hopf algebra admits a Frobenius structure (Larson-Sweedler Theorem).

Quantum family of algebra automorphisms

Let $\delta_F : A \to F \otimes A$ be a quantum family of algebra automorphisms of an algebra A parameterized by $\mathrm{Spec}(F)$, *i.e.*

- δ_F is an algebra map,
- the induced map $F \otimes A \to F \otimes A$, $f \otimes a \mapsto f\delta_F(a)$ is bijective.

Example. For any left H-comodule algebra A and any algebra map $\gamma : H \to F$ the algebra map

$$\delta_F : A \to F \otimes A, \quad a \mapsto \gamma(a_{(-1)}) \otimes a_{(0)}$$

induces a bijective map

$$F \otimes A \to F \otimes A, \quad f \otimes a \mapsto f\gamma(a_{(-1)}) \otimes a_{(0)}$$

with the inverse $f \otimes a \mapsto f\gamma(S(a_{(-1)})) \otimes a_{(0)}$.

Quantum family of twisted traces

Let $(\delta_F : A \to F \otimes A, \;\; \tau_F : A \to F)$ be a quantum family of twisted traces on an algebra A parameterized by $\mathrm{Spec}(F)$, i.e.

- δ_F be a quantum family of algebra automorphisms as above,
- $\tau_F(aa') = \tau_F(a'\delta(a))$,

where $a'(f \otimes a) := f \otimes a'a$ and τ_F on the right hand side is regarded as a left F-linear map $F \otimes A \to F$.

Support of a twisted trace

Definition

We say that a twisted trace $(\delta : A \to A, \quad \tau : A \to k)$ is supported on a quantum closed subspace corresponding to the ideal $I \subset A$, if $\tau(I) = 0$.
We define the support $\mathrm{Supp}(\tau)$ of this twisted trace as the maximal quantum closed subspace of $\mathrm{Spec}(A)$ on which that twisted trace is supported.

- It corresponds to the ideal $I(\tau) := \{a' \in A \mid \forall a \in A \quad \tau(aa') = 0\}$ and $\mathrm{Supp}(\tau) = \mathrm{Spec}(A/I(\tau))$.
- If $\mathrm{Supp}(\tau) = \mathrm{Spec}(A)$ τ is called entire.
- This means that $I(\tau) = 0$ and implies that the linear map

$$A \to A^* = \mathrm{Hom}(A, k), \quad a \mapsto (a' \mapsto \tau(aa')) \tag{1}$$

is injective.

- If A is finite dimensional the entire twisted trace is equivalent to a Frobenius structure on A and then the automorphism δ coincides with the Nakayama automorphism.
- The fact that $\tau(I(\tau)) = 0$ implies that τ defines a canonical Frobenius structure on $A/I(\tau)$.
  In particular, δ induces the Nakayama automorphism on $A/I(\tau)$.

Quantum Radon-Nikodym derivative with respect to a twisted trace

Let $(\delta : A \to A, \ \ \tau : A \to k)$ be a twisted trace on an algebra A.
We say that a quantum family of linear functionals $\varphi_F : A \to F$ parameterized by $\mathrm{Spec}(F)$ is Radon-Nikodym differentiable with respect to τ, if there exists an element $d\varphi_F/d\tau \in F \otimes A/I(\tau)$ such that for all $a \in A$

$$\varphi_F(a) = (F \otimes \tau)(a\frac{d\varphi_F}{d\tau}), \tag{2}$$

where on the right hand side $a(f \otimes a') := f \otimes aa'$.
Note that whenever $d\varphi_F/d\tau$ exists, it is unique (and well defined by (2)).
We call it Radon-Nikodym derivative of φ_F with respect to τ.

Radon-Nikodym differentiable structure on a quantum affine scheme

We define the quantum Radon-Nikodym differentiable structure on $\mathrm{Spec}(A)$ as a poset consisting of twisted traces on A, such that for any two traces τ, τ' in this category a morphism $\tau' \to \tau$ exists if and only if there exist a closed embedding $\mathrm{Supp}(\tau') \subset \mathrm{Supp}(\tau)$ (this means that $I(\tau) \subset I(\tau')$), and the Radon-Nikodym derivative $d(\tau' \mid_{\mathrm{Supp}(\tau)})/d\tau$.

The composition is defined in a natural way.

Fundamental cycle of a finite quantum space

After setting a quantum Radon-Nikodym differentiable structure on a finite dimensional algebra A we define the fundamental cycle on $\mathrm{Spec}(A)$ as an isomorphism class of a chosen entire trace in this poset.
For any entire trace τ in this isomorphism class we say that τ represents the fundamental cycle.

Quantum group Radon-Nikodym differentiable action

Given a left H-coaction $\alpha : A \to H \otimes A$ on a finite dimensional algebra A with a fundamental cycle we choose an entire trace representing the fundamental cycle and consider the family of functionals $\varphi_H := (H \otimes \tau)\alpha$, parameterized by the quantum group $\mathrm{Spec}(H)$, obtained as the composition

$$A \stackrel{\alpha}{\to} H \otimes A \stackrel{H\otimes\tau}{\longrightarrow} H. \tag{3}$$

It is easy to check that if τ is a trace (*i.e.* $\delta = \mathrm{id}$) and either H or A is commutative φ_H is a quantum family of traces, *i.e.* it is a trace with values in H.

Definition

We say that the above coaction is Radon-Nikodym differentiable if for some (hence any) entire twisted trace τ representing the fundamental cycle, the family of functionals φ_H is Radon-Nikodym differentiable with respect to τ.

Modular class of a quantum group Radon-Nikodym differentiable action

We want to understand the quantity $(S \otimes A)d\varphi_H/d\tau \in H \otimes A$. To reveal its algebraic status we have to invoke the canonical A-coring structure on $\mathfrak{C} = H \otimes A$, encoding the left coaction α. It is induced by the comultiplication $h \mapsto h_{(1)} \otimes h_{(2)}$ and the counit $h \mapsto \varepsilon(h)$ of the Hopf algebra H and its coaction α on A as follows. An A-bimodule structure $\mathfrak{C}$ is

$$a(h \otimes a') := a_{(-1)}h \otimes a_{(0)}a', \quad (h \otimes a')a := h \otimes a'a. \tag{4}$$

The comultiplication $\Delta : \mathfrak{C} \to \mathfrak{C} \otimes_A \mathfrak{C}$ and the counit $\varepsilon : \mathfrak{C} \to A$ are following

$$h \otimes a' \mapsto (h_{(1)} \otimes 1) \otimes_A (h_{(2)} \otimes a'), \quad h \otimes a' \mapsto \varepsilon(h)a'. \tag{5}$$

Modular class continued - group-likes in corings

Let us remind the reader that an element $\mathfrak{c}$ of a coring $\mathfrak{C}$ is called a group-like if it satisfies the following identities

$$\Delta\mathfrak{c} = \mathfrak{c} \otimes_A \mathfrak{c}, \quad \varepsilon(\mathfrak{c}) = 1. \tag{6}$$

Note that our coring has a distinguished group-like element $\mathfrak{c}_0 = 1 \otimes 1$. We call this group-like trivial.

Modular class continued - Radon-Nikodym derivative

Theorem

The element $\mathfrak{c} = \mathfrak{c}(\alpha, \tau) := (S \otimes A) d\varphi_H / d\tau$ *in* $\mathfrak{C}$ *is a group-like.*

Modular class continued - classical points of quantum groups

- To understand the geometric meaning of this group-like, we will evaluate it on the group scheme G of classical points of the quantum group scheme $\mathfrak{G} = \mathrm{Spec}(H)$.
- The element $\mathfrak{c}$ can be evaluated at classical points $g : H \to K$ of $\mathfrak{G} = \mathrm{Spec}(H)$ as follows

$$\mathfrak{c}(g) := g(\mathfrak{c}_{\langle -1 \rangle}) \cdot \mathfrak{c}_{\langle 0 \rangle} \in K \otimes A, \tag{7}$$

  where the dot denotes the multiplication in $K \otimes A$.
- To see what happens with conditions of being a group-like under this evaluation we need to invoke the fact that the left H-coaction on A defines the following canonical right action of the group of characters of H on A

$$ag := g(a_{(-1)}) \cdot a_{(0)} \in K \otimes A. \tag{8}$$

Modular class continued - crossed homomorphism

Theorem

The above group-like $\mathfrak{c}$ in $\mathfrak{C} = H \otimes A$ evaluated on the group scheme G of classical points of the quantum group scheme $\mathfrak{G} = \mathrm{Spec}(H)$ defines a crossed homomorphism to the multiplicative group $A^{\times}$, i.e.

$$\mathfrak{c}(g_1 g_2) = \mathfrak{c}(g_1)^{g_2} \cdot \mathfrak{c}(g_2)$$

The moral message of this proposition is that the condition of being a group-like for an element in the A-coring $\mathfrak{C} = H \otimes K$ implementing an coaction of H on A is a condition of being a quantum crossed homomorphism from the quantum group scheme $\mathfrak{G} = \mathrm{Spec}(H)$ to the point set geometry of the multiplicative group $A^{\times}$.

Modular class continued - cocycle condition

- As it is well known to be of fundamental importance, the crossed homomorphism condition is the cocycle condition leading to a cohomology class in the corresponding first cohomology of the group with values in the group of coefficients. This cohomology forms a set with a distinguished element, and if the group of coefficients is abelian it is an abelian group with the neutral element as the distinguished element.
- We want to understand a quantum counterpart of the relation of being cohomologous in the case of two quantum cocycles understood as group-likes in the coring $\mathfrak{C}$. First, we will propose a simple definition.
- Next, we will verify its classical meaning by evaluating it on the group scheme of classical points of a quantum group scheme.

Modular class continued - quantum nonabelian cohomology

Definition

Two group-likes $\mathfrak{c}, \mathfrak{c}'$ in $\mathfrak{C}$ are said to be cohomologous if there exists an invertible element a in A such that

$$\mathfrak{c}' = a \cdot \mathfrak{c} \cdot a^{-1}. \quad (9)$$

It is easy to see that it is a well defined equivalence relation on group-likes.

Theorem

If two group-likes $\mathfrak{c}, \mathfrak{c}'$ in $\mathfrak{C}$ are cohomologous their restrictions to the group scheme G of classical points of an affine quantum group scheme $\mathfrak{G} = \mathrm{Spec}(H)$ are cohomologous as 1-cocycles on G with values in the point set geometry of $A^{\times}$, i.e.

$$\mathfrak{c}'(g) = ag \cdot \mathfrak{c}(g) \cdot a^{-1}.$$

Modular class continued - independence of the choice of trace

Theorem

The cohomology class of the group-like $\mathfrak{c} = \mathfrak{c}(\alpha, \tau)$ *is independent of the choice of an entire trace* τ *representing the fundamental cycle.*

Definition

For any A-coring $\mathfrak{C}$ with a distinguished group-like $\mathfrak{c}_0$ we define $\mathrm{H}^1(\mathfrak{C}, \mathfrak{c}_0)$ as a set of cohomology classes of group-likes. The trivial cohomology class is by definition the class of the distinguished group-like.

Modular class continued - final definition

Note that what we obtain for our A-coring $\mathfrak{C} = H \otimes A$ with the group-like $\mathfrak{c}_0 = 1 \otimes 1$ should be denoted by $\mathrm{H}^1(\mathfrak{G}, A^\times)$, and regarded as the quantum first cohomology of the quantum group scheme $\mathfrak{G}$ with values in the point set geometry of $A^\times$.

Definition

We call the cohomology class of $\mathfrak{c}(\alpha, \tau)$ in $\mathrm{H}^1(\mathfrak{G}, A^\times)$ the modular class of the fundamental cycle preserving action of a quantum group scheme $\mathfrak{G} = \mathrm{Spec}(H)$ on the quantum space $\mathfrak{X} = \mathrm{Spec}(A)$ with a fundamental cycle.

Modular class continued - geometric meaning?

It is also well known that cohomology classes can be interpreted as obstructions to existence of solutions of many important problems. A crucial question which should now be addressed is following.

What is a kind of structure on a finite dimensional H-comodule algebra A with a fundamental cycle to which existence the modular class is an obstruction?

Modular class continued - geometric meaning

Theorem

Let $\mathfrak{G} = \mathrm{Spec}(H)$ *be a quantum affine group scheme. For any finite quantum scheme* $\mathfrak{X}$ *with a Radon-Nikodym differentiable* $\mathfrak{G}$*-action the modular class vanishes if and only if* $\mathfrak{X}$ *has a Radon-Nikodym differentiable* $\mathfrak{G}$*-invariant Frobenius structure.*

Quantum group actions preserving the fundamental class

- We will say that a given quantum group $\mathfrak{G}$ action preserves the fundamental cycle if there exist a quantum family δ_H of algebra automorphisms of A such that family of functionals $\tau_H := \varphi_H = (H \otimes \tau)\alpha$ parameterized by $\mathfrak{G}$ is a Radon-Nikodym differentiable family of entire twisted traces.
- This means that (δ_H, τ_H) is a twisted trace with values in H and admits invertible Radon-Nikodym derivative $d\tau_H/d\tau$ for some (hence any) entire trace τ supported on the fundamental cycle, i.e.

$$\frac{d\tau_H}{d\tau} = \frac{d((H \otimes \tau)\alpha)}{d\tau} \in (H \otimes A)^{\times}. \qquad (10)$$

- Note that any fundamental cycle preserving $\mathfrak{G}$-action is in particular Radon-Nikodym differentiable.

Universal quantum group action on a finite quantum scheme

Theorem (essentially Manin + Tambara)

For any finite dimensional algebra A There exists a universal Hopf algebra with bijective antipode coaction on the algebra A.

Corollary

The modular class of the universal Hopf algebra with bijective antipode coaction is an invariant of Frobenius algebras whose universal coaction preserves the fundamental cycle.

We will call it the universal modular class.

Universal quantum group with bijective antipode acting on a finite set

- Let $A = k^n$ be a split commutative k-algebra of rank n. It is generated by elements $a_1, \ldots, a_n$ subject to the relations

$$a_i a_k = \delta_{ik} a_k, \quad \sum_i a_i = 1. \tag{11}$$

- Note that A can be identified with the algebra of k-valued functions on a finite set of cardinality n with point-wise algebraic operations.

Universal quantum group with bijective antipode acting on a finite set

Theorem

The universal Hopf algebra with bijective antipode coacting on the algebra k^n is generated by generators h_{ij}, $u_{i[p]}, v_{i[p]}, u^*_{i[p]}, v^*_{i[p]}$, labeled by $i, j \in \{1, \ldots, n\}$, $p \in \{0, 1, \ldots\}$, subject to the relations

$$h_{ik} h_{jk} = \delta_{ij} h_{jk}, \tag{12}$$

$$\sum_i h_{ik} = 1, \tag{13}$$

$$h_{ki} u_{k[1]} h_{kj} = \delta_{ij} h_{kj}, \qquad h_{ki} u^*_{k[1]} h_{kj} = \delta_{ij} h_{kj}, \tag{14}$$

$$\sum_i h_{ki} = u^{-1}_{k[1]}, \qquad \sum_i h_{ki} = u^{*-1}_{k[1]}, \tag{15}$$

$$u_{i[0]} = 1, \qquad u^*_{i[0]} = 1, \tag{16}$$

$$v_{i[0]} = 1, \qquad v^*_{i[0]} = 1, \tag{17}$$

$$u_{i[p+1]} = \left(\sum_k v_{k[p]} h_{ik} \right)^{-1}, \quad u^*_{i[p+1]} = \left(\sum_k h_{ik} v^*_{k[p]} \right)^{-1}, \tag{18}$$

$$v_{i[p+1]} = \left(\sum_k h_{ki} u_{k[p+1]} \right)^{-1}, \quad v^*_{i[p+1]} = \left(\sum_k u^*_{k[p+1]} h_{ki} \right)^{-1} \tag{19}$$

with the Hopf algebra (with bijective antipode) structure

$$\Delta(h_{ik}) = \sum_j h_{ij} \otimes h_{jk}, \tag{20}$$

$$\Delta(u_{i[p]}) = \sum_k u_{i[1]} h_{ik} u_{i[p]} \otimes u_{k[p]}, \quad \Delta(u^*_{i[p]}) = \sum_k u^*_{i[p]} h_{ik} u^*_{i[1]} \otimes u^*_{k[p]}, \tag{21}$$

$$\Delta(v_{i[p]}) = \sum_k v_{k[p]} \otimes v_{i[p]} h_{ki}, \quad \Delta(v^*_{i[p]}) = \sum_k v^*_{k[p]} \otimes h_{ki} v^*_{i[p]}, \tag{22}$$

$$\varepsilon(h_{ik}) = \delta_{ki}, \tag{23}$$

$$\varepsilon(u_{i[p]}) = \varepsilon(v_{i[p]}) = 1, \qquad \varepsilon(u^*_{i[p]}) = \varepsilon(v^*_{i[p]}) = 1, \tag{24}$$

$$S(h_{ik}) = h_{ki} u_{k[1]}, \qquad S^{-1}(h_{ik}) = u^*_{k[1]} h_{ki}, \tag{25}$$

The universal modular class of a finite set

Theorem

The universal quantum group action on a finite set is Radon-Nikodym differentiable and preserves the classical fundamental cycle (coming from the trivial automorphism and the counting measure). For at least two elements the modular class of this coaction is nontrivial although it vanishes on classical permutations.

Idea of the proof

- Modular class trivial $\Rightarrow$
- exists an invariant Frobenius structure with respect to the universal coaction $\Rightarrow$
- the Frobenius structure invariant with respect to classical permutations $\Rightarrow$
- the Frobenius structure proportional to the classical one $\Rightarrow$
- universal quantum group action factors through quantum permutations in the sense of Wang $\Rightarrow$
- for two element set the universal quantum group action equal to the classical permutations.

However, we have constructed a quantum family (parameterized by $\mathrm{Spec}(k(\bullet \to \bullet)))$ of permutations of a two element set essentialy bigger than the classical permutations. $\square$