Quantization by categorification. Hopf cyclic cohomology

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Categorification of geometry. History

- Grothendieck (toposes, Grothendieck categories),
- Gabriel-Rosenberg (reconstruction of quasi-compact quasi-separated schemes from their Grothendieck categories of quasicoherent sheaves),
- Balmer, Lurie, Brandenburg-Chirvasitu (reconstruction theorems from monoidal categories).

Theorem (Brandenburg-Chirvasitu)

For a quasi-compact quasi-separated scheme X and an arbitrary scheme Y we show that the pullback construction $f \mapsto f^*$ implements an equivalence between the discrete category of morphisms $X \to Y$ and the category of cocontinuous strong opmonoidal functors $\mathrm{Qcoh}_Y \to \mathrm{Qcoh}_X$.

Local model

If A is a commutative associative unital ring then

$$\mathrm{Qcoh}_{\mathrm{Spec}(A)} = \mathrm{Mod}_A.$$

It is monoidal with respect to the usual tensor product

$$(M_1, M_2) \mapsto M_1 \otimes_A M_2$$

of A-modules balanced over A.
The morphism of affine schemes

$$f : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$$

induces the pull-back functor

$$f^* : \mathrm{Qcoh}_{\mathrm{Spec}(B)} = \mathrm{Mod}_B \to \mathrm{Mod}_A = \mathrm{Qcoh}_{\mathrm{Spec}(A)},$$

$$N \mapsto N \otimes_B A.$$

One can easily check that it is cocontinuous and strong opmonoidal, the latter meaning that

$$f^* B \xrightarrow{\cong} A,$$

$$f^*(N_1 \otimes_B N_2) \xrightarrow{\cong} f^* N_1 \otimes_A f^* N_2.$$

Corollary

Knowing the spectrum $\mathrm{Spec}(A)$ as a scheme is equivalent to knowing the monoidal category Mod_A of modules, and knowing a morphism of schemes $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$ is equivalent to knowing a cocontinuous strong opmonoidal functor $\mathrm{Mod}_B \to \mathrm{Mod}_A$.

Global sections

The identification

$$\mathrm{Qcoh}_{\mathrm{Spec}(A)} = \mathrm{Mod}_A$$

uses the global sections functor
$\Gamma(X, -) = \mathrm{Qcoh}_X(\mathscr{O}_X, -) : \mathrm{Qcoh}_X \to \mathrm{Ab}$, where

$$A = \Gamma(\mathrm{Spec}(A), \mathscr{O}_{\mathrm{Spec}(A)})$$

$$M = \Gamma(\mathrm{Spec}(A), \mathscr{F}),$$

for the structural sheaf $\mathscr{O}_{\mathrm{Spec}(A)}$ and any quasicoherent sheaf $\mathscr{F}$ on the spectrum.

What if A is not commutative

- Modules do not form a monoidal category
- Bimodules over a commutative ring do not reconstruct spectra
- Symmetric bimodules do not make sense
- Bimodule maps from A to any bimodule is the center construction, not the identity

So, maybe associative algebras are not good generalization of commutative ones?

Happily, both associative and commutative rings are special cases of bialgebroids, (A, A) for A commutative, $(A, A^{op} \otimes A)$ for A associative.

In both cases an additional structure is so canonical that it is invisible.

For bialgebroids all problems as above can be cured or better posed.

Cyclic schemes

Definition

A *cyclic scheme* X is a *monoidal abelian category* $(\mathrm{Qcoh}_X, \otimes, \mathscr{O}_X)$ equipped with a *cyclic functor* $\Gamma_X : \mathrm{Qcoh}_X \to \mathrm{Ab}$, *i.e.* an additive functor equipped with a natural isomorphism

$$\gamma_{\mathscr{F}_0,\mathscr{F}_1} : \Gamma_X(\mathscr{F}_0 \otimes \mathscr{F}_1) \to \Gamma_X(\mathscr{F}_1 \otimes \mathscr{F}_0)$$

satisfying the following identities

$$\gamma_{\mathscr{F}_1,\mathscr{F}_2\otimes\mathscr{F}_0} \circ \gamma_{\mathscr{F}_0,\mathscr{F}_1\otimes\mathscr{F}_2} = \gamma_{\mathscr{F}_0\otimes\mathscr{F}_1,\mathscr{F}_2},$$

$$\gamma_{\mathscr{O}_X,\mathscr{F}} = \gamma_{\mathscr{F},\mathscr{O}_X} = \mathrm{Id}_{\tau_X(\mathscr{F})},$$

$$\gamma_{\mathscr{F}_1,\mathscr{F}_0} = \gamma^{-1}_{\mathscr{F}_0,\mathscr{F}_1}$$

.

Cyclic symmetry

Lemma

$$\gamma_{\mathscr{F}_n,\mathscr{F}_0\otimes\cdots\otimes\mathscr{F}_{n-1}}\circ\gamma_{\mathscr{F}_{n-1},\mathscr{F}_n\otimes\mathscr{F}_0\otimes\cdots\otimes\mathscr{F}_{n-2}}\circ\cdots\circ\gamma_{\mathscr{F}_0,\mathscr{F}_1\otimes\cdots\otimes\mathscr{F}_n}$$

$$=\mathrm{Id}_{\tau_X(\mathscr{F}_0\otimes\cdots\otimes\mathscr{F}_n)}.$$

Example. Commutative schemes

- With every classical commutative scheme (quasi-compact, quasi-separated) X one can associate an abelian monoidal category $(\mathrm{Qcoh}_X, \otimes, \mathscr{O}_X)$ of quasi-coherent sheaves.
  It is equipped with a canonical cyclic functor of sections

$$\Gamma_X := \Gamma(X, -) : \mathrm{Qcoh}_X \to \mathrm{Ab}$$

  where the cyclic structure comes from the symmetry of the monoidal structure.
- For an affine scheme $X = \mathrm{Spec}(A)$, A being a commutative ring there is a strong monoidal equivalence

$$(\mathrm{Qcoh}_X, \otimes, \mathscr{O}_X) \stackrel{\sim}{\to} (\mathrm{Mod}_A, \otimes_A, A),$$

  and the cyclic functor forgets the A-module structure.

Example: Cyclic spectra of associative rings

Let R be a unital associative ring. We define a cyclic scheme X so that is the monoidal abelian category of R-bimodules

$$(\mathrm{Qcoh}_X, \otimes, \mathscr{O}_X) := (\mathrm{Bim}_R, \otimes_R, R)$$

with the tensor product balanced over R.
If $\mathscr{F} = M$ is an R-bimodule, we have a canonical cyclic functor

$$\Gamma_X(\mathscr{F}) = \Gamma_R(M) := M \otimes_{R^o \otimes R} R$$

obtained by tensoring balanced over the enveloping ring $R^o \otimes R$.
The natural transformation γ is the flip

$$(M_0 \otimes_R M_1) \otimes_{R^o \otimes R} R \to (M_1 \otimes_R M_0) \otimes_{R^o \otimes R} R,$$

$$(m_0 \otimes m_1) \otimes r \mapsto (m_1 \otimes m_0) \otimes r,$$

well defined and satisfying axioms of a cyclic functor thanks to balancing over $R^o \otimes R$.
We call this cyclic scheme the cyclic spectrum of an associative ring R.

Traces

We want to unravel the natural origin of traces. First, we want to understand the *character*

$$S/[S,S] \to R/[R,R]. \qquad (1)$$

of a representation $S \to \mathrm{End}_R(P)$ of the ring S on a finitely generated projective right R-module P.
The point is that in general it *is not* induced by any ring homomorphism $S \to R$, but merely by some *mild correspondence* from S to R.

Mild correspondences

Basic principles of *mild correspondences* we derive from classical algebraic geometry.There a correspondence f from a scheme X to a scheme Y is a diagram of (quasi-compact and quasi-separated) schemes

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & Y \\ \pi \downarrow & & \\ X & & \end{array}$$

and we call it *mild* if its domain projection π is finite and flat.

Induced adjunction

Although a correspondence f is not a honest morphism of schemes $f : X \to Y$, it still defines a monoidal functor of a direct image $f_* := \widetilde{f}_* \pi^* : \mathrm{Qcoh}_X \to \mathrm{Qcoh}_Y$ between categories of quasi-coherent sheaves. It is monoidal because $\widetilde{f}_*$ is monoidal and π^* is strong opmonoidal, hence monoidal as well.
If in addition f is mild f_* has a left adjoint (hence canonically opmonoidal) functor $f^* \dashv f_*$

Coalgebra

Moreover, there exist an $\mathscr{O}_X$-coalgebra D equipped with a structure of an $\pi_*\mathscr{O}_{\widetilde{X}}$-module s.t.

$$f^* := \pi_* \widetilde{f}^*(-) \otimes_{\pi_*\mathscr{O}_{\widetilde{X}}} D : \mathrm{Qcoh}_Y \to \mathrm{Qcoh}_X,$$

$$f_* = \widetilde{f}_*(\mathscr{H}om_X(D, -)^{\sim}) : \mathrm{Qcoh}_X \to \mathrm{Qcoh}_Y$$

where $(-)^{\sim}$ denotes sheafifying by localisation of a $\pi_*\mathscr{O}_{\widetilde{X}}$-module to obtain a quasi-coherent sheaf on $\widetilde{X} = \mathrm{Spec}_X(\pi_*\mathscr{O}_{\widetilde{X}})$, the relative spectrum of a commutative quasi-coherent $\mathscr{O}_X$-algebra $\pi_*\mathscr{O}_{\widetilde{X}}$.

Mild correspondences of affine schemes

Thus for affine schemes $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ a mild correspondence f from X to Y can be written as a homomorphism of commutative rings

$$S \to \mathrm{Hom}_R(D, R), \ s \mapsto (d \mapsto s(d))$$

where the ring on the right hand side is a convolution ring dual to some cocommutative R-coalgebra D, *i.e.* its unit is a counit $\varepsilon : D \to R$ and multiplication comes from the comultiplication $D \to D \otimes_R D$, $d \mapsto d_{(1)} \otimes d_{(2)}$ (Heyneman-Sweedler notation) via dualization, *i.e.*

$$\mathrm{Hom}_R(D, R) \otimes \mathrm{Hom}_R(D, R) \to \mathrm{Hom}_R(D, R),$$

$$\rho_1 \otimes \rho_2 \mapsto (d \mapsto \rho_1(d_{(1)})\rho_2(d_{(2)})).$$

Adjunction for affine schemes

The corresponding adjunction between monoidal categories of modules $\mathrm{Qcoh}_X = \mathrm{Mod}_R$ and $\mathrm{Qcoh}_Y = \mathrm{Mod}_S$ is given as follows

$$f_* M = \mathrm{Hom}_R(D, M),$$

$$f^* N = (N \otimes_S \mathrm{Hom}_R(D, R)) \otimes_{\mathrm{Hom}_R(D,R)} D = N \otimes_S D.$$

A monoidal structure of f_* (or equivalently, an opmonoidal structure of f^*) is related to the coalgebra structure of D as follows.

The morphism $\mathscr{O}_Y \to f_* \mathscr{O}_X$ is defined as $S \to \mathrm{Hom}_R(D, R)$, $s \mapsto (d \mapsto s(d))$, with respect to which the image of the unit of S is equal to the counit of D, and the natural transformation $f_* \mathscr{F}_0 \otimes f_* \mathscr{F}_1 \to f_*(\mathscr{F}_0 \otimes \mathscr{F}_1)$ is defined by means of the comultiplication of D as

$$\mathrm{Hom}_R(D, M_1) \otimes_S \mathrm{Hom}_R(D, M_2) \to \mathrm{Hom}_R(D, M_1 \otimes_R M_2),$$

$$\mu_1 \otimes \mu_2 \mapsto (d \mapsto \mu_1(d_{(1)}) \otimes \mu_2(d_{(2)})).$$

Mild correspondences of noncommutative rings

This can be easily extended to noncommutative rings by noticing that, for R being commutative, R itself and any coalgebra D over R are symmetric R-bimodules, hence

$$\mathrm{Hom}_R(D, R) = \mathrm{Hom}_{R^o \otimes R}(D, R)$$

where on the right hand side we have homomorphisms of R-bimodules regarded as right modules over the enveloping ring $R^o \otimes R$. This still makes sense if one takes noncommutative rings R and S, and an arbitrary R-coring D instead of a cocommutative R-coalgebra over a commutative ring R.

Mild correspondences of rings. Definition

Then we say that a *mild correspondence from a ring S to a ring R* is given if there is given a ring homomorphism

$$S \to \mathrm{Hom}_{R^o \otimes R}(D, R), \; s \mapsto (d \mapsto s(d))$$

where the structure of the convolution ring on $\mathrm{Hom}_{R^o \otimes R}(D, R)$ is induced from the R-coring structure of D.

Adjunction for noncommutative rings

A mild correspondence $S \to \mathrm{Hom}_{R^o\otimes R}(D, R)$ from a ring S to a ring R defines an adjunction between monoidal categories of bimodules $\mathrm{Qcoh}_X = \mathrm{Bim}_R$ and $\mathrm{Qcoh}_Y = \mathrm{Bim}_S$ as follows

$$f_*M = \mathrm{Hom}_{R^o\otimes R}(D, M),\ \ f^*N = N \otimes_{S^o\otimes S} D.$$

A monoidal structure of f_* (or equivalently, an opmonoidal structure of f^*) generalizes the structure of the convolution ring.

What mild correspondences have to do with traces?

$\mathrm{End}_R(P)$ is a convolution ring $\mathrm{Hom}_{R^o\otimes R}(D, R)$ of an R-coring $D = P^* \otimes P$ whose canonical counit $\varepsilon : D \to R$ is the evaluation of elements of $P^* = \mathrm{Hom}_R(P, R)$ on elements of P,

$$P^* \otimes P \to R,$$

$$p^* \otimes p \to p^*(p),$$

its canonical comultiplication $D \to D \otimes_R D$, $d \mapsto d_{(1)} \otimes d_{(2)}$ can be written in terms of any dual basis $(p_i, p_i^*)_{i\in I}$ for P as

$$P^* \otimes P \to (P^* \otimes P) \otimes_R (P^* \otimes P),$$

$$p^* \otimes p \mapsto \sum_{i\in I} (p^* \otimes p_i) \otimes (p_i^* \otimes p),$$

What is the corresponding adjunction?

The morphism $\mathscr{O}_Y \to f_* \mathscr{O}_X$ is defined as above and the natural transformation $f_* \mathscr{F}_1 \otimes f_* \mathscr{F}_2 \to f_*(\mathscr{F}_1 \otimes \mathscr{F}_2)$ is defined by means of the comultiplication of D as

$$\mathrm{Hom}_{R^o \otimes R}(D, M_1) \otimes_S \mathrm{Hom}_{R^o \otimes R}(D, M_2) \to \mathrm{Hom}_{R^o \otimes R}(D, M_1 \otimes_R M_2),$$

$$\mu_1 \otimes \mu_2 \mapsto (d \mapsto \mu_1(d_{(1)}) \otimes \mu_2(d_{(2)})).$$

What is the character from this categorical perspective?

It is an R-component of a natural isomorphism of additive functors $\mathrm{Bim}_R \to \mathrm{Ab}$ whose M-component is

$$\mathrm{Hom}_{R^o \otimes R}(D, M) \otimes_{S^o \otimes S} S \to M \otimes_{R^o \otimes R} R,$$

$$\mu \otimes s \mapsto (\mu \otimes R)(\delta(1))$$

where $\delta \in \mathrm{Hom}_{S^o \otimes S}(S, D \otimes_{R^o \otimes R} R)$ is a canonical element which can be written in terms of any dual basis as

$$S \to (P^* \otimes P) \otimes_{R^o \otimes R} R,$$

$$s \mapsto \sum_{i \in I} (p_i^* \otimes s \cdot p_i) \otimes 1.$$

Character as a natural transformation

Finally, the character of the above representation can be written as a natural transformation

$$\Gamma_Y f_* \to \Gamma_X$$

where X and Y are cyclic spectra of rings R and S, respectively.

The trace property

It is easy to check that the trace property is equivalent to commutativity of all natural diagrams

$$\begin{array}{ccccc}
\Gamma_Y(f_*\mathscr{F}_0 \otimes f_*\mathscr{F}_1) & \longrightarrow & \Gamma_Y(f_*(\mathscr{F}_0 \otimes \mathscr{F}_1)) & \longrightarrow & \Gamma_X(\mathscr{F}_0 \otimes \mathscr{F}_1) \\
{\scriptstyle \gamma_{f_*\mathscr{F}_0, f_*\mathscr{F}_1}}\downarrow & & & & \downarrow{\scriptstyle \gamma_{\mathscr{F}_0, \mathscr{F}_1}} \\
\Gamma_Y(f_*\mathscr{F}_1 \otimes f_*\mathscr{F}_0) & \longrightarrow & \Gamma_Y(f_*(\mathscr{F}_1 \otimes \mathscr{F}_0)) & \longrightarrow & \Gamma_X(\mathscr{F}_1\mathscr{F}_0),
\end{array}$$

Categorical back-bone of cyclic (co)homology

Motivated by this we consider now (large) abelian groups of natural transformations

$$c^{\mathscr{F}_0,\cdots,\mathscr{F}_n} : \Gamma_Y(f_*\mathscr{F}_0 \otimes \cdots \otimes f_*\mathscr{F}_n) \longrightarrow \Gamma_X(\mathscr{F}_0 \otimes \cdots \otimes \mathscr{F}_n) ,$$

$$c_{\mathscr{G}_0,\cdots,\mathscr{G}_n} : \Gamma_Y(\mathscr{G}_0 \otimes \cdots \otimes \mathscr{G}_n) \longrightarrow \Gamma_X(f^*\mathscr{G}_0 \otimes \cdots \otimes f^*\mathscr{G}_n) .$$

All this collection of abelian of natural transformations groups forms a cocyclic object.

- Cofaces come from the composition with natural transformations $f_*\mathscr{F}_0 \otimes f_*\mathscr{F}_1 \to f_*(\mathscr{F}_0 \otimes \mathscr{F}_1)$ defining the monoidal structure of f_*,
- codegeneracies come from the structural morphism $\mathscr{O}_Y \to f_*\mathscr{O}_X$,
- cyclic operators come from the natural transformations γ of the cyclic functors.

Example: Cyclic cohomology of an algebra

For an algebra A over a field k we prepare the following categorical environment.

$$\mathrm{Qcoh}_X = \mathrm{Vect}^{op}, \Gamma_X(V) = V^*, \ \mathrm{Qcoh}_Y = \mathrm{Vect}, \Gamma_Y(V) = V,$$

$$f_* V = Hom(V, A).$$

The cocyclic object of natural transformations:

$$\Gamma_Y(f_*\mathscr{F}_0 \otimes \cdots \otimes f_*\mathscr{F}_n) \to \Gamma_X(\mathscr{F}_0 \otimes \cdots \otimes \mathscr{F}_n)$$

reads as

$$Hom(V_0, A) \otimes \cdots \otimes Hom(V_n, A) \to Hom(V_0 \otimes \cdots \otimes V_n, k)$$

whose component corresponding to $V_0 = \cdots = V_n = k$ is

$$A \otimes \cdots \otimes A \to k,$$

the classical cocyclic object $A^\natural$ of Connes.

A cyclic Eilenberg-Moore construction

Let R be a ring in a monoidal category Qcoh_Y, and Bim_R be its monoidal category of bimodules equipped with an opmonoidal monad a^*. For any opmonoidal monad a^* on the monoidal category Bim_R of R-bimodules over a ring R in a monoidal category Qcoh_Y, with structural natural transformations

$$\mu_{a^*}^M : a^*a^*M \to a^*M,\ \ \eta_{a^*}^M : M \to a^*M,$$
$$\delta_{a^*}^{M_0,M_1} : a^*(M_0 \otimes_R M_1) \to a^*M_0 \otimes_R a^*M_1,$$

and a structural morphism

$$\varepsilon : a^*R \to R,$$

one defines a natural transformation of *right fusion*

$$\varphi_{a^*}^{M_0,M_1} : a^*(M_0 \otimes_R a^*M_1) \to a^*M_0 \otimes_R a^*M_1$$

as a composition

$$a^*(M_0 \otimes_R a^*M_1) \xrightarrow{\delta_{a^*}^{M_0,a^*M_1}} a^*M_0 \otimes_R a^*a^*M_1 \xrightarrow{a^*M_0 \otimes_R \mu_{a^*}^{M_1}} a^*M_0 \otimes_R a^*M_1\ .$$

Monoidal Eilenberg-Moore construction for Hopf monads on bimodule categories

The Eilenberg-Moore category $(\mathrm{Bim}_R)^{a^*}$ of a^* consists of objects M equipped with with morphisms

$$\alpha_M : a^* M \to M,$$

satisfying some properties (commutative diagrams). What is important, they form a monoidal category as follows.

$$\alpha_{M_0 \otimes_R M_1} : a^*(M_0 \otimes_R M_1) \to M_0 \otimes_R M_1$$

$$a^*(M_0 \otimes_R M_1) \xrightarrow{\delta_{a^*}^{M_0, M_1}} a^* M_0 \otimes_R a^* M_1 \xrightarrow{\alpha_{M_0} \otimes_R \alpha_{M_1}} M_0 \otimes_R M_1,$$

We will denote by A the pair (R, a^*), and by $\mathrm{Spec}_Y(A)$ the Eileberg-Moore category $(\mathrm{Bim}_R)^{a^*}$.

Back to commutative algebras

Commutative rings can be regarded as Hopf monads and they have their cyclic spectra as such. Let $R = A$ where A is a commutative ring. The category Bim_R of R-bimodules admits an endofunctor a^* of symmetrization

$$a^* M := M/[M, R] \tag{2}$$

well defined thanks to the fact that for commutative R the commutator $[M, R] \subset M$ is an R-subbimodule.It is a Hopf monad making the cyclic functor of Bim_R a twisted cyclic functor on the Eilenberg-Moore category $(\mathrm{Bim}_R)^{a^*}$.

Theorem

The cyclic spectrum for the above Hopf monad and the twisted cyclic functor is equivalent to $\mathrm{Qcoh}_{\mathrm{Spec}(A)} = \mathrm{Mod}_A$.

Stable anti-Yetter-Drinfeld conditions for twisted cyclic functors

We say that a functor $\tau_R : \mathrm{Bim}_R \to \mathrm{Ab}$ is a *twisted cyclic functor*, if it is equipped with two natural transformations, the *twisted transposition*

$$\tau_R(M_0 \otimes_R M_1) \xrightarrow{t_R^{M_0,M_1}} \tau_R(M_1 \otimes_R a^* M_0)$$

and the *right action* of the opmonoidal monad a^*

$$\tau_R a^* \xrightarrow{\alpha_{\tau_R}} \tau_R$$

satisfying the following conditions.

SAYD-type conditions. Preparation

First, for the composition $\tau_R a^*$ we define an analogical twisted transposition, a natural transformation

$$\tau_R a^*(M_0 \otimes_R M_1) \xrightarrow{t_{R,a^*}^{M_0,M_1}} \tau_R a^*(M_1 \otimes_R a^* M_0)$$

being a composition

$$\begin{aligned}
\tau_R a^*(M_0 \otimes_R M_1) &\xrightarrow{\tau_R(\delta^{M_0,M_1})} \tau_R(a^* M_0 \otimes_R a^* M_1) \xrightarrow{t_R^{a^* M_0, a^* M_1}} \tau_R(a^* M_1 \otimes_R a^* a^* M_0)) \\
&\xrightarrow{\tau_R(M_0 \otimes \mu_{a^*}(M_1))} \tau_R(a^* M_1 \otimes_R a^* M_0) \xrightarrow{\tau_R(\varphi_{a^*}^{M_1,M_0})^{-1}} \tau_R a^*(M_1 \otimes_R a^* M_0)
\end{aligned}$$

Anti-Yetter-Drinfeld condition

The first condition for τ_R to be a twisted cyclic functor consists in commutativity of the following diagram

$$\begin{array}{ccc} \tau_R a^*(M_0 \otimes_R M_1) & \xrightarrow{t_{R,a^*}^{M_0,M_1}} & \tau_R a^*(M_1 \otimes_R a^* M_0) \\ {\scriptstyle \alpha_{\tau_R}^{M_0 \otimes_R M_1}} \big\downarrow & & \big\downarrow {\scriptstyle \alpha_{\tau_R}^{M_0 \otimes_R a^* M_1}} \\ \tau_R(M_0 \otimes_R M_1) & \xrightarrow{t_R^{M_0,M_1}} & \tau_R(M_1 \otimes_R a^* M_0). \end{array}$$

which means that $t_{R,a^*}^{M_0,M_1}$ lifts $t_R^{M_0,M_1}$ along the Hopf monad a^* action α_{τ_R} on τ_R.

Stability condition

The second condition for τ_R to be a twisted cyclic functor consists in commutativity of the following diagram

$$\begin{array}{ccc} \tau_R(M) & \xrightarrow{t_R^{M,R}} & \tau_R a^*(M) \\ & \searrow\!\!\!\!= & \downarrow \alpha^M_{\tau_R} \\ & & \tau_R(M) \end{array}$$

where the horizontal arrow utilizes identifications via tensoring by the monoidal unit R as follows

$$\tau_R(M) = \tau_R(M \otimes_R R) \xrightarrow{t_R^{M,R}} \tau_R(R \otimes_R a^* M) = \tau_R a^*(M).$$

This means that the Hopf monad a^* action α_{τ_R} on τ_R neutralizes the twisted transposition with the monoidal unit R.

Cyclic functor on the monoidal Eilenberg-Moore category from SAYD conditions

The following coequalizer diagram

$$\tau_R a^* M \underset{\tau_R(\alpha_M)}{\overset{\alpha^M_{\tau_R}}{\rightrightarrows}} \tau_R M \longrightarrow \tau_A M$$

defines an additive functor $\tau_A : \mathrm{Qcoh}_{\mathrm{Spec}_Y(A)} \to \mathrm{Ab}$.

Theorem

τ_A makes $\mathrm{Spec}_Y(A)$ a cyclic scheme.

Example: Hopf-cyclic cohomology of an algebra

For a left H-module algebra A over a Hopf algebra H over a field k and a right-left stable anti-Yetter-Drinfeld H-module Γ we can consider the Hopf bialgebroid or $B = (k, b^* = H \otimes (-))$ and

$$\mathrm{Qcoh}_X = \mathrm{Vect}^{op}, \Gamma_X(V) = Hom(V, k),$$

$$\mathrm{Qcoh}_{\mathrm{Spec}(B)} = H - \mathrm{Mod}, \Gamma_{\mathrm{Spec}(B)}(V) = \Gamma \otimes_H V,$$

$$f_* V = Hom(V, A), \;\; f_* M = \;_H Hom(M, A).$$

The cocyclic object of natural transformations:

$$\Gamma_Y(f_*\mathscr{F}_0 \otimes \cdots \otimes f_*\mathscr{F}_n) \to \Gamma_X(\mathscr{F}_0 \otimes \cdots \otimes \mathscr{F}_n)$$

reads as

$$\Gamma \otimes_H (Hom(V_0, A) \otimes \cdots \otimes Hom(V_n, A)) \to Hom(V_0 \otimes \cdots \otimes V_n, k)$$

whose component corresponding to $V_0 = \cdots = V_n = k$ is

$$\Gamma \otimes_H (A \otimes \cdots \otimes A) \to k,$$

the cocyclic object of Hajac-Khalkhali-Rangipour-Sommerhäuser.