The Bohr-Sommerfeld groupoid of quantum $\mathbb{CP}^n$
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Let \((M, \pi)\) be an integrable Poisson manifold with symplectic groupoid\[G(M) \xrightarrow{r} M : m : G_2(M) \rightarrow G(M)\]

**Karasev-Weinstein-Zakrzewski**

Apply geometric quantization to \(G(M)\) and compare the outcome with deformation quantization of \((M, \pi)\).
Symplectic integration

For a Poisson manifold \((M, \pi)\) the cotangent bundle \(T^*M\) has a natural structure of Lie algebroid (i.e. Lie bracket between 1–forms + Lie map between 1-forms and vector fields).

A symplectic groupoid is a Lie groupoid integrating this Lie algebroid (much as Lie groups integrate Lie algebras - but... possible obstructions).

If the obstruction is not present (meaning of the word *integrable*) then the groupoid has also a symplectic manifold *compatible* with the Lie groupoid structure.
1. Prequantum line bundle \((L, \nabla) + \sigma\) covariantly constant normalized 2–cocyce in \(L\);

2. **Multiplicative** polarization \(\mathcal{F}\): set of leaves \(\mathcal{G}(M)/\mathcal{F}\) is a groupoid inheriting (reduced) 2–cocycle \(\sigma_0\);

3. Bohr-Sommerfeld condition identifying a subgroupoid \((\mathcal{G}(M)/\mathcal{F})^{bs}\);

4. (Twisted) convolution \(C^\ast\)–algebra \(C^\ast((\mathcal{G}(M)/\mathcal{F})^{bs}; \sigma_0)\).
Motivating example

Let $M = \mathbb{T}^2$ with constant symplectic structure

$$\pi = \theta \partial_1 \wedge \partial_2$$

$\mathcal{G}(\mathbb{T}^2) = T^*\mathbb{T}^2$ (change in grpd + sympl.)

Prequantum bundle = trivial line bundle + 2–cocycle;

- Horizontal polarization $\Rightarrow C^* (\mathbb{Z}^2; \sigma_0)$ with $\sigma_0 = e^{\pi}$ (Weyl);
- Cylindrical polarization $\Rightarrow C^* (\mathbb{Z} \rtimes \mathbb{S}^1)$ action groupoid with trivial cocycle (irrational rotation algebra).

Outcome

Quantum torus $p \star q = e^{\hbar} q \star p$. 
A groupoid polarization $\mathcal{F} \subseteq T^C G$ is multiplicative (Hawkins JSG 2008) if, letting

$$\mathcal{F}_2 = (\mathcal{F} \times \mathcal{F}) \cap T^C G_2$$

then

$$m_*(\mathcal{F}_2(\gamma, \eta)) = \mathcal{F}(m(\gamma, \eta))$$

for any composable pair $(\gamma, \eta) \in G_2$.

**Problem:** there are topological obstructions to the existence of real multiplicative polarizations
Let $\pi$ be any integrable Poisson structure on $\mathbb{CP}^1$, then there are no real multiplicative polarizations on its symplectic groupoid (linked to non existence of rank 1 foliations on $\mathbb{CP}^1$).

**Bruhat-Poisson** structure on $\mathbb{CP}^1$:

$$\pi_B = \begin{cases} 
-\nu(1 + |z|^2) \partial_z \wedge \partial_{\overline{z}} & \text{on } \mathbb{CP}^1 \setminus [1, 0] \\
-\nu|w|^2(1 + |w|^2) \partial_w \wedge \partial_{\overline{w}} & \text{on } \mathbb{CP}^1 \setminus [0, 1] 
\end{cases}$$

Still possibile to perform KWZ procedure with a singular multiplicative polarization (Bonechi, C., Staffolani, Tarlini JGP 2012).
Loosening requirements

What do we really need for a $C^*$–groupoid convolution algebra?

- $\mathcal{G} \to \mathcal{G}_F$ Lagrangian fibration of topological groupoids;
- $\mathcal{G}^{bs}_F$ Bohr–Sommerfeld subgroupoid carrying a left Haar measure;
- the prequantization cocycle descending to $\mathcal{G}^{bs}_F$;
- the *modular* $1$–cocycle descending to $\mathcal{G}^{bs}_F$;
Intermezzo – the modular cocycle

$(M, \pi)$ Poisson, $V$ volume form on $M \Rightarrow \chi_V$ modular vector field (divergence of $\pi$ w.r. to $V$) defines a class in $H^1_\pi(M)$. $\chi_V \Rightarrow f_V$

(van Est map) 1–cocycle on $G$; $f_V$ should be quantizable, coincide with the modular function of the quasi invariant measure on the base space, implement KMS condition.
integrable

A family $F = \{f_1, \ldots, f_N\}$ of functions, $N = \frac{1}{2} \dim \mathcal{G}$, is an integrable system if are in involution $\{f_i, f_j\} = 0$ and $df_1 \wedge \ldots \wedge df_N \neq 0$ on a dense open subset of $M$.

multiplicative

The integrable system is called multiplicative if the distribution $\mathcal{F} = \langle X_{f_1}, \ldots, X_{f_N} \rangle$ is multiplicative, or, more generically, if the topological space of level sets of $f_1, \ldots, f_N$ inherits a topological groupoid structure from $\mathcal{G}$.

modular

The integrable system is called modular if the modular function $f_V$ is in involution with all $f_i$’s.
Consider the level sets of a multiplicative integrable system

\[ \mathcal{G}_F(M) = \mathcal{G}(M) / \mathcal{F} \]

It is well behaved if:

1. \( \mathcal{G}_F(M) \) is a topological groupoid and \( \mathcal{G}(M) \rightarrow \mathcal{G}_F(M) \) a topological groupoid epimorphism;
2. For each pair \( l_1, l_2 \) of composable leaves \( m : l_1 \times l_2 \rightarrow l_1 l_2 \) induces a surjective map in homology (⇒ subgroupoid \( \mathcal{G}_F^{bs}(M) \)).
3. \( \mathcal{G}_F^{bs}(M) \) admits a left Haar system (guaranteed if it is étale).
Let $SU(n+1)$ be given the *standard* Poisson–Lie structure $\pi_{std}$.

There is a one–parameter family of *covariant* $(\mathbb{C}P^n, \pi_t)$, non symplectic when $t \in [0, 1]$.

Non symplectic are all quotient by coisotropic subgroups:

$$U_t(n) = \sigma_t S(U(1) \times U(n))\sigma_t^{-1} \subseteq SU(n+1)$$

where

$$\sigma_t = \begin{pmatrix}
\sqrt{1-t} & 0 & \sqrt{t} \\
0 & \text{id}_{n-1} & 0 \\
-\sqrt{t} & 0 & \sqrt{1-t}
\end{pmatrix}$$
Some equivalences. In fact:

\[ \psi : \mathbb{CP}^n \to \mathbb{CP}^n; \quad \psi(\pi_t) = -\pi_{1-t} \]

- \( \pi_0, \pi_1 \), standard or Bruhat–Poisson
- \( \pi_t, t \in ]0, 1[ \), non standard.

### Poisson pencil

Let \( \pi_\lambda \) be the Fubini-Study bivector. Then [\( \pi_\lambda, \pi_0 \) = 0 (Koroshkin-Radul-Rubtsov CMP ‘93) and \( \pi_t = \pi_0 + t\pi_\lambda \).
Projecting the chain of Poisson subgroups

\[ SU(1) \subseteq SU(2) \subseteq \ldots \subseteq SU(n) \]

one gets the chain of Poisson submanifolds

\[ \{*\} \subseteq \mathbb{CP}^1 \subseteq \ldots \subseteq \mathbb{CP}^{n-1} \]

In homogeneous coordinates

\[ P_k = \{[X_1, \ldots, X_k, 0, \ldots, 0]\} \]

is a Poisson submanifold. All symplectic leaves are contractible and symplectomorphic to standard \( \mathbb{C}^k \).
Non standard $\mathbb{CP}^n$: symplectic foliation

**singular locus**

Let

$$ P_k(t) = \left\{ F_{k,t} = t \sum_{i=1}^{k} |X_i|^2 - (1 - t) \sum_{i=k+1}^{n} |X_i|^2 = 0 \right\} $$

Then $\bigcup_{i=1}^{n} P_i(t)$ is the singular part; complement has $n + 1$ connected contractible leaves $\simeq \mathbb{C}^n$.

Scheme of the singular part for $\mathbb{CP}^3$: 

\[ \mathbb{S}^5 \quad \mathbb{S}^3 \quad \mathbb{S}^3 \times \mathbb{S}^3 \quad \mathbb{S}^1 \quad \mathbb{S}^5 \]

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symplectic foliation of $\mathbb{C}P^2$
The symplectic groupoid of \((\mathbb{C}P^n, \pi_t)\)

The symplectic groupoid

\[
\mathcal{G}(\mathbb{C}P^n, \pi_t) = \{ [g\gamma] : g \in SU(n+1), \gamma \in SB(n+1, \mathbb{C}), g\gamma \in U_t(n)^\perp \}
\]

is a fibre bundle over \(\mathbb{C}P^n\) with contractible fibre \(U_t(n)^\perp\).

It is an exact symplectic manifold.

It carries a hamiltonian \(\mathbb{T}^n\)–action with momentum map

\[
h([g\gamma]) = \log p_{A_{n+1}}(\gamma)
\]
Bihamiltonian

torus action

The Cartan $\mathbb{T}^{n} \subseteq SU(n + 1)$ acts on $(\mathbb{C}P^{n}, \pi_{\lambda})$ with momentum map

$$c : \mathbb{C}P^{n} \rightarrow t_{n}^{*}; \quad \text{Im} \ c = \Delta_{n}$$

The action is Poisson w. r. to $\pi_{t}$.

Suitable basis $H_{k}$ of $t_{n}$ such that

1. infinitesimal vector fields $\sigma_{H_{k}}$ are eigenvalues of the Nijenhuis operator with eigenvector $(c_{k} - 1)$;
2. $\sigma_{H_{k}} = \{b_{k}, -\}$, with $b_{k} = \log |c_{k} - 1 + t|$. 

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Multiplicative integrability - $\mathbb{C}P^{n}$
Summarizing actions

- Hamiltonian $\mathbb{T}^n$–action on $\mathbb{CP}^n$ with momentum map $c : \mathbb{CP}^n \to \mathbb{R}^n$;

- Hamiltonian $\mathbb{T}^n$–action on $\mathcal{G}(\mathbb{CP}^n, \pi_t)$ with momentum map $h : \mathcal{G}(\mathbb{CP}^n) \to \mathbb{R}^n$ by groupoid 1–cocycles;

Let us consider

$$\mathcal{F} = \{ l^* c_i, h_i \ldots i = 1, \ldots, n \}$$
Theorem

$\mathcal{F}$ is a multiplicative modular integrable system on $\mathcal{G}(\mathbb{CP}^n, \pi_t)$ with:

$$f_{FS} = \sum_{i=1}^{n} h_i$$

**Aim:** prove this integrable system is well behaved.
Let $\mathbb{R}^n$ act on $\mathbb{R}^n$ via

$$c \cdot h = (1 - t + e^{-h}(c + t - 1))$$

and let $\mathbb{R}^n \times \mathbb{R}^n|_{\Delta_n}$ be the action groupoid restricted to the standard simplex. Then:

$$G_{\mathcal{F}}(t) = \{(c, h) \in \mathbb{R}^n \times \mathbb{R}^n|_{\Delta_n} : c_i = c_{i+1} = 1 - t \Rightarrow h_i = h_{i+1}\}$$

is the topological groupoid of level sets.
Bohr-Sommerfeld conditions

Level sets $L_{ch}$ are connected with: $H_1(L_{ch}; \mathbb{Z})$ generated by hamiltonian flows of $h_j, l^* c_j$;

**Theorem**

BS conditions select a discret subset of lagrangians

$$
\mathcal{G}_{bs}(t) = \{(c, h) \in \mathcal{G}_{F}(t) : h_k \in \hbar \mathbb{Z}, \log |c_k - 1 + t| \in \hbar \mathbb{Z}\}
$$

This is an étale subgroupoid with a unique left Haar system.

The modular function $f_{FS}$ is quantized to

$$
f_{FS}(c, h) = \sum_{i=1}^{n} h_i
$$
The space of units is
\[ \Delta_n^\mathbb{Z}(t) = \{ c \in \Delta_n : c_k = 1 - t + e^{-\hbar n_k} \} \]

The quasi invariant measure associated to \( f_{FS} \) is:
\[ \mu_{fs}(c) = \exp(-\hbar \sum_{k=1}^{n} n_k) \]

Groupoid orbits are labelled by \((r, s) : r + s \leq n\). Each is a transitive subgroupoid over
\[ \Delta_{r,s}^\mathbb{Z}(t) = \left\{ (m, \infty, n) \in \mathbb{Z}^r \times \infty \times \mathbb{Z}^s : \frac{-\log(1-t)}{\hbar} \leq \frac{m_i}{n_i} \leq \frac{m_{i+1}}{n_{i+1}} \right\} \]
1. The Poisson antiautomorphism $\psi$ lifts to a groupoid isomorphism;

2. Poisson submanifolds are quantized by topological subgroupoids

$$P_k(t) = \{(c, h) \mid c_k = 1 - t\}$$

3. Groupoids thus obtained coincide with:
   - Sheu for $(\mathbb{C}P^n, \pi_0)$;
   - Sheu for $S^{2n-1}$ as Poisson submanifold of $\mathbb{C}P^n$, $\pi_t$, $t \neq 0, 1$;
   - Sheu for $(\mathbb{C}P^1, \pi_t)$.
Example: $\mathbb{C}P^2$

Exponentially separated BS leaves

two copies of $S^3$
1–cocycle $c = \log D \in Z^1(\mathcal{G}; \mathbb{R})$

$D$ modular function w.r. to $\mu$

Modular class in $H^1_\pi(M)$

$A_c(t) = e^{itc}$ map in $\text{Aut}(C^*\mathcal{G})$

$\mu$ quasi-invariant measure on $\mathcal{G}_0$

$\phi_\mu : C^*(\mathcal{G}) \to \mathbb{R}$

Van den Bergh bimodule
Functoriality: \((\mathbb{CP}^1, \pi_t)\) are all Poisson–Morita equivalent for \(0 < t < 1\) and the nonstandard groupoid does not depend on \(t^1\).

- Are groupoids for \(n > 1\) independent of \(t\)?
- Is \((\mathbb{CP}^n, \pi_t)\) Poisson-Morita to \((\mathbb{CP}^n, \pi_s)\) for \(t, s \in ]0, 1[\)?
- Can we characterize Poisson submanifolds which functorially quantize?

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\(^1\)Bursztyn-Radko, Ann. Inst. Fourier 2003
Nonstandard quantum $\mathbb{CP}^1$ is not only a groupoid but also a graph $C^*$–algebra $^2$.

Is nonstandard quantum $\mathbb{CP}^n$ the graph $C^*$–algebra of the following graph?

$^2$Hong–Szymański, CMP 2002