BRAIDED NONCOMMUTATIVE JOIN ALGEBRA OF GALOIS OBJECTS

Ludwik Dąbrowski
(SISSA, Trieste)

Joint work with T. Hadfield, P. M. Hajac, E. Wagner

IMPAN, 21 August 2014
Goal and plan

Motivation: Extend the noncommutative join for compact quantum groups (Hopf algebras) to include Galois objects (quantum torsors). Then to quantum principal bundles.

Applications:
1. Quantum coverings from anti-Drinfeld doubles that are used in Hopf-cyclic theory with coefficients.
2. Quantum torus-bundles with potential for constructing new Dirac operators [L.D., A. Sitarz, A. Zucca].

Plan:
1. Recall the basics: classical joins, braidings, Galois objects.
2. Show that the diagonal coaction of noncommutative Hopf algebras on the braided tensor product of Galois objects is a homomorphism of algebras.
3. Construct a braided noncommutative join algebra of Galois objects, and show that it is a principal comodule algebra for the diagonal coaction.
4. Apply to noncommutative tori (& tackle *-structure) and to anti-Drinfeld doubles.
The join $X \ast Y$ of compact Cartan principal $G$-bundles $X$ and $Y$ (local triviality not assumed) is again a compact Cartan principal $G$-bundle for the diagonal $G$-action on $X \ast Y$:

In particular $G \ast G$ is a non-trivializable principal $G$-bundle over $\Sigma G$ for any compact Hausdorff topological group $G \neq 1$.

For example, we get this way $S^1 \to \mathbb{R}P^1$, $S^3 \to S^2$ and $S^7 \to S^4$, using $G = \mathbb{Z}/2\mathbb{Z}$, $U(1)$ and $SU(2)$, respectively.
Definition

Let $A_1$ and $A_2$ be unital C*-algebras. We call the unital C*-algebra

$$A_1 \ast A_2 := \left\{ x \in C([0, 1]) \otimes_{\min} A_1 \otimes_{\min} A_2 \mid (ev_0 \otimes \text{id})(x) \in \mathbb{C} \otimes A_2, (ev_1 \otimes \text{id})(x) \in A_1 \otimes \mathbb{C} \right\}$$

join C*-algebra of $A_1$ and $A_2$.

Quantum group actions?
Oops... no diagonal action, i.e. (dually) the diagonal coaction

$$\Delta(a \otimes a') = a_0 \otimes a'_0 \otimes a_1 a'_1$$

is not a homomorphism of algebras.

Two possible ways out (classically insignificant):

1. gauge coaction
2. braid multiplication

Today about the second option:
\[ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \]
\[ 1 \leq i \leq n - 2 \]

\[ \sigma_i \sigma_j = \sigma_j \sigma_i \]
\[ |i - j| > 1 \]
A factorization of two algebras $A$ and $A'$ is a linear map $\sigma : A' \otimes A \rightarrow A \otimes A'$ such that

1. $\forall a \in A, a' \in A' : \sigma(1 \otimes a) = a \otimes 1$ and $\sigma(a' \otimes 1) = 1 \otimes a'$,

2. $\sigma \circ (m' \otimes \text{id}) = (\text{id} \otimes m') \circ \sigma_{12} \circ \sigma_{23}$,
   $\sigma \circ (\text{id} \otimes m) = (m \otimes \text{id}) \circ \sigma_{23} \circ \sigma_{12}$.

Here $m$ and $m'$ are multiplications in $A$ and $A'$ respectively. If in addition $A' = A$ and the braid equation

$$\sigma_{12} \circ \sigma_{23} \circ \sigma_{12} = \sigma_{23} \circ \sigma_{12} \circ \sigma_{23}$$

is satisfied, we call $\sigma$ a braiding.

Factorizations classify all associative multiplications on $A \otimes A'$ s.t. $A$ and $A'$ are included in $A \otimes A'$ as unital subalgebras:

$$m_\sigma = (m \otimes m') \circ (\text{id} \otimes \sigma \otimes \text{id}).$$
Let $H$ be a Hopf algebra, and $P$ a left (right) $H$-comodule algebra with coaction

$P \Delta(x) = x(-1) \otimes x(0)$ (left),

$\Delta_P(x) = x(0) \otimes x(1)$ (right).

Def. of the left (right) coaction-invariant subalgebra:

$B := co^H P := \{ x \in P \mid P \Delta(x) = 1 \otimes x \}$ (left),

$B := P^{co^H} := \{ x \in P \mid \Delta_P(x) = x \otimes 1 \}$ (right).

Def. of the canonical maps:

$can_L : P \otimes P \ni x \otimes y \mapsto x(-1) \otimes x(0)y \in H \otimes P$ (left),

$can_R : P \otimes P \ni x \otimes y \mapsto xy(0) \otimes y(1) \in P \otimes H$ (right).

**Definition**

$P$ is called **left (right) $H$-Galois extension of $B$** iff $can_L$ ($can_R$) is a bijection.
**Theorem (M. Durdevic)**

Let $P$ be a left $H$-Galois extension of $B$. Then the linear map

$$\sigma: P \otimes_B P \ni x \otimes y \mapsto y(-1)^{[1]} \otimes y(-1)^{[2]} x y(0) \in P \otimes_B P$$

is a braiding. Here $h^{[1]} \otimes h^{[2]} := \text{can}_L^{-1}(h \otimes 1)$.

- $\sigma$ is called Durdevic braiding.
- $\sigma$ becomes a flip when $P$ is commutative.

**Special cases:**

1. $B = \mathbb{C}$ (i.e. left Galois object).
   This is the case we are to explore.

2. $P = H$ (a Hopf algebra).
   Then the Durdevic $\sigma$ coincides with the Yetter-Drinfeld $\sigma$:
   $$\sigma(a \otimes b) = b^{(1)} \otimes S(b^{(2)})ab^{(3)},$$
   where $S$ is the antipode of $H$. 
Let $\sigma : A \otimes A \rightarrow A \otimes A$ be a braiding. We call $A \otimes A$ with multiplication $m_\sigma$ braided tensor product algebra and denote it $A \underline{\otimes} A$.

**Lemma (Key lemma)**

Let $H$ be a Hopf algebra and $A$ a bicomodule algebra over $H$ (left and right coactions commute). Assume that $A$ is a left Galois object over $H$, and that $A \underline{\otimes} A$ is the tensor product algebra braided by the Durdevic braiding. Then the right diagonal coaction

$$\Delta_{A \underline{\otimes} A} : A \underline{\otimes} A \ni a \otimes a' \mapsto a(0) \otimes a'_0 \otimes a(1) a'_1 \in A \underline{\otimes} A \otimes H$$

is an algebra homomorphism.

**Pf.** In fact $m_{A \underline{\otimes} A}$ is just the 'pullback' by $can_L$ of the tensor multiplication on $H \otimes A$, and so $can_L : A \underline{\otimes} A \rightarrow A \otimes H$ becomes a colinear algebra isomorphism.
**Definition**

Let $H$ be a Hopf algebra and $A$ a bicomodule algebra over $H$. Assume that $A$ is a left Galois object over $H$. We call

$$A*A := \left\{ x \in C([0,1]) \otimes A \otimes A \left| \begin{array}{l}
(ev_0 \otimes \text{id})(x) \in \mathbb{C} \otimes A, \\
(ev_1 \otimes \text{id})(x) \in A \otimes \mathbb{C}
\end{array} \right. \right\}$$

the $H$-braided noncommutative join algebra of $A$.

**Lemma**

*The map*

$$C([0,1]) \otimes A \otimes A \rightarrow C([0,1]) \otimes A \otimes A \otimes H,$$

$$f \otimes a \otimes b \mapsto f \otimes a(0) \otimes b(0) \otimes a(1)b(1),$$

restricts and corestricts to $\delta: A*A \rightarrow (A*A) \otimes H$ making $A*A$ a right $H$-comodule algebra.
Main theorem

**Theorem**

Let $A \ast A$ be the $H$-braided noncommutative join algebra of $A$. Assume that the antipode of $H$ is bijective and that $A$ is also a right Galois object. Then the coaction

$$\delta : A \ast A \longrightarrow (A \ast A) \otimes H$$

is principal, i.e. the canonical map it induces is bijective and $P$ is $H$-equivariantly projective as a left $B$-module. Furthermore, the coaction-invariant subalgebra $B$ is the unreduced suspension $\Sigma H$.

Pf. goes by exhibiting $A \ast A$ to be isomorphic to the pullback of two pieces which are shown to be principal, and using the fact [HKMZ11] that pullbacks preserve the principality.
Quantum-torus example

Take $A := O(\mathbb{T}_\theta^2)$, generated by unitaries $U$ and $V$; and the Hopf algebra $H := O(\mathbb{T}^2)$ generated by (commuting) unitaries $u$ and $v$. With the usual coactions, $A$ is an $H$-bicomodule and a left Galois object. Setting

$$U_L := U \otimes 1, \quad V_L := V \otimes 1, \quad U_R := 1 \otimes U, \quad V_R := 1 \otimes V,$$

we can write the linear basis of $A \otimes A$ as $\{ U^k_L V^l_L U^m_R V^n_R \}_{k,l,m,n \in \mathbb{Z}}$.

The $H$-braided join comodule algebra of $A$

$$A_* A = \left\{ \sum_{\text{finite}} f_{klmn} \otimes U^k_L V^l_L U^m_R V^n_R \in C([0,1]) \otimes A \otimes A \right\}$$

is a $\theta$-deformation of a nontrivial $\mathbb{T}^2$-principal bundle $\mathbb{T}^2 \ast \mathbb{T}^2$ preserving the structure group, the base space, and principality.

The $*$-structure $U^* = U^{-1}, V^* = V^{-1}$ of $O(\mathbb{T}_\theta^2)$ matches too:
If $H$ is a *-Hopf algebra, we call a *-algebra $A$ right $H$ *-comodule algebra iff

$$(\ast \otimes \ast) \circ \Delta_A = \Delta_A \circ \ast.$$ 

Then on $A \otimes A$ we use the pullback by $\text{can}_L$ of $\ast \otimes \ast$ on $H \otimes A$

$$(a \otimes b)^* := (\text{can}_L^{-1} \circ (\ast \otimes \ast) \circ \text{can}_L)(a \otimes b)$$

$$= a^* (-1)^{[1]} \otimes a^* (-1)^{[2]} b^* a^* (0) = (1 \otimes b^*) \cdot (a^* \otimes 1).$$

This, combined with the c.c. on $C([0,1])$, restricts to $A_\ast A$.

Furthermore, since

$$\Delta_{A \otimes A} = A \text{can}^{-1} \circ (\text{id} \otimes \Delta_A) \circ A \text{can},$$

is a composition of *-homomorphisms, so is $\Delta_{A \otimes A}$, as well as $\Delta_{A_\ast A}$ as a restriction of $\text{id} \otimes \Delta_{A \otimes A}$. Thus

**Theorem**

*If $A$ is an $H$ bicomodule and right *-comodule algebra, and a left $H$-Galois object, then the braided join algebra $A_\ast A$ is a right $H$ *-comodule algebra for the diagonal coaction.*
Non-cosemisimple example

Let $q \in \mathbb{C}$ such that $q^3 = 1$, and let $H$ denote the (9-dim) Hopf algebra generated by $a$ and $b$ with relations

$$ab = qba, \quad a^3 = 1, \quad b^3 = 0.$$ 

The comultiplication $\Delta$, counit $\varepsilon$, and antipode $S$ are

$$\Delta(a) = a \otimes a, \quad \varepsilon(a) = 1, \quad S(a) = a^2,$$

$$\Delta(b) = a \otimes b + b \otimes a^2, \quad \varepsilon(b) = 0, \quad S(b) = -q^2 b.$$

Set $\alpha_L := a \otimes 1$, $\beta_L := b \otimes 1$, $\alpha_R := 1 \otimes a$, $\beta_R := 1 \otimes b$.

The $H$-braided join of $H$

$$H \ast H = \left\{ \sum_{k,l,m,n=0}^{2} f_{klmn} \otimes \alpha_L^k \beta_L^l \alpha_R^m \beta_R^n \in C([0,1]) \otimes H \otimes H \midight.$$ 

$$f_{klmn}(0) = 0 \quad \text{for} \quad (k,l) \neq (0,0),$$

$$f_{klmn}(1) = 0 \quad \text{for} \quad (m,n) \neq (0,0) \right\}$$

is a finite quantum covering encapsulating the nontrivial $\mathbb{Z}/3\mathbb{Z}$-principal bundle $(\mathbb{Z}/3\mathbb{Z}) \ast (\mathbb{Z}/3\mathbb{Z})$ over $\Sigma(\mathbb{Z}/3\mathbb{Z})$. 
Let $H$ be a finite-dimensional Hopf algebra. The multiplication of the anti-Drinfeld double algebra $AD(H) := H^* \otimes H$ is

$$(\varphi \otimes h)(\varphi' \otimes h') = \varphi'_{(1)}(S^{-1}(h_{(3)}))\varphi'_{(3)}(S^2(h_{(1)})) \varphi \varphi'_{(2)} \otimes h_{(2)}h'.$$

$D(H)$ is a Hopf algebra with $\Delta(\varphi \otimes h) = \varphi_{(2)} \otimes h_{(1)} \otimes \varphi_{(1)} \otimes h_{(2)}$.

$AD(H)$-modules $\leftrightarrow$ anti-Yetter-Drinfeld modules over $H$.

**Theorem**

Let $H$ be a finite-dimensional Hopf algebra. Then the anti-Drinfeld double $AD(H)$ is a bicomodule algebra and a left and right Galois object over the Drinfeld double $D(H)$ for coactions, respectively,

$$\Delta(\psi \otimes k) = \psi_{(2)} \otimes S^2(k_{(1)}) \otimes \psi_{(1)} \otimes k_{(2)},$$

$$\Delta(\varphi \otimes h) = \varphi_{(2)} \otimes h_{(1)} \otimes \varphi_{(1)} \otimes h_{(2)}.$$

With $H$ as before, $\text{dim } D(H) = \text{dim } AD(H) = 81$, and we get a neat example of $AD(H)^* AD(H)$ as a $D(H)$-bundle over $\Sigma D(H)$. 
Are the semiclassical aspects of the above interesting?

Thanks!

&

JOIN!