Quantization of Poisson-Lie Hamiltonian systems

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Outline

Hamiltonian actions
  Hamiltonian actions in canonical setting
  Hamiltonian actions in Poisson-Lie setting

Quantization
  Formal approach
  Drinfeld approach
Symmetries and Conserved quantities

How to obtain conserved quantities for systems with symmetries?

- system?
- symmetries?
- conserved quantity?
Semi-classical Step

Let’s put a Poisson structure on our Lie group!

New structures:

▶ Poisson Lie groups
▶ Lie bialgebras

What is a Hamiltonian action in this context?
Poisson action

Definition
The action of $(G, \pi_G)$ on $(M, \pi)$ is called Poisson action if the map $\Phi : G \times M \to M$ is Poisson, where $G \times M$ is a Poisson manifold with structure $\pi_G \oplus \pi$.

Generalization of canonical action! If $\pi_G = 0$, the action is Poisson if and only if it preserves $\pi$. 
Momentum map

Definition (Lu)
A momentum map for the Poisson action $\Phi : G \times M \to M$ is a map $\mu : M \to G^*$ such that

$$\hat{X} = \pi^\#(\mu^*(\theta_X))$$

where $\theta_X$ is the left invariant 1-form on $G^*$ defined by the element $X \in \mathfrak{g} = (T_e G^*)^*$ and $\mu^*$ is the cotangent lift $T^* G^* \to T^* M$. 
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A Hamiltonian action is a Poisson action induced by an equivariant momentum map.
Infinitesimal momentum map

The forms $\alpha_X = \mu^*(\theta_X)$ satisfy

$$\alpha_{[X,Y]} = [\alpha_X, \alpha_Y]_\pi \quad \text{and} \quad d\alpha_X + \alpha \wedge \alpha \circ \delta(X) = 0$$

**Definition**

Let $M$ be a Poisson manifold and $G$ a Poisson Lie group. An infinitesimal momentum map is a morphism of Gerstenhaber algebras

$$\alpha : (\wedge^\bullet \mathfrak{g}, \delta, [ , ]) \longrightarrow (\Omega^\bullet(M), d_{DR}, [ , ]_\pi).$$
Steps in formal approach

Goal: quantize Hamiltonian actions

1. Quantize structures
2. Quantize Poisson action
3. Quantize Momentum map
Quantum action

How can we define a quantum action of $U\hbar(g)$ on $A\hbar$?

- Hopf algebra action
- $\hbar \to 0$ Poisson action
Quantum action

How can we define a quantum action of $\mathcal{U}_\hbar(g)$ on $A_\hbar$?

- Hopf algebra action
- $\hbar \to 0$ Poisson action

Definition
The quantum action is a linear map

$$\Phi_\hbar : \mathcal{U}_\hbar(g) \to End \ A_\hbar : X \mapsto \Phi_\hbar(X)(f)$$

such that

1. Hopf algebra action
2. Algebra homomorphism
Quantum Hamiltonian action

1. Quantum momentum map which, as in the classical case, generates the quantum action
2. $\hbar \to 0$ classical momentum map
Quantum Hamiltonian action

1. Quantum momentum map which, as in the classical case, generates the quantum action
2. $\hbar \to 0$ classical momentum map

Definition
A quantum momentum map is defined to be a linear map

$$\mu_\hbar : \mathcal{U}_\hbar(g) \to \Omega^1(A_\hbar) : X \mapsto a_X db_X.$$
General idea

joint with R. Nest and P. Bieliavsky

- Formal Drinfeld twist
- Non-formal Drinfeld twists (Bielavsky, Gayral)
Triangular Lie biagebras

Consider a particular class of Lie bialgebras \((\mathfrak{g}, \delta)\) with

\[ \delta(x) = [r, x] \]
Triangular Lie biagebras

Consider a particular class of Lie bialgebras \((g, \delta)\) with

\[
\delta(x) = [r, x]
\]

**Theorem (Drinfeld)**

*Let \(g\) be a finite dimensional real Lie algebra, with \(r\)-matrix \(r \in g \otimes g\). There exists a deformation \(U_h(g)\) of \(U(g)\) whose classical limit is \(g\) with Lie bialgebra structure defined by \(r\). Furthermore, \(U_h(g)\) is a triangular Hopf algebra and isomorphic to \(U(g)[[\hbar]]\)*
Drinfeld Twist

- giving a twist on $\mathcal{U}_\hbar(g)$ is equivalent to give an associative star product on $C^\infty(G)$

$$ f \star g := m(\tilde{F}(f \otimes g)) $$
Drinfeld Twist

- Giving a twist on $\mathcal{U}_\hbar(\mathfrak{g})$ is equivalent to give an associative star product on $\mathcal{C}^\infty(G)$

$$f \ast g := m(\tilde{F}(f \otimes g))$$

- Given a twist, every $\mathcal{U}(\mathfrak{g})$-module-algebra $\mathcal{A}$ may then be formally deformed into an associative algebra $\mathcal{A}[[\hbar]]$

$$m^F := m \circ F.$$
Drinfeld Twist

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Question: does twist produce quantum Hamiltonian action?
Beliavsky-Gayral construction

Triangular structures associated to Kähler Lie groups: non formal approach!

Explicit construction of families of kernels

\[ \{ \kappa_t \in C^\infty(G \times G) \}_t \]

such that for “any” action of $G$ on a $C^*$-algebra $\mathcal{A}$ by $C^*$-algebra automorphisms, $\kappa_t$ defines an star product on $\mathcal{A}$
Non formal Twist?

If $\mathcal{A}$ is the algebra of (complex valued continuous) functions on $G$, which $G$ acts on via the right-regular representation, then asymptotic expansion automatically yields a left-invariant formal $\star$-product on $(G, \omega^G)$:

$$ f_1 \star_t f_2 := f_1 f_2 + \sum_{k \geq 1} \left( \frac{t}{2i} \right)^k \tilde{F}_k^{(\kappa)}(f_1, f_2) \quad (f_1, f_2 \in C_0^\infty(G)) $$

$F$ defines formal twist quantization of our triangular Lie bialgebra!