Piecewise Principal Coactions of Co-Commutative Hopf Algebras

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Motivation

Principal comodule algebras can be thought of as objects representing principal bundles in non-commutative geometry. A crucial component of a principal comodule algebra is a **strong connection map**.

- Sometimes it suffices to prove that strong connection exists,
- Computing the associated bundle projectors or Chern-Galois characters requires an explicit formula for a strong connection.
- It is known how to construct a strong connection map on a multi-pullback comodule algebra from strong connections on multi-pullback components (in particular we know that it exists):
  - Hajac P.M., Krähmer U., Matthes R., Zieliński B., *Piecewise principal comodule algebras*, J. Noncomm. Geom. **5** (2011), 591–614.
  - Hajac P.M., Wagner E., *The Pullbacks of Principal Coactions* Documenta Math. 19 (2014) 1025–1060.
- Unfortunately, the known explicit general formula is unwieldy.

The Goal and the Plan

- Here we derive a much easier to use formula for strong connection on a mulitipullback comodule algebra, but applicable only in the case when a Hopf algebra is co-commutative.
- As certain linear splittings of projections in multi-pullback comodule algebras play a crucial role in the construction, we also present some derivations of the explicit formulas for such a splittings.
- Finally, we utilize our results to derive a strong connection formula for a recently constructed quantum sphere viewed as a quantum $\mathbb{Z}_2$-principal bundle.

Principal Comodule Algebras and Strong Connections

Let H be a Hopf algebra with bijective antipode, and let P be a right H-comodule algebra.

P is a principal comodule algebra iff

there exists a linear map $\ell : H \to P \otimes P$, $\quad \ell(h) =: \ell(h)^{\langle 1\rangle} \otimes \ell(h)^{\langle 2\rangle}$ satisfying the following conditions:

$$\ell(1_H) = 1_P \otimes 1_P$$

$$\ell(h)^{\langle 1\rangle}\ell(h)^{\langle 2\rangle} = \epsilon(h),$$

$$\ell(h_{(1)})^{\langle 1\rangle} \otimes \ell(h_{(1)})^{\langle 2\rangle} \otimes h_{(2)} = \ell(h)^{\langle 1\rangle} \otimes \ell(h)^{\langle 2\rangle}{}_{(0)} \otimes \ell(h)^{\langle 2\rangle}{}_{(1)},$$

$$S(h_{(1)}) \otimes \ell(h_{(2)})^{\langle 1\rangle} \otimes \ell(h_{(2)})^{\langle 2\rangle} = \ell(h)^{\langle 1\rangle}{}_{(1)} \otimes \ell(h)^{\langle 1\rangle}{}_{(0)} \otimes \ell(h)^{\langle 2\rangle}.$$

Such a map, if it exists, is called a **strong connection** on P. Strong connections are usually non-unique.

Piecewise Principal Comodule Algebras

Definition

A family of surjective algebra homomorphisms $\{\pi_i : P \to P_i\}_{i\in\{1,\dots,N\}}$ is called a **covering** iff

1. $\bigcap_{i\in\{1,\dots,N\}} \ker \pi_i = \{0\}$,
2. The family of ideals $(\ker \pi_i)_{i\in\{1,\dots,N\}}$ generates a distributive lattice with $+$ and $\cap$ as meet and join respectively.

Definition

An H-comodule algebra P is called **piecewise principal** iff there exists a finite family $\{\pi_i : P \to P_i\}_{i\in J}$ of surjective H-comodule algebra morphisms such that:

1. The restrictions $\pi_i\big|_{P^{\mathrm{co}H}} : P^{\mathrm{co}H} \to P_i^{\mathrm{co}H}$ form a covering.
2. The P_i's are principal H-comodule algebras.

Theorem

A piecewise principal comodule algebra is principal.

A Strong Connection Formula

Theorem

Let H be a cocomutative Hopf algebra. Let $\{\pi_i : P \to P_i\}_{i\in\{0,\dots,n\}}$ be a piecewise principal H-comodule algebra, and let $\{\ell_i : H \to P_i \otimes P_i\}_{i\in\{0,\dots,n\}}$ denote a family of strong connections on P_i's. Let V_i, $i \in \{0,\dots,n\}$, be an H sub-comodule of P_i such that $\ell_i(H) \subseteq V_i \otimes V_i$ and let $\alpha_i : V_i \to P$ be a unital, colinear splitting of π_i, i.e., $\pi_i \circ \alpha_i = \mathrm{id}_{V_i}$.
For brevity, denote for $i \in \{0,\dots,n\}$, $h \in H$

$$\theta_i(h) := \epsilon(h) - \alpha_i(\ell_i(h)^{\langle 1\rangle})\alpha_i(\ell_i(h)^{\langle 2\rangle}),$$

$$T_i(h) := \theta_i(h_{(1)})\theta_{i+1}(h_{(2)})\cdots\theta_n(h_{(n-i+1)}), \quad T_{n+1}(h) := \epsilon(h).$$

Then the linear map $\ell : H \to P \otimes P$ defined for all $h \in H$ by the formula

$$\ell(h) = \sum_{i=0}^{n} \alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle}) \otimes \alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle})T_{i+1}(h_{(2)})$$

is a strong connection on P.

Outline of the Proof. Part I

$$\theta_i(h) := \epsilon(h) - \alpha_i(\ell_i(h)^{\langle 1 \rangle})\alpha_i(\ell_i(h)^{\langle 2 \rangle}), \quad T_i(h) := \theta_i(h_{(1)})\theta_{i+1}(h_{(2)})\cdots\theta_n(h_{(n-i+1)}), \quad T_{n+1}(h) := \epsilon(h),$$

$$\ell(h) = \sum_{i=0}^{n} \alpha_i(\ell_i(h_{(1)})^{\langle 1 \rangle}) \otimes \alpha_i(\ell_i(h_{(1)})^{\langle 2 \rangle}) T_{i+1}(h_{(2)})$$

- First we prove that $\alpha_i(\ell_i(h)^{\langle 1 \rangle})\alpha_i(\ell_i(h)^{\langle 2 \rangle})$'s are coaction invariant, using the bi-colinearity of ℓ_i's, colinearity of α_i's and the co-commutativity of H.
- Hence $T_i(h)$'s are coaction invariant as well.
- The bi-colinearity of ℓ easily follows. In case of right H-colinearity it is necessary to use co-commutativity of H again.
- The unitality of ℓ follows from the unitality of ℓ_i's and α_i's.

Outline of the Proof. Part II

Prove that $\ell(h)^{\langle 1\rangle}\ell(h)^{\langle 2\rangle} = \epsilon(h)$

$$\theta_i(h) := \epsilon(h) - \alpha_i(\ell_i(h)^{\langle 1\rangle})\alpha_i(\ell_i(h)^{\langle 2\rangle}), \quad T_i(h) := \theta_i(h_{(1)})\theta_{i+1}(h_{(2)})\cdots\theta_n(h_{(n-i+1)}), \quad T_{n+1}(h) := \epsilon(h),$$
$$\ell(h) = \sum_{i=0}^{n} \alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle}) \otimes \alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle}) T_{i+1}(h_{(2)})$$

Note now that for all $i \in \{0, \dots, n\}$, and $h \in H$

$$\begin{aligned} T_i(h) &= \theta_i(h_{(1)}) T_{i+1}(h_{(2)}) \\ &= \epsilon(h_{(1)}) T_{i+1}(h_{(2)}) - \alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle})\alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle}) T_{i+1}(h_{(2)}) \\ &= T_{i+1}(h) - \alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle})\alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle}) T_{i+1}(h_{(2)}). \end{aligned}$$

By applying this formula to $T_0(h)$ and keeping to expand with it the leftmost summand of the resulting expansion we obtain easily:

$$T_0(h) = \epsilon(h) - \sum_{i=0}^{n} \alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle})\alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle}) T_{i+1}(h_{(2)}).$$

Outline of the Proof. Part III

Prove that $\ell(h)^{\langle 1\rangle}\ell(h)^{\langle 2\rangle} = \epsilon(h)$ cd.

$$\theta_i(h):=\epsilon(h)-\alpha_i(\ell_i(h)^{\langle 1\rangle})\alpha_i(\ell_i(h)^{\langle 2\rangle}), \quad T_i(h):=\theta_i(h_{(1)})\theta_{i+1}(h_{(2)})\cdots\theta_n(h_{(n-i+1)}), \quad T_{n+1}(h):=\epsilon(h),$$

$$\ell(h)=\sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle})\otimes\alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle})T_{i+1}(h_{(2)}),$$

$$T_0(h)=\epsilon(h)-\sum_{i=0}^n \alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle})\alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle})T_{i+1}(h_{(2)}).$$

On the other hand, as α_i is the splitting of π_i it follows that:

$$\begin{aligned}\pi_i(\theta_i(h)) &= \epsilon(h)-\pi_i\Big(\alpha_i(\ell_i(h)^{\langle 1\rangle})\Big)\pi_i\Big(\alpha_i(\ell_i(h)^{\langle 2\rangle})\Big)\\ &= \epsilon(h)-\ell_i(h)^{\langle 1\rangle}\ell_i(h)^{\langle 2\rangle}=0.\end{aligned}$$

Hence

$$\pi_i(T_j(h))=0, \quad \text{for all } i\geq j,\ i\in\{0,\ldots,n\},\ h\in H.$$

In particular, $\pi_i(T_0(h))=0$ for all $i\in\{0,\ldots,n\}$ and $h\in H$. It follows that $T_0(h)=0$ for all $h\in H$ because $\bigcap_{i=0}^n \ker\pi_i=\{0\}$, as $\{\pi_i : P\longrightarrow P_i\}_{i\in\{0,\ldots,n\}}$ is a covering by the rezults of [HKMZ11].

Outline of the Proof. Part IV

Prove that $\ell(h)^{\langle 1\rangle}\ell(h)^{\langle 2\rangle} = \epsilon(h)$ cd.

$$\theta_i(h):=\epsilon(h)-\alpha_i(\ell_i(h)^{\langle 1\rangle})\alpha_i(\ell_i(h)^{\langle 2\rangle}),\quad T_i(h):=\theta_i(h_{(1)})\theta_{i+1}(h_{(2)})\cdots\theta_n(h_{(n-i+1)}),\quad T_{n+1}(h):=\epsilon(h),$$

$$\ell(h)=\textstyle\sum_{i=0}^{n}\alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle})\otimes\alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle})T_{i+1}(h_{(2)}),$$

$$T_0(h)=\epsilon(h)-\textstyle\sum_{i=0}^{n}\alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle})\alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle})T_{i+1}(h_{(2)})=0.$$

Combining $T_0(h) = 0$ with the formula for $\ell(h)$ we obtain that for all $h \in H$

$$\ell(h)^{\langle 1\rangle}\ell(h)^{\langle 2\rangle} = \sum_{i=0}^{n}\alpha_i(\ell_i(h_{(1)})^{\langle 1\rangle})\alpha_i(\ell_i(h_{(1)})^{\langle 2\rangle})T_{i+1}(h_{(2)}) = \epsilon(h).$$

Further Development

- Our expression for a strong connection requires the unital and colinear splittings of projections π_i to be given.
- The lemma below guarantees the existence of such a splitting, but the construction assumes ℓ is already known.
- In many cases, the appropriate splittings will be easily guessable.
- However we will examine methods of constructing the splittings in cases when the piecewise principal extension is given as a multimullback comodule algebra, without using ℓ.

Lemma [HKMZ11]

Let $\pi : P \to Q$ be a surjection of right H-comodule algebras. If P is principal, then:

1. The induced map $\pi^{\mathrm{co}H} : P^{\mathrm{co}H} \to Q^{\mathrm{co}H}$ is surjective.
2. There exists a unital H-colinear splitting of π.

The splitting is given by $\alpha(q) := \alpha^{\mathrm{co}H}(q_{(0)}\pi(\ell(q_{(1)})^{\langle 1\rangle}))\ell(q_{(1)})^{\langle 2\rangle})$, where $\alpha^{\mathrm{co}H}$ is any unital splitting of $\pi^{\mathrm{co}H}$.

Multi-pullbacks of Algebras

Let J be a finite set. and let the following be the family of algebra homomorphisms referred to as as "gluing maps":

$$\{\pi^i_j : A_i \longrightarrow A_{ij} = A_{ji}\}_{i,j\in J, i\neq j} \qquad (*)$$

Definition

A family (*) of surjective algebra homomorphisms is called **distributive** iff their kernels generate distributive lattices of ideals.

Definition

The **multi-pullback algebra** A^π of a family (*) of algebra homomorphisms is defined as

$$A^\pi := \left\{ (a_i)_{i\in J} \in \prod_{i\in J} A_i \;\middle|\; \pi^i_j(a_i) = \pi^j_i(a_j),\ \forall\, i,j \in J, i \neq j \right\}.$$

Cocycle Condition

Let $(\pi^i_j : A_i \to A_{ij})_{i,j\in J,\, i\neq j}$ be a family of surjective algebra homomorphisms. For any distinct i, j, k we put $A^i_{jk} := A_i/(\ker\pi^i_j + \ker\pi^i_k)$ and take $[\cdot]^i_{jk} : A_i \to A^i_{jk}$ to be the canonical surjections. Next, we introduce the family of maps

$$\pi^{ij}_k : A^i_{jk} \longrightarrow A_{ij}/\pi^i_j(\ker\pi^i_k), \qquad [a_i]^i_{jk} \longmapsto \pi^i_j(a_i) + \pi^i_j(\ker\pi^i_k).$$

They are isomorphisms when π^i_j's are surjective homomorphisms.

Definition

We say that a family $(\pi^i_j : A_i \to A_{ij})_{i,j\in J,\, i\neq j}$ of surjective algebra homomorphisms satisfies the **cocycle condition** if and only if, for all distinct $i, j, k \in J$,

1. $\pi^i_j(\ker\pi^i_k) = \pi^j_i(\ker\pi^j_k)$,
2. isomorphisms $\phi^{ij}_k := (\pi^{ij}_k)^{-1} \circ \pi^{ji}_k : A^j_{ik} \to A^i_{jk}$ satisfy $\phi^{ik}_j = \phi^{ij}_k \circ \phi^{jk}_i$.

Cocycle Condition cont.

One can prove

that the cocycle condition together with distributivity guarantees that all projections on components of a multipullback are surjective (in fact all projections on submultipullbacks are surjective, but we will not make use of that fact).

An observation

Observe that, for all distinct $i, j, k \in J$ and any $a_i \in A_i$, $a_j \in A_j$,

$$[a_i]^i_{jk} = \phi^{ij}_k([a_j]^j_{ik}) \Leftrightarrow \pi^{ji}_k([a_j]^j_{ik}) = \pi^{ij}_k([a_i]^i_{jk})$$
$$\Leftrightarrow \pi^i_j(a_i) - \pi^j_i(a_j) \in \pi^i_j(\ker \pi^i_k).$$

Assumptions

Suppose that a distributive family $(\pi^i_j : A_i \to A_{ij})_{i,j\in J, i\neq j}$ satisfies the cocycle condition and that there exists two families $\alpha^i_j, \beta^i_j : A_{ij} \to A_i$, $i, j \in J,\ j \neq i$ of linear (colinear) splittings of π^i_j's such that all β^i_j's are unital and for all distinct $i, j, k \in J$ we have

$$\alpha^i_j(\pi^i_j(\ker \pi^i_k)) \subseteq \ker \pi^i_k. \qquad (**)$$

Construction of Colinear Splittings cont.

Theorem

Let $i \in J$, $|J| = n+1$ *and let* $\kappa : \{0, \dots, n\} \to J$ *be a bijection s.t.* $\kappa_0 = i$*, where* $\kappa_j := \kappa(j)$*. Then*

$$\alpha_i : A_i \to A^\pi, \quad a \mapsto (a_j)_{j \in J},$$

where $a_i := a$ *and* $a_{\kappa_{m+1}} := a^m_{\kappa_{m+1}}$ *for any* $0 \leq m < n$*, is a unital and linear (colinear) splitting of* $\pi_i : A^\pi \to A_i$*.*
The collections $\{a^k_{\kappa_{m+1}}\}_{0 \leq k \leq m} \subseteq A_{\kappa_{m+1}}$*, for* $0 \leq m < n$ *are defined by:*

$$a^0_{\kappa_{m+1}} := \beta^{\kappa_{m+1}}_{\kappa_0}(\pi^{\kappa_0}_{\kappa_{m+1}}(a_{\kappa_0})),$$
$$a^{k+1}_{\kappa_{m+1}} := a^k_{\kappa_{m+1}} - \alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right)$$

for $0 \leq k < m$*.*

Outline of the Proof. Part I

$$\alpha^i_j, \beta^i_j : A_{ij} \to A_i, \quad \pi^i_j \circ \alpha^i_j = \pi^i_j \circ \beta^i_j = \mathrm{id}_{A_{ij}}, \quad \alpha^i_j(\pi^i_j(\ker \pi^i_k)) \subseteq \ker \pi^i_k$$
$$\alpha_i : A_i \to A^\pi, \quad a \mapsto (a_j)_{j\in J}, \quad \text{where} \quad a_{\kappa_0} := a, \quad a_{\kappa_{m+1}} := a^m_{\kappa_{m+1}}, \quad \text{for all } 0 \le m < n,$$
$$a^0_{\kappa_{m+1}} := \beta^{\kappa_{m+1}}_{\kappa_0}(\pi^{\kappa_0}_{\kappa_{m+1}}(a_{\kappa_0})), \quad 0 \le m < n,$$
$$a^{k+1}_{\kappa_{m+1}} := a^k_{\kappa_{m+1}} - \alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right), \quad 0 \le k < m < n.$$

- Because all the maps involved in the definition of α_i are unital and linear (colinear if need be) it follows that also α_i is (co)-linear.
- Unitality of α_i follows easily from the unitality of β^j_k's.
- Now it remains to show that $\alpha_i(a) \in A^\pi$ for all $a \in A_i$. The inductive proof is a constructive version of the proof of Proposition 9 in
  - Calow, D., Matthes, R. (2000). *„Covering and gluing of algebras and differential algebras"*. Journal of Geometry and Physics, **32**(4), 364-396.

We will show that for any $0 \le m \le n$ we have

$$\pi^{\kappa_j}_{\kappa_l}(a_{\kappa_j}) = \pi^{\kappa_l}_{\kappa_j}(a_{\kappa_l}), \quad \text{for all } j, l \in \{0, \ldots, m\},\ j \ne l. \qquad (***)$$

For $m = 0$ this condition is emptily satisfied.

Outline of the Proof. Part II

$$\beta^i_j : A_{ij} \to A_i, \quad \pi^i_j \circ \beta^i_j = \mathrm{id}_{A_{ij}},$$

$$\alpha_i : A_i \to A^\pi, \quad a \mapsto (a_j)_{j\in J}, \quad \text{where} \quad a_{\kappa_0} := a, \quad a_{\kappa_{m+1}} := a^m_{\kappa_{m+1}}, \quad \text{for all } 0 \le m < n,$$

$$a^0_{\kappa_{m+1}} := \beta^{\kappa_{m+1}}_{\kappa_0}(\pi^{\kappa_0}_{\kappa_{m+1}}(a_{\kappa_0})), \quad 0 \le m < n.$$

$$\pi^{\kappa_j}_{\kappa_l}(a_{\kappa_j}) = \pi^{\kappa_l}_{\kappa_j}(a_{\kappa_l}), \quad \text{for all } j,l \in \{0,\dots,m\},\ j \neq l. \tag{***}$$

- Suppose we have proven (***) for some m. In order to demonstrate it for $m+1$, we prove by induction that for any $0 \le k \le m < n$,

$$\pi^{\kappa_j}_{\kappa_{m+1}}(a_{\kappa_j}) = \pi^{\kappa_{m+1}}_{\kappa_j}(a^k_{\kappa_{m+1}}), \quad \text{for all } 0 \le j \le k. \tag{****}$$

If $k = 0$ then substituting the definition of $a^0_{\kappa_{m+1}}$ yields

$$\pi^{\kappa_{m+1}}_{\kappa_0}(a^0_{\kappa_{m+1}}) = \pi^{\kappa_{m+1}}_{\kappa_0}\Big(\beta^{\kappa_{m+1}}_{\kappa_0}(\pi^{\kappa_0}_{\kappa_{m+1}}(a_{\kappa_0}))\Big) = \pi^{\kappa_0}_{\kappa_{m+1}}(a_{\kappa_0}).$$

Outline of the Proof. Part III

For any distinct i,j,k: $A^i_{jk} := A_i/(\ker\pi^i_j + \ker\pi^i_k)$, $[\cdot]^i_{jk} : A_i \to A^i_{jk}$ — canonical surjections

$$\phi^{ij}_k : A^j_{ik} \to A^i_{jk}, \quad \phi^{ik}_j = \phi^{ij}_k \circ \phi^{jk}_i$$

For distinct $i,j,k \in J$ and all $a_i \in A_i$, $a_j \in A_j$, $[a_i]^i_{jk} = \phi^{ij}_k([a_j]^j_{ik}) \Leftrightarrow \pi^i_j(a_i) - \pi^j_i(a_j) \in \pi^i_j(\ker\pi^i_k)$.

$$\pi^{\kappa_j}_{\kappa_l}(a_{\kappa_j}) = \pi^{\kappa_l}_{\kappa_j}(a_{\kappa_l}), \quad \text{for all } j,l \in \{0,\ldots,m\},\ j \neq l, \tag{***}$$

$$\pi^{\kappa_j}_{\kappa_{m+1}}(a_{\kappa_j}) = \pi^{\kappa_{m+1}}_{\kappa_j}(a^k_{\kappa_{m+1}}), \quad \text{for all } 0 \le j \le k. \tag{****}$$

Suppose now that we have proven Condition (****) for some $0 \le k < m$. Pick any $0 \le j \le k$. Then by (inductively assumed) Condition (***) $[a_{\kappa_j}]^{\kappa_j}_{\kappa_{k+1}\kappa_{m+1}} = \phi^{\kappa_j\kappa_{k+1}}_{\kappa_{m+1}}\left([a_{\kappa_{k+1}}]^{\kappa_{k+1}}_{\kappa_j\kappa_{m+1}}\right)$. Then it follows that

$$\begin{aligned}
[a^k_{\kappa_{m+1}}]^{\kappa_{m+1}}_{\kappa_j\kappa_{k+1}} &= \phi^{\kappa_{m+1}\kappa_j}_{\kappa_{k+1}}\left([a_{\kappa_j}]^{\kappa_j}_{\kappa_{m+1}\kappa_{k+1}}\right)\\
&= \phi^{\kappa_{m+1}\kappa_j}_{\kappa_{k+1}}\left(\phi^{\kappa_j\kappa_{k+1}}_{\kappa_{m+1}}\left([a_{\kappa_{k+1}}]^{\kappa_{k+1}}_{\kappa_j\kappa_{m+1}}\right)\right)\\
&= \phi^{\kappa_{m+1}\kappa_{k+1}}_{\kappa_j}\left([a_{\kappa_{k+1}}]^{\kappa_{k+1}}_{\kappa_j\kappa_{m+1}}\right).
\end{aligned}$$

Outline of the Proof. Part IV

For any distinct i,j,k: $A^i_{jk} := A_i/(\ker\pi^i_j + \ker\pi^i_k)$, $[\cdot]^i_{jk} : A_i \to A^i_{jk}$ — canonical surjections

$$\phi^{ij}_k : A^j_{ik} \to A^i_{jk}, \quad \phi^{ik}_j = \phi^{ij}_k \circ \phi^{jk}_i$$

For distinct $i,j,k \in J$ and all $a_i \in A_i$, $a_j \in A_j$, $[a_i]^i_{jk} = \phi^{ij}_k([a_j]^j_{ik}) \Leftrightarrow \pi^i_j(a_i) - \pi^j_i(a_j) \in \pi^i_j(\ker\pi^i_k)$.

The equality $[a^k_{\kappa_{m+1}}]^{\kappa_{m+1}}_{\kappa_j\kappa_{k+1}} = \phi^{\kappa_{m+1}\kappa_{k+1}}_{\kappa_j}\left([a_{\kappa_{k+1}}]^{\kappa_{k+1}}_{\kappa_j\kappa_{m+1}}\right)$ is equivalent to

$$\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}}) \in \pi^{\kappa_{m+1}}_{\kappa_{k+1}}(\ker\pi^{\kappa_{m+1}}_{\kappa_j}).$$

Because the above relation "is an element of" holds for an arbitrary $0 \le j \le k$ it implies immediately that

$$\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}}) \in \bigcap_{0\le j\le k} \pi^{\kappa_{m+1}}_{\kappa_{k+1}}(\ker\pi^{\kappa_{m+1}}_{\kappa_j}).$$

Outline of the Proof. Part V

$$\alpha^i_j : A_{ij} \to A_i, \quad \alpha^i_j(\pi^i_j(\ker \pi^i_k)) \subseteq \ker \pi^i_k,$$

$$\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}}) \in \bigcap_{0\le j\le k} \pi^{\kappa_{m+1}}_{\kappa_{k+1}}(\ker \pi^{\kappa_{m+1}}_{\kappa_j}).$$

Then

$$\begin{aligned} \alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right) \in \quad & \alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\bigcap_{0\le j\le k} \pi^{\kappa_{m+1}}_{\kappa_{k+1}}(\ker \pi^{\kappa_{m+1}}_{\kappa_j})\right) \\ \in \quad & \bigcap_{0\le j\le k} \alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(\ker \pi^{\kappa_{m+1}}_{\kappa_j})\right) \\ \subseteq \quad & \bigcap_{0\le j\le k} \ker \pi^{\kappa_{m+1}}_{\kappa_j} \end{aligned}$$

that is

$$\alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right) \in \bigcap_{0\le j\le k} \ker \pi^{\kappa_{m+1}}_{\kappa_j}.$$

Outline of the Proof. Part VI

$$\pi^{\kappa_j}_{\kappa_l}(a_{\kappa_j}) = \pi^{\kappa_l}_{\kappa_j}(a_{\kappa_l}), \quad \text{for all } j,l \in \{0,\dots,m\},\ j \neq l, \tag{***}$$

$$\pi^{\kappa_j}_{\kappa_{m+1}}(a_{\kappa_j}) = \pi^{\kappa_{m+1}}_{\kappa_j}(a^k_{\kappa_{m+1}}), \quad \text{for all } 0 \leq j \leq k. \tag{****}$$

$$\alpha^i_j : A_{ij} \to A_i, \quad \pi^i_j \circ \alpha^i_j = \mathrm{id}_{A_{ij}}, \quad \alpha^i_j(\pi^i_j(\ker \pi^i_k)) \subseteq \ker \pi^i_k$$

$$\alpha_i : A_i \to A^\pi, \quad a \mapsto (a_j)_{j\in J}, \quad \text{where} \quad a_{\kappa_0} := a, \quad a_{\kappa_{m+1}} := a^m_{\kappa_{m+1}}, \quad \text{for all } 0 \leq m < n,$$

$$a^{k+1}_{\kappa_{m+1}} := a^k_{\kappa_{m+1}} - \alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right), \quad 0 \leq k < m < n,$$

$$\alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right) \in \bigcap_{0\leq j\leq k} \ker \pi^{\kappa_{m+1}}_{\kappa_j}.$$

Then for all $0 \leq l \leq k$

$$\begin{aligned}
\pi^{\kappa_{m+1}}_{\kappa_l}(a^{k+1}_{\kappa_{m+1}}) &= \pi^{\kappa_{m+1}}_{\kappa_l}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{m+1}}_{\kappa_l}\left(\alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right)\right) \\
&= \pi^{\kappa_{m+1}}_{\kappa_l}(a^k_{\kappa_{m+1}}) \\
&= \pi^{\kappa_l}_{\kappa_{m+1}}(a_{\kappa_l}).
\end{aligned}$$

Outline of the Proof. Part VII

$$\pi^{\kappa_j}_{\kappa_l}(a_{\kappa_j}) = \pi^{\kappa_l}_{\kappa_j}(a_{\kappa_l}), \quad \text{for all } j,l \in \{0,\dots,m\},\ j \neq l, \qquad (***)$$

$$\pi^{\kappa_j}_{\kappa_{m+1}}(a_{\kappa_j}) = \pi^{\kappa_{m+1}}_{\kappa_j}(a^k_{\kappa_{m+1}}), \quad \text{for all } 0 \leq j \leq k. \qquad (****)$$

$$\alpha^i_j : A_{ij} \to A_i, \quad \pi^i_j \circ \alpha^i_j = \mathrm{id}_{A_{ij}}, \quad \alpha^i_j(\pi^i_j(\ker \pi^i_k)) \subseteq \ker \pi^i_k$$

$$\alpha_i : A_i \to A^\pi, \quad a \mapsto (a_j)_{j\in J}, \quad \text{where} \quad a_{\kappa_0} := a, \quad a_{\kappa_{m+1}} := a^m_{\kappa_{m+1}}, \quad \text{for all } 0 \leq m < n,$$

$$a^{k+1}_{\kappa_{m+1}} := a^k_{\kappa_{m+1}} - \alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right), \quad 0 \leq k < m < n.$$

Moreover, using the fact that $\alpha^{\kappa_{m+1}}_{\kappa_{k+1}}$ is a splitting of $\pi^{\kappa_{m+1}}_{\kappa_{k+1}}$ we obtain

$$\begin{aligned}
\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^{k+1}_{\kappa_{m+1}}) &= \pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\alpha^{\kappa_{m+1}}_{\kappa_{k+1}}\left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right)\right) \\
&= \pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \left(\pi^{\kappa_{m+1}}_{\kappa_{k+1}}(a^k_{\kappa_{m+1}}) - \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}})\right) \\
&= \pi^{\kappa_{k+1}}_{\kappa_{m+1}}(a_{\kappa_{k+1}}),
\end{aligned}$$

which ends the proof.

A Comment on the Applicabilty of the Result

- At this point, the skeptical reader might be excused for doubting the applicability of the above theorem.
- Unital and linear splittings β_j^i's of π_j^i's exist because of the surjectivity of π_j^i's, and the colinear ones can be constructed using strong connections on A_i's.
- But it is not clear how to find the linear splittings α_j^i satisfying $\alpha_j^i(\pi_j^i(\ker \pi_k^i)) \subseteq \ker \pi_k^i$, nor that they exist at all.
- Fortunately, the results from the subsequent slides assure the existence of splittings α_j^i and provide the method of their (semi)-explicit construction.

Partitions of Sets

Let A be a set and let A_i, $i \in J$ be a fixed finite family of subsets of A. For any $\Gamma \in 2^J$ we denote for brevity:

$$A_\Gamma := \bigcap_{i \in \Gamma} A_i.$$

Obviously $A_{\Gamma_1} \cap A_{\Gamma_2} = A_{\Gamma_1 \cup \Gamma_2}$. Also $A_\emptyset = A$ by convention.

It is easy to see that A_i's generate a partition $\{B_\Gamma\}_{\Gamma \in 2^J}$ of A (i.e., all B_Γ's are disjoint and $A = \bigcup_{\Gamma \in 2^J} B_\Gamma$) such that

$$A_\Gamma = \bigcup_{\Gamma' \in 2^J \mid \Gamma \subseteq \Gamma'} B_{\Gamma'}, \quad \text{for all } \Gamma \in 2^J.$$

The partition can be described explicitly, for all $\Gamma \in 2^J$ by

$$B_\Gamma \quad := \quad \{x \in A \mid \forall i \in J \,.\, x \in A_i \Leftrightarrow i \in \Gamma\}.$$

Partitions of Vector Spaces

Let A be a vector space and let A_i, $i \in J$ be a fixed finite family of vector subspaces of A. We define

$$A_\Gamma := \bigcap_{i \in \Gamma} A_i.$$

- We want to define a linear counterpart of the associated partition.
- Similarly to plain sets, vector sub-spaces can be ordered by the set inclusion, and the resulting ordered set is a lattice with
  - $V_1 \cap V_2$ serving as infimum
  - and subspace sum $(V_1 + V_2)$ playing the role of supremum.
- The problem is that this lattice is not, in general, distributive.
- It turns out that the assumption that the subspaces A_i, $i \in J$ generate a distributive lattice is pivotal for proving the desired result.

Existence of the Partition

Lemma

Let A be a vector space and let A_i, $i \in I$ be a finite family of vector subspaces of A generating a distributive lattice. A has a linear basis $\mathcal{B} = \bigcup_{\Gamma \in 2^I} \mathcal{B}_\Gamma$, where $\mathcal{B}_\Gamma \subseteq A_\Gamma$, $\Gamma \in 2^I$, such that subsets $\mathcal{B}_\Gamma$ are all disjoint and satisfy the following property:

$$A_\Gamma = Span\left(\bigcup_{\Gamma' \in 2^I,\ \Gamma' \supseteq \Gamma} \mathcal{B}_{\Gamma'} \right) \qquad (*)$$

for all $\Gamma \in 2^I$.

Outline of the Proof. Part I

Fix a linear order $\leq$ on 2^I subject to the condition

$$\Gamma_1 \supseteq \Gamma_2 \quad \Rightarrow \quad \Gamma_1 \leq \Gamma_2, \qquad \text{for all } \Gamma_1, \Gamma_2 \in 2^I.$$

It is immediate that the minimal element in this order is I and maximal is $\emptyset$. Note the following property of $\leq$:

$$\Gamma > \Gamma' \quad \Rightarrow \quad \Gamma \cup \Gamma' \supset \Gamma, \quad \text{for all } \Gamma, \Gamma' \in 2^I.$$

The sets $\mathcal{B}_\Gamma$, $\Gamma \in 2^I$ can be generated inductively (with respect to $\leq$):

1. $\mathcal{B}_I$ is some linear basis of A_I.
2. $\mathcal{B}_\Gamma$, for $\Gamma > I$, is chosen as a maximal subset of A_Γ such that $\bigcup_{\Gamma' \leq \Gamma} \mathcal{B}_{\Gamma'}$ is linearly independent.

It is immediate by construction of $\mathcal{B}_\Gamma$'s that $\mathcal{B} := \bigcup_{\Gamma \in 2^I} \mathcal{B}_\Gamma$ is a linear basis of A and that all $\mathcal{B}_\Gamma$'s are disjoint.

Outline of the Proof. Part II

$\mathcal{B}_I$ is some linear basis of A_I.
$\mathcal{B}_\Gamma$, for $\Gamma > I$, is chosen as a maximal subset of A_Γ such that $\bigcup_{\Gamma' \leq \Gamma} \mathcal{B}_{\Gamma'}$ is linearly independent

We want to prove $\quad A_\Gamma = \mathrm{Span}\left(\bigcup_{\Gamma' \in 2^I,\ \Gamma' \supseteq \Gamma} \mathcal{B}_{\Gamma'}\right)$ (*)

Also by construction, $\mathcal{B}_{\Gamma'} \subseteq A_\Gamma$, $\Gamma \in 2^I$ whenever $\Gamma \subseteq \Gamma'$, which implies that half of Property (*) is trivially satisfied:

$$\mathrm{Span}\left(\bigcup_{\Gamma' \in 2^I,\ \Gamma' \supseteq \Gamma} \mathcal{B}_{\Gamma'}\right) \subseteq A_\Gamma, \quad \text{for all } \Gamma \in 2^I.$$

It also is immediate that

$$A_\Gamma \subseteq \mathrm{Span}\left(\bigcup_{\Gamma' \in 2^I,\ \Gamma' \leq \Gamma} \mathcal{B}_{\Gamma'}\right). \qquad (**)$$

We will prove the second half of Property (*) by induction on $\leq$.

Outline of the Proof. Part III

$\mathcal{B}_I$ is some linear basis of A_I.
$\mathcal{B}_\Gamma$, for $\Gamma > I$, is chosen as a maximal subset of A_Γ such that $\bigcup_{\Gamma' \leq \Gamma} \mathcal{B}_{\Gamma'}$ is linearly independent

We want to prove $\quad A_\Gamma = \mathrm{Span}\left(\bigcup_{\Gamma' \in 2^I,\ \Gamma' \supseteq \Gamma} \mathcal{B}_{\Gamma'}\right) \qquad (*)$

Induction base: I is minimal in 2^I with respect to $\leq$. Then by definition of B_I we have

$$A_I = \mathrm{Span}(\mathcal{B}_I) = \mathrm{Span}\left(\bigcup_{\Gamma' \in 2^I,\ \Gamma' \supseteq I} \mathcal{B}_{\Gamma'}\right).$$

Outline of the Proof. Part IV

$\mathcal{B}_I$ is some linear basis of A_I.
$\mathcal{B}_\Gamma$, for $\Gamma > I$, is chosen as a maximal subset of A_Γ such that $\bigcup_{\Gamma' \le \Gamma} \mathcal{B}_{\Gamma'}$ is linearly independent

We want to prove $\quad A_\Gamma = \mathrm{Span}\left(\bigcup_{\Gamma' \in 2^I,\ \Gamma' \supseteq \Gamma} \mathcal{B}_{\Gamma'}\right), \qquad (*)$

$$A_\Gamma \subseteq \mathrm{Span}\left(\bigcup_{\Gamma' \in 2^I,\ \Gamma' \le \Gamma} \mathcal{B}_{\Gamma'}\right). \qquad (**)$$

Induction step: Suppose we have proven Eq. (*) for all $\Gamma < \Gamma_0$.
For any $a \in A$, denote by $\{\alpha_\Gamma(a)\}_{\Gamma \in 2^I}$ the unique family of vectors such that $a = \sum_{\Gamma \in 2^I} \alpha_\Gamma(a)$ and that $\alpha_\Gamma(a) \in \mathrm{Span}(\mathcal{B}_\Gamma)$.
By (**) $\alpha_{\Gamma'}(a) = 0$ whenever $a \in A_\Gamma$ and $\Gamma' > \Gamma$, i.e.,

$$a = \sum_{\Gamma' \in 2^I,\ \Gamma' \le \Gamma} \alpha_{\Gamma'}(a), \quad \text{for all } a \in A_\Gamma. \qquad (***)$$

Let $a \in A_{\Gamma_0}$. Define $v := a - \alpha_{\Gamma_0}(a)$. By Eq. (***)

$$A_{\Gamma_0} \ni v = \sum_{\Gamma' \in 2^I,\ \Gamma' < \Gamma_0} \alpha_{\Gamma'}(a) \in \sum_{\Gamma' \in 2^I,\ \Gamma' < \Gamma_0} A_{\Gamma'}.$$

Outline of the Proof. Part V

$$\text{We want to prove} \quad A_\Gamma = \mathrm{Span}\big(\textstyle\bigcup_{\Gamma'\in 2^I,\ \Gamma'\supseteq\Gamma} \mathcal{B}_{\Gamma'}\big), \qquad (*)$$

$$A_\Gamma \subseteq \mathrm{Span}\big(\textstyle\bigcup_{\Gamma'\in 2^I,\ \Gamma'\leq\Gamma} \mathcal{B}_{\Gamma'}\big), \qquad (**)$$

$$A_{\Gamma_0} \ni v = \textstyle\sum_{\Gamma'\in 2^I,\,\Gamma'<\Gamma_0} \alpha_{\Gamma'}(a) \in \sum_{\Gamma'\in 2^I,\,\Gamma'<\Gamma_0} A_{\Gamma'}, \quad \Gamma\subset\Gamma\cup\Gamma' \text{ if } \Gamma'<\Gamma, \quad \Gamma'<\Gamma \text{ if } \Gamma'\supset\Gamma.$$

Hence

$$v \in A_{\Gamma_0} \cap \left(\sum_{\Gamma'\in 2^I,\,\Gamma'<\Gamma_0} A_{\Gamma'} \right) = \sum_{\Gamma'\in 2^I,\,\Gamma'<\Gamma_0} A_{\Gamma'\cup\Gamma_0}$$

$$\subseteq \sum_{\Gamma'\in 2^I,\,\Gamma'\supset\Gamma_0} A_{\Gamma'} \subseteq \mathrm{Span}\left(\bigcup_{\Gamma'\in 2^I,\,\Gamma'\supset\Gamma_0} \mathcal{B}_{\Gamma'} \right).$$

It follows that

$$a = \alpha_{\Gamma_0}(a) + v \in \mathrm{Span}(\mathcal{B}_{\Gamma_0}) + \mathrm{Span}\left(\bigcup_{\Gamma'\in 2^I,\,\Gamma'\supset\Gamma_0} \mathcal{B}_{\Gamma'} \right) = \mathrm{Span}\left(\bigcup_{\Gamma'\in 2^I,\,\Gamma'\supseteq\Gamma_0} \mathcal{B}_{\Gamma'} \right).$$

Lemma

Let $\pi : A \to B$ be a linear surjection, and let $\{A_i\}_{i \in I}$ be a finite family of vector subspaces of A such that $\{A_i\}_{i \in I} \cup \{\ker \pi\}$ generates a distributive lattice of vector subspaces. Then there exists a linear splitting $\alpha : B \to A$ of π such that $\alpha(\pi(A_i)) \subseteq A_i$ for all $i \in I$.

There exists a linear splitting $\alpha : B \to A$ of π such that $\alpha(\pi(A_i)) \subseteq A_i$ for all $i \in I$.

Auxilliary lemma

Let $\pi : A \to B$ be a linear map, and let $\{A_i\}_{i\in I}$ be a finite family of vector subspaces of A. Assume that $\ker\pi \cap (\sum_{i\in I} A_i) = \sum_{i\in I}(\ker\pi \cap A_i)$. Then $\pi(\bigcap_{i\in I} A_i) = \bigcap_{i\in I} \pi(A_i)$.

Let $\mathcal{B} := \bigcup_{\Gamma\in 2^I} \mathcal{B}_\Gamma$ be a linear basis of B defining a partition of B with respect to the family $\{B_i\}_{i\in I}$, where $B_i := \pi(A_i)$.

Note that the auxilliary lemma implies that B_i's generate distributive lattice of ideals because A_i's generate distributive lattice of ideals, and also that $B_\Gamma = \pi(A_\Gamma)$.

We define the splitting $\alpha : B \to A$ on basis elements. For all $b \in \mathcal{B}$ we define $\alpha(b)$ to be an arbitrary element of $\pi^{-1}(b) \cap A_\Gamma$, where $b \in \mathcal{B}_\Gamma$. Let $b \in B_i$, $i \in I$. Then $b \in \mathrm{Span}\left(\bigcup_{\Gamma\in 2^I \,|\, i\in\Gamma} \mathcal{B}_\Gamma\right)$ and hence

$$\alpha(b) \in \sum_{\Gamma\in 2^I \,|\, i\in\Gamma} \sum_{b'\in\mathcal{B}_\Gamma} \left(\pi^{-1}(b') \cap A_\Gamma\right) \subseteq \sum_{\Gamma\in 2^I \,|\, i\in\Gamma} A_\Gamma \subseteq A_i.$$

Lemma

Let A be a principal H-comodule algebra, let $\pi : A \to B$ be an H-comodule algebra surjection, and let $\{A_i\}_{i\in I}$ be a finite family of ideals in A which are subcomodules, such that $\{A_i\}_{i\in I} \cup \{\ker\pi\}$ generates a distributive lattice. Define for all $i \in I$: $A_i^{coH} := A_i \cap A^{coH}$, $B_i := \pi(A_i)$, $B_i^{coH} := B^{coH} \cap B_i$. Suppose that there exists a linear map $\alpha^{coH} : B^{coH} \to A^{coH}$ such that

$$\pi \circ \alpha^{coH} = \mathrm{id}_{B^{coH}}, \quad \alpha^{coH}(B_i^{coH}) \subseteq A_i^{coH}, \text{ for all } i \in I.$$

Let $\ell : H \to A \otimes A$ be a strong connection on A. Then the following formula:

$$\alpha : B \longrightarrow A, \quad b \longmapsto \alpha^{coH}\Big(b_{(0)}\pi(\ell(b_{(1)})^{\langle 1\rangle})\Big)\ell(b_{(1)})^{\langle 2\rangle}$$

defines a right H-colinear map satisfying

$$\pi \circ \alpha = \mathrm{id}_B, \quad \alpha(B_i) \subseteq A_i, \text{ for all } i \in I.$$

Example

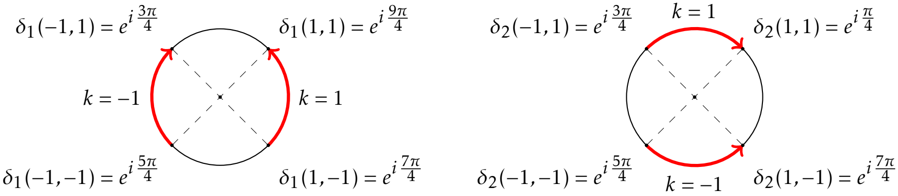
- Recently a new non-commutative real projective space $\mathbb{R}P^2_{\mathcal{T}}$ and a non-commutative sphere $S^2_{\mathbb{R}\mathcal{T}}$ were introduced, by defining $C(\mathbb{R}P^2_{\mathcal{T}})$ and $C(S^2_{\mathbb{R}\mathcal{T}})$ as a particular triple pullbacks of, respectively, three copies of the Toeplitz algebra $\mathcal{T}$ and the tensor product $\mathcal{T} \otimes C(\mathbb{Z}_2)$.
- The algebra $C(S^2_{\mathbb{R}\mathcal{T}})$ has a natural (component-wise) diagonal coaction of the Hopf algebra $C(\mathbb{Z}_2)$, and the subspace of invariants of this coaction is isomporphic with $C(\mathbb{R}P^2_{\mathcal{T}})$.
- Moreover, $C(S^2_{\mathbb{R}\mathcal{T}})$ is a piecewise principal (hence principal) $C(\mathbb{Z}_2)$-comodule algebra.
- Because $C(\mathbb{Z}_2)$ is co-commutative and $C(S^2_{\mathbb{R}\mathcal{T}})$ is defined as a triple pullback algebra, our main result is applicable here.

Hajac P.M., Rudnik J., Zieliński B.,
Reductions of piecewise trivial principal comodule algebras.

Toeplitz algebra $\mathcal{T}$ is the universal C^*-algebra generated by an isometry s. The symbol map is given by $\sigma : \mathcal{T} \ni s \mapsto \widetilde{u} \in C(S^1)$, where $\widetilde{u}$ is the unitary function generating $C(S^1)$. The following maps

$$\delta_1 : \mathbb{Z}_2 \times I \to S^1, \quad \delta_2 I \times \mathbb{Z}_2 \to S^1,$$

are defined as the parametrisation of two appropriate quarters of S^1:



Squaring the Toeplitz Algebra II

We denote the pullbacks of δ_1 and δ_2 by

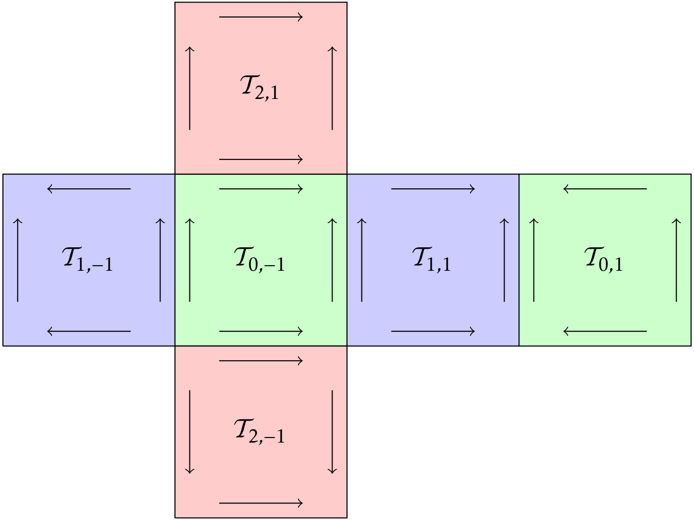
$$\delta_1^*\colon C(S^1) \longrightarrow C(\mathbb{Z}_2)\otimes C(I), \quad \delta_2^*\colon C(S^1) \longrightarrow C(I)\otimes C(\mathbb{Z}_2).$$

We denote for brevity $\sigma_i := \delta_i^* \circ \sigma$, $i = 1, 2$.

- We view S^1 and I as $\mathbb{Z}_2$-spaces via multiplication by ± 1. Then $\mathbb{Z}_2 \times I$ and $I \times \mathbb{Z}_2$ are $\mathbb{Z}_2$-spaces with the diagonal action.
- Accordingly, $C(I)$, $C(S^1)$, $C(\mathbb{Z}_2)\otimes C(I)$ and $C(I)\otimes C(\mathbb{Z}_2)$ are right $C(\mathbb{Z}_2)$-comodule algebras with coactions given by the pullbacks of respective $\mathbb{Z}_2$-actions.
- Denote by u the generator $C(\mathbb{Z}_2)$ given by $u(\pm 1) := \pm 1$. Then the assignment $s \mapsto s \otimes u$ makes $\mathcal{T}$ a $C(\mathbb{Z}_2)$-comodule algebra. (This coaction corresponds to the $\mathbb{Z}_2$-action given by $\alpha_{-1}^{\mathcal{T}}(s) = -s$.)
- The maps δ_i, $i = 1, 2$, are $\mathbb{Z}_2$-equivariant, so that δ_i^*'s are right $C(\mathbb{Z}_2)$-comodule maps. Also, since the symbol map σ is a right $C(\mathbb{Z}_2)$-comodule map, so are σ_i's.

The construction of $C(S^2_{\mathbb{R}\mathcal{T}})$

The quantum version of constructing the topological 2-sphere by assembling three pairs of squares to the boundary of a cube. $\mathcal{T} \otimes C(\mathbb{Z}_2)$ replaces the pair of squares.



The Multi-Pullback Presentation of $C(S^2_{\mathbb{R}\mathcal{T}})$. Part I

The algebra $C(S^2_{\mathbb{R}\mathcal{T}})$ is defined to be the following triple pullback of three copies of $\mathcal{T} \otimes C(\mathbb{Z}_2)$:

$$\begin{array}{ccc} \mathcal{T}_0 \otimes C(\mathbb{Z}_2) & & \mathcal{T}_1 \otimes C(\mathbb{Z}_2) \\ \downarrow{\scriptstyle \sigma_1\otimes\mathrm{id}} & & \downarrow{\scriptstyle \sigma_1\otimes\mathrm{id}} \\ C(\mathbb{Z}_2)\otimes C(I)\otimes C(\mathbb{Z}_2) & \xleftarrow[\Phi_{01}]{} & C(\mathbb{Z}_2)\otimes C(I)\otimes C(\mathbb{Z}_2)\,, \end{array}$$

$$\begin{array}{ccc} \mathcal{T}_0 \otimes C(\mathbb{Z}_2) & & \mathcal{T}_2 \otimes C(\mathbb{Z}_2) \\ \downarrow{\scriptstyle \sigma_2\otimes\mathrm{id}} & & \downarrow{\scriptstyle \sigma_1\otimes\mathrm{id}} \\ C(I)\otimes C(\mathbb{Z}_2)\otimes C(\mathbb{Z}_2) & \xleftarrow[\Phi_{02}]{} & C(\mathbb{Z}_2)\otimes C(I)\otimes C(\mathbb{Z}_2)\,, \end{array}$$

$$\begin{array}{ccc} \mathcal{T}_1 \otimes C(\mathbb{Z}_2) & & \mathcal{T}_2 \otimes C(\mathbb{Z}_2) \\ \downarrow{\scriptstyle \sigma_2\otimes\mathrm{id}} & & \downarrow{\scriptstyle \sigma_2\otimes\mathrm{id}} \\ C(I)\otimes C(\mathbb{Z}_2)\otimes C(\mathbb{Z}_2) & \xleftarrow[\Phi_{12}]{} & C(I)\otimes C(\mathbb{Z}_2)\otimes C(\mathbb{Z}_2)\,. \end{array}$$

The Multi-Pullback Presentation of $C(S^2_{\mathbb{R}\mathcal{T}})$. Part II

The isomorphisms Φ_{ij} are defined by the following formulas, for all $h, k \in C(\mathbb{Z}_2)$ and $p \in C(I)$:

$$\Phi_{01}(h \otimes p \otimes k) := k \otimes p \otimes h,$$
$$\Phi_{02}(h \otimes p \otimes k) := p \otimes k \otimes h,$$
$$\Phi_{12}(p \otimes h \otimes k) := p \otimes k \otimes h.$$

We view the algebras $\mathcal{T} \otimes C(\mathbb{Z}_2)$, $C(I) \otimes C(\mathbb{Z}_2) \otimes C(\mathbb{Z}_2)$ and $C(\mathbb{Z}_2) \otimes C(I) \otimes C(\mathbb{Z}_2)$ as right $C(\mathbb{Z}_2)$-comodules with the diagonal $C(\mathbb{Z}_2)$-coaction. The coaction of $C(\mathbb{Z}_2)$ is defined on $C(S^2_{\mathbb{R}\mathcal{T}})$ componentwise.

Auxilliary Elements of $\mathcal{T}$

The construction of a strong connection will require the existence of elements $\phi_1 \in \sigma_1^{-1}(u \otimes 1_{C(I)}) \subseteq \mathcal{T}$, $\phi_2 \in \sigma_2^{-1}(1_{C(I)} \otimes u) \subseteq \mathcal{T}$ with certain additional properties. These elements will play the crucial role in the construction of appropriate splittings.

Lemma

There exist elements $\phi_1, \phi_2 \in \mathcal{T}$ satisfying:

$$\rho(\phi_1) = \phi_1 \otimes u, \quad \rho(\phi_2) = \phi_2 \otimes u, \tag{1a}$$

$$\sigma_1(\phi_1) = u \otimes 1_{C(I)}, \quad \sigma_2(\phi_1) = \imath_I \otimes 1_{C(\mathbb{Z}_2)}, \tag{1b}$$

$$\sigma_2(\phi_2) = 1_{C(I)} \otimes u, \quad \sigma_1(\phi_2) = 1_{C(\mathbb{Z}_2)} \otimes \imath_I, \tag{1c}$$

$$(1 - \phi_2^2)(1 - \phi_1^2) \neq 0. \tag{1d}$$

where $\imath_I \in C(I)$ is an an identity map $\imath_I(t) = t$ and $\rho : \mathcal{T} \to \mathcal{T} \otimes C(\mathbb{Z}_2)$ is a right coaction.

A Strong Connection Formula for $C(S^2_{\mathbb{R}\mathcal{T}})$. Part I

The strong connections on the three copies of $C(\mathbb{Z}_2)$-comodule algebra (with diagonal coaction) $\mathcal{T} \otimes C(\mathbb{Z}_2)$ are chosen as

$$\ell_1(u) = \ell_2(u) = \ell_3(u) = (1_{\mathcal{T}} \otimes u) \otimes (1_{\mathcal{T}} \otimes u),$$
$$\ell_1(1_{C(\mathbb{Z}_2)}) = \ell_2(1_{C(\mathbb{Z}_2)}) = \ell_3(1_{C(\mathbb{Z}_2)}) = (1_{\mathcal{T}} \otimes 1_{C(\mathbb{Z}_2)}) \otimes (1_{\mathcal{T}} \otimes 1_{C(\mathbb{Z}_2)}).$$

In order to use our main result we need the appropriate colinear and unital splittings from the linear subspaces generated by the legs of ℓ_i's into $C(S^2_{\mathbb{R}\mathcal{T}})$: the maps $\alpha_i : \mathrm{Span}\{1_{\mathcal{T}} \otimes u, 1_{\mathcal{T}} \otimes 1_{C(\mathbb{Z}_2)}\} \to C(S^2_{\mathbb{R}\mathcal{T}})$, $i = 0, 1, 2$ which can be defined by

$$\alpha_0(1_{\mathcal{T}} \otimes u) := (1_{\mathcal{T}} \otimes u, \phi_1 \otimes 1_{C(\mathbb{Z}_2)}, \phi_1 \otimes 1_{C(\mathbb{Z}_2)}),$$
$$\alpha_1(1_{\mathcal{T}} \otimes u) := (\phi_1 \otimes 1_{C(\mathbb{Z}_2)}, 1_{\mathcal{T}} \otimes u, \phi_2 \otimes 1_{C(\mathbb{Z}_2)}),$$
$$\alpha_2(1_{\mathcal{T}} \otimes u) := (\phi_2 \otimes 1_{C(\mathbb{Z}_2)}, \phi_2 \otimes 1_{C(\mathbb{Z}_2)}, 1_{\mathcal{T}} \otimes u).$$

A Strong Connection Formula for $C(S^2_{\mathbb{R}\mathcal{T}})$. Part II

Let us denote for brevity $\alpha_i := \alpha_i(1_{\mathcal{T}} \otimes u)$. Because $u^2 = 1$ we have

$$1-\alpha_1^2 = \Big((1-\phi_1^2)\otimes 1, 0, (1-\phi_2^2)\otimes 1\Big), \quad 1-\alpha_1^2 = \Big((1-\phi_2^2)\otimes 1, (1-\phi_2^2)\otimes 1, 0\Big).$$

The straightforward application of the formula from the main theorem yields:

$$\begin{aligned}
\ell(u) &:= \alpha_0 \otimes \alpha_0(1-\alpha_1^2)(1-\alpha_2^2) + \alpha_1 \otimes \alpha_1(1-\alpha_2^2) + \alpha_2 \otimes \alpha_2 \\
&= (1\otimes u, \phi_1 \otimes 1, \phi_1 \otimes 1) \otimes \Big((1-\phi_1^2)(1-\phi_2^2)\otimes u, 0, 0\Big) \\
&\quad + (\phi_1 \otimes 1, 1 \otimes u, \phi_2 \otimes 1) \otimes \Big(\phi_1(1-\phi_2^2)\otimes 1, (1-\phi_2^2)\otimes u, 0\Big) \\
&\quad + (\phi_2 \otimes 1_{C(\mathbb{Z}_2)}, \phi_2 \otimes 1_{C(\mathbb{Z}_2)}, 1_{\mathcal{T}} \otimes u) \otimes (\phi_2 \otimes 1_{C(\mathbb{Z}_2)}, \\
&\qquad\quad \phi_2 \otimes 1_{C(\mathbb{Z}_2)}, 1_{\mathcal{T}} \otimes u).
\end{aligned}$$

Both left and right legs of the above strong connection are linearly independent (when taken separately).