Quantum correlations on quantum spaces

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Quantum spaces

Definition

A *quantum space* is an object $\mathbb{X}$ of the category dual to the category of C^*-algebras.

Notation

The C^*-algebra corresponding to the quantum space $\mathbb{X}$ is denoted by $\mathrm{C}(\mathbb{X})$ (resp. $\mathrm{C}_0(\mathbb{X})$) for the unital (resp. nonunital) case.

Definition

A quantum space $\mathbb{X}$ is called

- *compact* if the corresponding C^*-algebra is unital,
- *finite* if the corresponding C^*-algebra is finite-dimensional.

Quantum correlations

Definition

Let P and O be finite sets. A *quantum correlation* (or *quantum strategy*) on P and O is a collection of non-negative numbers $\{p(a,b|x,y) \mid a,b \in O,\ x,y \in P\}$ such that for each (x,y) the maps

$$a \longmapsto \sum_b p(a,b|x,y), \qquad b \longmapsto \sum_a p(a,b|x,y)$$

are probability distributions on P.

Remark

The above notion is closely related to the theory of non-local games.

Quantum correlations

Definition

Quantum commuting correlations (qc-correlations) are those of the form

$$p(a, b|x, y) = \langle \xi | E_{x,a} F_{y,b} \xi \rangle, \qquad x, y \in P,\ a, b \in O,$$

where ξ is a unit vector in a Hilbert space H and

$$\{E_{x,a} |\, x \in P,\ a \in O\} \text{ and } \{F_{y,b} |\, y \in P,\ b \in O\}$$

are families of projections in $\mathrm{B(H)}$ such that

- for all $(x, y, a, b) \in P \times P \times O \times O$ we have $E_{x,a} F_{y,b} = F_{y,b} E_{x,a}$,
- for all $x \in P$ we have $\sum_a E_{x,a} = \mathbb{1}_{\mathrm{H}}$,
- for all $y \in P$ we have $\sum_b F_{y,b} = \mathbb{1}_{\mathrm{H}}$.

Our goal

Study quantum correlations with the classical finite sets P and O replaced by their quantum analogues $\mathbb{P}$ and $\mathbb{O}$.

Inspired by previous works:

- [Brannan-Ganesan-Harris '20]
- [Todorov-Turowska '20]

Quantum-to-classical graph homomorphism game [Brannan-Ganesan-Harris '20]

The crucial object

The universal C^*-algebra $\mathcal{P}_{n,c}$ (with $n, c \in \mathbb{N}$) generated by entries of orthogonal projections $P_1, \ldots, P_c \in \mathrm{Mat}_n(\mathcal{P}_{n,c})$ s.t. $P_1 + \ldots + P_c = \mathbb{1}$.

Alternative point of view

- $P_1, \ldots, P_c$ define a $*$-homomorphism:

$$\Phi : C(\{1, \ldots, c\}) \ni f \longmapsto \sum_{a=1}^{c} f(a) P_a \in \mathrm{Mat}_n(\mathbb{C}) \otimes \mathcal{P}_{n,c}.$$

- Φ is universal: given a $*$-homomorphism

$$\Psi : C(\{1, \ldots, c\}) \to \mathrm{Mat}_n(\mathbb{C}) \otimes \mathrm{B(H)}$$

there exist unique $*$-homomorphism $\Lambda : \mathcal{P}_{n,c} \to \mathrm{B(H)}$ such that $\Psi = (\mathrm{id} \otimes \Lambda)\Phi$.

Quantum spaces of maps

Definition

Let $\mathbb{P}, \mathbb{O}$ and $\mathbb{X}$ be quantum spaces. A *quantum family of maps* from $\mathbb{P}$ to $\mathbb{O}$ indexed by $\mathbb{X}$ is a morphism $\Phi \in \mathrm{Mor}(\mathrm{C}_0(\mathbb{O}), \mathrm{C}_0(\mathbb{P}) \otimes \mathrm{C}_0(\mathbb{X}))$.

Definition

Let $\mathbb{P}$ and $\mathbb{O}$ be quantum spaces. We say that $\boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}} \in \mathrm{Mor}\left(\mathrm{C}_0(\mathbb{O}), \mathrm{C}_0(\mathbb{P}) \otimes \mathrm{C}_0(\mathbb{M}_{\mathbb{P},\mathbb{O}})\right)$ is the *quantum family of all maps* from $\mathbb{P}$ to $\mathbb{O}$ if for any quantum space $\mathbb{X}$ and any quantum family $\Psi \in \mathrm{Mor}\left(\mathrm{C}_0(\mathbb{X}), \mathrm{C}_0(\mathbb{P}) \otimes \mathrm{C}_0(\mathbb{X})\right)$ there exists a unique $\Lambda \in \mathrm{Mor}\left(\mathrm{C}_0(\mathbb{M}_{\mathbb{P},\mathbb{O}}), \mathrm{C}_0(\mathbb{X})\right)$ such that the following diagram is commutative

$$\begin{array}{ccc}
\mathrm{C}_0(\mathbb{O}) & \xrightarrow{\boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}}} & \mathrm{C}_0(\mathbb{P}) \otimes \mathrm{C}_0(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \\
\Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\Lambda} \\
\mathrm{C}_0(\mathbb{O}) & \xrightarrow{\Psi} & \mathrm{C}_0(\mathbb{P}) \otimes \mathrm{C}_0(\mathbb{X})
\end{array}$$

Quantum spaces of maps

Observation

If $(\mathbb{M}_{\mathbb{P},\mathbb{O}}, \boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}})$ exists then it is unique (up to isomorphism).

Definition

$\mathbb{M}_{\mathbb{P},\mathbb{O}}$ is called the *quantum space of all maps* from $\mathbb{P}$ to $\mathbb{O}$.

Theorem [Skalski-Sołtan '16]

Let $\mathbb{P}$ be a finite quantum space and $\mathbb{O}$ a compact quantum space. Then the quantum space $\mathbb{M}_{\mathbb{P},\mathbb{O}}$ of all maps from $\mathbb{P}$ to $\mathbb{O}$ exists and is compact. Moreover the C^*-algebra $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is generated by the set

$$\left\{(\omega \otimes \mathrm{id})\boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}}(a) \,\middle|\, a \in \mathrm{C}(\mathbb{O}),\, \omega \in \mathrm{C}(\mathbb{P})^*\right\}.$$

Remark

The C^*-algebra $\mathcal{P}_{n,c}$ of [Brannan-Ganesan-Harris '20] is precisely $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ with $\mathrm{C}(\mathbb{P}) = \mathrm{Mat}_n(\mathbb{C})$ and $\mathrm{C}(\mathbb{O}) = \mathbb{C}^c$.

Residual finite dimensionality

Definition

A C^*-algebra is *residually finite dimensional (RDF)* if it possesses a separating family of finite-dimensional representations

Theorem [Choi '80]

$\mathrm{C}^*(\mathbb{F}_2)$ is RDF.

Theorem [Brannan-Ganesan-Harris '20]

$\mathcal{P}_{n,c}$ is RDF.

Disjoint sums of quantum spaces

Definition

Let $\mathbb{P}_1, \mathbb{P}_2$ be compact quantum spaces. The quantum space $\mathbb{P}_1 \sqcup \mathbb{P}_2$ is defined by $\mathrm{C}(\mathbb{P}_1 \sqcup \mathbb{P}_2) = \mathrm{C}(\mathbb{P}_1) \oplus \mathrm{C}(\mathbb{P}_2)$.

Proposition

Let $\mathbb{P}_1, \mathbb{P}_2$ be finite quantum spaces and $\mathbb{O}$ be a compact quantum space. Then the C^*-algebra $\mathrm{C}(\mathbb{M}_{\mathbb{P}_1 \sqcup \mathbb{P}_2, \mathbb{O}})$ is isomorphic to the universal free product $\mathrm{C}(\mathbb{M}_{\mathbb{P}_1, \mathbb{O}}) * \mathrm{C}(\mathbb{M}_{\mathbb{P}_2, \mathbb{O}})$.

Lemma

Let C be a unital C^*-algebra, $\gamma : \mathrm{Mat}_n(\mathbb{C}) \to \mathsf{C}$ be a unital $*$-homomorphism and

$$\mathsf{D} = \{c \in \mathsf{C} \,|\, c\gamma(x) = \gamma(x)c \;\text{ for all } x \in \mathrm{Mat}_n(\mathbb{C})\}\,.$$

Then D is a unital C^*-algebra and C is isomorphic to $\mathrm{Mat}_n(\mathbb{C}) \otimes \mathsf{D}$.

Disjoint sums of quantum spaces

Proposition

- Let $\mathrm{C}(\mathbb{P}) = \mathrm{Mat}_n(\mathbb{C})$ and let $\mathbb{O}$ be a compact quantum space. Then $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is the relative commutant of $\mathrm{C}(\mathbb{P})$ in $\mathrm{C}(\mathbb{P}) * \mathrm{C}(\mathbb{O})$, and $\mathbf{\Phi}_{\mathbb{P},\mathbb{O}}$ is the composition of the inclusion $\mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) * \mathrm{C}(\mathbb{O})$ with the isomorphism from the previous Lemma.
- Let $\mathbb{O}$ be a compact quantum space and $\mathbb{P}$ a finite quantum space with

$$\mathrm{C}(\mathbb{P}) = \bigoplus_{i=1}^{m} \mathrm{Mat}_{n_i}(\mathbb{C}).$$

  Then $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \cong \mathsf{D}_1 * \ldots * \mathsf{D}_m$, where D_i is the relative commutant of $\mathrm{Mat}_{n_i}(\mathbb{C})$ in $\mathrm{Mat}_{n_i}(\mathbb{C}) * \mathrm{C}(\mathbb{O})$.

Residual finite dimensionality

Theorem

Let $\mathbb{O}$ be a compact quantum space such that $\mathrm{C}(\mathbb{O})$ is RDF, and let $\mathbb{P}$ be a finite quantum space. Then $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is RDF.

Proof.

- $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is a free product of algebras being subalgebras of free products of the form $\mathrm{Mat}_n(\mathbb{C}) * \mathrm{C}(\mathbb{O})$,
- RDF passes to free products [Exel-Loring '92] and to subalgebras.

□

Corollary

For any finite quantum spaces $\mathbb{P}, \mathbb{O}$ the C^*-algebra $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ possesses a faithful trace.

Functorial properties of $\mathbb{M}_{\mathbb{P},\mathbb{O}}$

Notation

- $\mathfrak{QS}_{\mathrm{fin}}$ - the full subcategory of the category of quantum spaces consisting of the finite quantum spaces
- $\mathfrak{QS}_{\mathrm{cpt}}$ - the full subcategory of the category of quantum spaces consisting of the compact quantum spaces

Functorial properties of $\mathbb{M}_{\mathbb{P},\mathbb{O}}$

Proposition

The following mapping is a bi-functor:

$$\mathfrak{QS}_{\mathrm{fin}} \times \mathfrak{QS}_{\mathrm{cpt}} \ni (\mathbb{P}, \mathbb{O}) \longmapsto \mathbb{M}_{\mathbb{P},\mathbb{O}} \in \mathfrak{QS}_{\mathrm{cpt}}.$$

Given $\mathbb{P}_1, \mathbb{P}_2 \in \mathfrak{QS}_{\mathrm{fin}}, \mathbb{O}_1, \mathbb{O}_2 \in \mathfrak{QS}_{\mathrm{cpt}}$, and $\rho : \mathrm{C}(\mathbb{P}_2) \to \mathrm{C}(\mathbb{P}_1)$, $\pi : \mathrm{C}(\mathbb{O}_1) \to \mathrm{C}(\mathbb{O}_2)$, the associated map $\mathbb{M}_{\rho,\pi} : \mathrm{C}(\mathbb{M}_{\mathbb{P}_1,\mathbb{O}_1}) \to \mathrm{C}(\mathbb{M}_{\mathbb{P}_2,\mathbb{O}_2})$ is the unique Λ making the following diagram commutative:

$$\begin{array}{ccc}
\mathrm{C}(\mathbb{O}_1) & \xrightarrow{\boldsymbol{\Phi}_{\mathbb{P}_1,\mathbb{O}_1}} & \mathrm{C}(\mathbb{P}_1) \otimes \mathrm{C}(\mathbb{M}_{\mathbb{P}_1,\mathbb{O}_1}) \\
\Big\| & & \Big\downarrow {\scriptstyle \mathrm{id}\otimes\Lambda} \\
\mathrm{C}(\mathbb{O}_1) & \xrightarrow{(\rho\otimes\mathrm{id})\circ\boldsymbol{\Phi}_{\mathbb{P}_2,\mathbb{P}_2}\circ\pi} & \mathrm{C}(\mathbb{P}_1) \otimes \mathrm{C}(\mathbb{M}_{\mathbb{P}_2,\mathbb{O}_2})
\end{array}$$

$\mathbb{M}_{\bullet,\bullet}$ is contravariant wrt the first variable and covariant wrt the second one.

Functorial properties of $\mathbb{M}_{\mathbb{P},\mathbb{O}}$

Theorem

Let $\mathbb{P}, \mathbb{P}_1, \mathbb{P}_2 \in \mathfrak{QS}_{\text{fin}}$, $\mathbb{O}, \mathbb{O}_1, \mathbb{O}_2 \in \mathfrak{QS}_{\text{cpt}}$, $\rho : \mathrm{C}(\mathbb{P}_2) \to \mathrm{C}(\mathbb{P}_1)$ and $\pi : \mathrm{C}(\mathbb{O}_1) \to \mathrm{C}(\mathbb{O}_2)$. Then:

- π - surjective $\Rightarrow \mathbb{M}_{\mathrm{id},\pi}$ - surjective,
- π - injective $\Rightarrow \mathbb{M}_{\mathrm{id},\pi}$ - injective,
- ρ - injective $\Rightarrow \mathbb{M}_{\rho,\mathrm{id}}$ - surjective,
- ρ - surjective $\Rightarrow \mathbb{M}_{\rho,\mathrm{id}}$ - injective.

The opposite algebra

Proposition

For a finite quantum space $\mathbb{P}$ and a compact quantum space $\mathbb{O}$ the pair $(\mathbb{M}_{\mathbb{P}^{\mathrm{op}},\mathbb{O}^{\mathrm{op}}}, \boldsymbol{\Phi}_{\mathbb{P}^{\mathrm{op}},\mathbb{O}^{\mathrm{op}}})$ is naturally isomorphic to $(\mathbb{M}_{\mathbb{P},\mathbb{O}}^{\mathrm{op}}, \boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}})$.

Corollary

Let $\mathbb{P}$ be a finite quantum space and $\mathbb{O}$ be a compact quantum space s.t. $\mathrm{C}(\mathbb{O})^{\mathrm{op}} \cong \mathrm{C}(\mathbb{O})$. Then $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})^{\mathrm{op}} \cong \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$.

Corollary

The assumption in the above Corollary is satisfied in particular for $\mathbb{O}$ finite.

Completeness - reminder

Definition

Let A and B be C^*-algebras and a map $\varphi : \mathsf{A} \to \mathsf{B}$. For any n we have also maps $\varphi_n : \mathrm{Mat}_n(\mathsf{A}) \to \mathrm{Mat}_n(\mathsf{B})$ given by $\varphi_n((a_{i,j})) = (\varphi(a_{i,j}))$. We say that

- φ is completely positive (c.p) if φ_n is positive for any n,
- φ is completely bounded if φ_n is bounded for any n,
- similarly: completely isometric, completely contractive, . . .

Fact

A u.c.p. map $\psi : \mathsf{A} \to \mathsf{B}$ defines a non-degenerate c.p. map $\mathrm{id} \otimes \psi : \mathcal{K} \otimes \mathsf{A} \to \mathcal{K} \otimes \mathsf{B}$, where $\mathcal{K} = \mathcal{K}(\ell_2)$, which extends uniquely to a u.c.p. map $\mathrm{M}(\mathcal{K} \otimes \mathsf{A}) \to \mathrm{M}(\mathcal{K} \otimes \mathsf{B})$.

Multiplicative domains

Theorem[Choi '74]

Let $\varphi : \mathrm{A} \to \mathrm{B}$ be a u.c.p. map between unital C^*-algebras. Then

$$\begin{aligned}\mathcal{C}_\varphi &:= \{a \in \mathrm{A} : \varphi(a)^*\varphi(a) = \varphi(a^*a) \text{ and } \varphi(a)\varphi(a)^* = \varphi(aa^*)\} \\ &= \{a \in \mathrm{A} : \varphi(ab) = \varphi(a)\varphi(b) \text{ and } \varphi(ba) = \varphi(b)\varphi(a) \text{ for all } b \in \mathrm{A}\}\end{aligned}$$

is a C^*-subalgebra of A and $\varphi|_{\mathcal{C}_\varphi}$ is a $*$-homomorphism.

Lemma

Let $\varphi : \mathrm{A} \to \mathrm{B}$ be a u.c.p. map between unital C^*-algebras, and let $a \in \mathrm{M}(\mathcal{K} \otimes \mathrm{A})$ belongs to $\mathcal{C}_{\mathrm{id}\otimes\varphi}$. Then $(\omega \otimes \mathrm{id})(a) \in \mathcal{C}_\varphi$ for any $\omega \in \mathcal{K}^*$.

The crucial consequence of Kasparov's dilation theorem

Lemma

For a finite quantum space $\mathbb{P}$, a compact quantum space $\mathbb{O}$ with $\mathrm{C}(\mathbb{O})$ separable, a separable unital C^*-algebra B and a u.c.p. map $\psi : \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathsf{B}$ there exists $\Psi \in \mathrm{Mor}\,(\mathrm{C}(\mathbb{O}), \mathrm{C}(\mathbb{P}) \otimes \mathcal{K} \otimes \mathsf{B})$ such that

$$\psi(x) = (\mathrm{id} \otimes \omega_{1,1} \otimes \mathrm{id})(\Psi(x)) \equiv \Psi_{1,1}(x), \qquad x \in \mathrm{C}(\mathbb{O}),$$

where $\omega_{1,1}(a) = \langle e_1 | a e_1 \rangle$ with e_1 being the first vector of the standard basis of ℓ_2.

The universal operator system

Definition [Brannan-Ganesan-Harris '20]

$\mathcal{Q}_{n,c}$ = the universal operator system generated by the matrix elements $\{q_{a,ij} : \ 1 \leq i, j \leq n\}_{a=1}^{c}$ s.t. $Q_a = [q_{a,ij}] \geq 0$ and satisfy $Q_1 + \ldots + Q_c = \mathbb{1}$.

Universal property

The universality of $\mathcal{Q}_{n,c}$ can be expressed in a similar way as we did it for $\mathcal{P}_{n,c}$.

Concrete realization

$\mathcal{Q}_{n,c} = \mathrm{span}\{p_{a,ij} : 1 \leq a \leq c, \ 1 \leq i, j \leq n\}$.

The universal operator system

Definition

$$\mathbb{S}_{\mathbb{P},\mathbb{O}} := \overline{\operatorname{span}} \left\{ (\omega \otimes \mathrm{id}) \boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}}(x) \,|\, x \in \mathrm{C}(\mathbb{O}),\, \omega \in \mathrm{C}(\mathbb{P})^* \right\}.$$

Lemma

$\mathbb{S}_{\mathbb{P},\mathbb{O}}$ is an operator system (equipped with a u.c.p. map $\boldsymbol{\varphi}_{\mathbb{P},\mathbb{O}} : \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathbb{S}_{\mathbb{P},\mathbb{O}}$).

Theorem: *universality*

For any operator system S and any u.c.p. map $\psi : \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathsf{S}$ there exists a unique u.c.p map $\lambda : \mathbb{S}_{\mathbb{P},\mathbb{O}} \to \mathsf{S}$ such that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{C}(\mathbb{O}) & \xrightarrow{\boldsymbol{\varphi}_{\mathbb{P},\mathbb{O}}} & \mathrm{C}(\mathbb{P}) \otimes \mathbb{S}_{\mathbb{P},\mathbb{O}} \\
\Big\| & & \Big\downarrow{\scriptstyle \mathrm{id}\otimes\lambda} \\
\mathrm{C}(\mathbb{O}) & \xrightarrow{\psi} & \mathrm{C}(\mathbb{P}) \otimes \mathsf{S}
\end{array}$$

The universal operator system

Sketch of the proof

- $\mathsf{B} := \mathrm{C}^* \left\langle (\phi \otimes \mathrm{id})(\psi(x)) \,|\, x \in \mathrm{C}(\mathbb{O}),\, \phi \in \mathrm{C}(\mathbb{P})^* \right\rangle$
- B - unital, separable
- $\psi : \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathsf{B}$ - u.c.p.
- $\psi = \boldsymbol{\Psi}_{1,1}$ for some $\boldsymbol{\Psi} \in \mathrm{Mor}\left(\mathrm{C}(\mathbb{O}), \mathrm{C}(\mathbb{P}) \otimes \mathcal{K} \otimes \mathsf{B}\right)$
- $\boldsymbol{\Psi} = (\mathrm{id} \otimes \Lambda)\boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}}$ for some $\Lambda \in \mathrm{Mor}\left(\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}), \mathcal{K} \otimes \mathsf{B}\right)$
- $\widetilde{\lambda} := (\omega_{1,1} \otimes \mathrm{id})\Lambda$ - u.c.p.: $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \to \mathsf{B}$
- For $\lambda := \widetilde{\lambda}|_{\mathbb{S}_{\mathbb{P},\mathbb{O}}}$ we have $\psi = (\mathrm{id} \otimes \lambda)\boldsymbol{\Phi}_{\mathbb{P},\mathbb{O}}$
- slicing...

The embedding $\mathbb{S}_{\mathbb{P},\mathbb{O}} \subset C(\mathbb{M}_{\mathbb{P},\mathbb{O}})$

Definition [Hamana]

The C^*-envelope for a operator system S is the C^*-algebra $C^*_{env}(S)$ which is generated by S, and S is essential in this C^*-algebra, i.e. complete order embeddings $C^*_{env}(S) \to T$ are detected by restriction S.

Theorem [Brannan-Ganesan-Harris '20]

$\mathcal{P}_{n,c} = C^*_{env}(\mathcal{Q}_{n,c})$.

Hyperrigidity

Definition [Arveson]

The embedding $\mathsf{S} \subset \mathsf{A}$ of an operator system S in a C^*-algebra A is *hyperrigid* if for every $*$-homomorphism $\pi : \mathsf{A} \to \mathrm{B}(\mathrm{H})$ and a u.c.p. map $\eta : \mathsf{A} \to \mathrm{B}(\mathrm{H})$ (with H a Hilbert space) satisfying $\pi|_{\mathsf{S}} = \eta|_{\mathsf{S}}$, we have $\pi = \eta$.

Hyperrigidity

Theorem

If $\mathsf{S} \subset \mathsf{A}$ is hyperrigid and $\mathrm{C}^*\langle \mathsf{S} \rangle = \mathsf{A}$, then $\mathrm{C}^*_{\mathrm{env}}(\mathsf{S}) = \mathsf{A}$.

Remark

The theorem remains true without the assumption that $\mathrm{C}^*\langle \mathsf{S} \rangle = \mathsf{A}$ [Harris - Kim ' 19].

Theorem

Let A, B be unital C^*-algebras, C be a C^*-algebra, $\Phi \in \mathrm{Mor}(\mathsf{A}, \mathsf{C} \otimes \mathsf{B})$, and

$$\mathsf{S} := \overline{\mathrm{span}} \left\{ (\omega \otimes \mathrm{id})(\Phi(x)) \,|\, x \in \mathsf{C}^*, x \in \mathsf{A} \right\}.$$

If $\mathrm{C}^*\langle \mathsf{S} \rangle = \mathsf{B}$, then $\mathsf{S} \subset \mathsf{B}$ is hyperigid (and, as a result, $\mathrm{C}^*_{\mathrm{env}}(\mathsf{S}) = \mathsf{B}$).

Corollary

$\mathbb{S}_{\mathbb{P},\mathbb{O}} \subset \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is hyperrigid and $\mathrm{C}^*_{\mathrm{env}}(\mathbb{S}_{\mathbb{P},\mathbb{O}}) = \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$, for every finite quantum space $\mathbb{P}$ and every compact quantum space $\mathbb{O}$.

Strong extension property

Definition

The embedding $\mathsf{S} \subset \mathsf{A}$ of an operator system S in a C^*-algebra A has *strong extension property* if for every C^*-algebra B and a u.c.p. map $\psi : \mathsf{S} \to \mathsf{B}$ there exists a u.c.p. map $\eta : \mathsf{A} \to \mathsf{B}$ s.t. $\eta|_{\mathsf{S}} = \psi$.

Theorem

For a finite quantum space $\mathbb{P}$ and a compact quantum space $\mathbb{O}$ such that $\mathrm{C}(\mathbb{O})$ is separable, the embedding $\mathbb{S}_{\mathbb{P},\mathbb{O}} \subset \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ has the strong extension property.

The universal operator system - basic properties

Fact

For any operator system S there is a canonical u.c.p. map $\mathbb{S}_{\mathbb{P},\mathbb{O}} \otimes_c \mathsf{S} \to \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_c \mathsf{S}$.

Fact

$\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_c \mathsf{S} = \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_{\max} \mathsf{S}$

Corollary

There exists a canonical u.c.p. map $\mathbb{S}_{\mathbb{P},\mathbb{O}} \otimes_c \mathsf{S} \to \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_{\max} \mathsf{S}$.

Lemma

For a finite quantum space $\mathbb{P}$, a compact quantum space $\mathbb{O}$ such that $\mathrm{C}(\mathbb{O})$ is separable, and an operator system S, the above map is a complete order embedding.

The universal operator system

Corollary

For a finite quantum space $\mathbb{P}$, a compact quantum space $\mathbb{O}$ such that $\mathrm{C}(\mathbb{O})$ is separable, the canonical map $\mathbb{S}_{\mathbb{P},\mathbb{O}} \otimes_c \mathbb{S}_{\mathbb{P},\mathbb{O}} \to \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_{\max} \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is a complete order embedding.

The lifting property

Definition

A C^*-algebra A is said to have the *lifting property* if given any C^*-algebra B with an ideal $\mathsf{J} \subset \mathsf{B}$, any c.c.p. map $\varphi : \mathsf{A} \to \mathsf{B}/\mathsf{J}$ admits a c.c.p. lift $\widetilde{\varphi} : \mathsf{A} \to \mathsf{B}$.

Remark

In the unital case: c.c.p. $\rightsquigarrow$ u.c.p.

Theorem [Brannan-Ganesan-Harris '20]

$\mathcal{P}_{n,c}$ has the lifting property.

Theorem [Choi-Effros '76]

Any separable nuclear C^*-algebra has the lifting property.

$\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is almost never nuclear ...

If $|\mathrm{Char}(\mathrm{C}(\mathbb{O}))| \geq 2$ and dim $\mathrm{C}(\mathbb{P}) \in \{3, 5, 6, \ldots\}$, then $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ surjects onto $\mathrm{C}(\mathbb{M}_{3,2}) \cong \mathrm{C}^*(\mathbb{Z}_2 * \mathbb{Z}_3)$.

If $|\mathrm{Char}(\mathrm{C}(\mathbb{O}))| \geq 3$ and dim $\mathrm{C}(\mathbb{P}) > 1$, then $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ surjects onto $\mathrm{C}(\mathbb{M}_{2,3}) \cong \mathrm{C}^*(\mathbb{Z}_2^{*3})$.

If $\mathrm{C}(\mathbb{O})$ has $\mathrm{Mat}_n(\mathbb{C})$ with $n \geq 2$ as a quotient and dim $\mathrm{C}(\mathbb{P}) > 1$, then $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ surjects onto $\mathrm{C}(\mathbb{M}_{2,\mathbb{O}}) \cong \mathrm{Mat}_n(\mathbb{C}) * \mathrm{Mat}_n(\mathbb{C})$.

The lifting property

Theorem

For a finite quantum space $\mathbb{P}$, a compact quantum space $\mathbb{O}$ s.t. $\mathrm{C}(\mathbb{O})$ has the lifting property, the C^*-algebra $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ has the lifting property.

Theorem

For a finite quantum space $\mathbb{P}$, a compact quantum space $\mathbb{O}$ s.t. $\mathrm{C}(\mathbb{O})$ has the lifting property, the operator system $\mathbb{S}_{\mathbb{P},\mathbb{O}}$ has the lifting property.

Quantum correlations

- From now on both $\mathbb{P}$ and $\mathbb{O}$ are finite quantum spaces.

Definition

A u.c.p. map $\mathrm{C}(\mathbb{O}) \to \mathrm{B}(\mathrm{H})$ is called a quantum positive operator-valued measure (q-POVM) on $\mathbb{O}$.

Definition

A u.c.p. map $\mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{B}(\mathrm{H})$ is called a *quantum family* of POVMs on $\mathbb{O}$ indexed by $\mathbb{P}$.

Remark

One can replace $\mathrm{B}(\mathrm{H})$ by a fixed unital C^*-algebra A.

Remark

A quantum family of POVMs on $\mathbb{O}$ is a $\mathrm{C}(\mathbb{P}) \otimes \mathrm{B}(\mathrm{H})$-valued quantum POVM on $\mathbb{O}$.

Quantum correlations

Definition

A *quantum correlation* with quantum set of questions $\mathbb{P}$ and quantum set of answers $\mathbb{O}$ (a $(\mathbb{P}, \mathbb{O})$-correlation) is a u.c.p. map $T : \mathrm{C}(\mathbb{O}) \otimes \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{C}(\mathbb{P})$.

Definition

A $(\mathbb{P}, \mathbb{O})$-correlation T is called *non-signalling* if

$$T\left(\mathrm{C}(\mathbb{O}) \otimes \mathbb{1}_{\mathrm{C}(\mathbb{O})}\right) \subset \mathrm{C}(\mathbb{P}) \otimes \mathbb{1}_{\mathrm{C}(\mathbb{P})}, \quad T\left(\mathbb{1}_{\mathrm{C}(\mathbb{O})} \otimes \mathrm{C}(\mathbb{O})\right) \subset \mathbb{1}_{\mathrm{C}(\mathbb{P})} \otimes \mathrm{C}(\mathbb{P}).$$

Realizable correlations

Proposition

Let H be a Hilbert space, ω a state on $\mathrm{B}(\mathrm{H})$ and $\varphi_1, \varphi_2 : \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{B}(\mathrm{H})$ quantum families of POVMs on $\mathbb{O}$ indexed by $\mathbb{P}$ s.t.

$$\varphi_1(x)_{13}\varphi_2(y)_{23} = \varphi_2(y)_{23}\varphi_1(x)_{13}, \quad x, y \in \mathrm{C}(\mathbb{O}).$$

Then there exists a unique linear map $T : \mathrm{C}(\mathbb{O}) \otimes \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{C}(\mathbb{P})$ s.th:

$$T(x \otimes y) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega)\left(\varphi_1(x)_{13}\varphi_2(y)_{23}\right), \quad x, y \in \mathrm{C}(\mathbb{O})$$

and T is a non-signalling $(\mathbb{P}, \mathbb{O})$-correlation.

Realizable correlations

Theorem

For a $(\mathbb{P}, \mathbb{O})$-correlation $T : \mathrm{C}(\mathbb{O}) \otimes \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{C}(\mathbb{P})$ TFAE:

(a) there exists a Hilbert space H, a pair of u.c.p. maps $\varphi_1, \varphi_2 : \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{B}(\mathrm{H})$ satisfying

$$\varphi_1(x)_{13}\varphi_2(y)_{23} = \varphi_2(y)_{23}\varphi_1(x)_{13},$$

and a norm-one vector $\xi \in \mathrm{H}$ s.t.

$$T(x \otimes y) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_\xi)\left(\varphi_1(x)_{13}\varphi_2(y)_{23}\right).$$

Realizable correlations

Theorem

For a $(\mathbb{P}, \mathbb{O})$-correlation $T : \mathrm{C}(\mathbb{O}) \otimes \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{C}(\mathbb{P})$ TFAE:

(b) there exists a Hilbert space H, a pair of unital $*$-homomorphisms $\Phi_1, \Phi_2 : \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{B}(\mathrm{H})$ satisfying

$$\Phi_1(x)_{13}\Phi_2(y)_{23} = \Phi_2(y)_{23}\Phi_1(x)_{13},$$

and a norm-one vector $\xi \in \mathrm{H}$ s.t.

$$T(x \otimes y) = (\mathrm{id} \otimes \mathrm{id} \otimes \omega_\xi)\left(\Phi_1(x)_{13}\Phi_2(y)_{23}\right).$$

Realizable correlations

Theorem

For a $(\mathbb{P}, \mathbb{O})$-correlation $T : \mathrm{C}(\mathbb{O}) \otimes \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{C}(\mathbb{P})$ TFAE:

(c) there exists a state σ on $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_{\max} \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ s.t.

$$T(x \otimes y) = (\mathrm{id} \otimes \mathrm{id} \otimes \sigma) \left(\mathbf{\Phi}_{\mathbb{P},\mathbb{O}}(x)_{13} \mathbf{\Phi}_{\mathbb{P},\mathbb{O}}(y)_{24} \right).$$

(d) there exists a state σ on $\mathbb{S}_{\mathbb{P},\mathbb{O}} \otimes_c \mathbb{S}_{\mathbb{P},\mathbb{O}}$ s.t.

$$T(x \otimes y) = (\mathrm{id} \otimes \mathrm{id} \otimes \sigma) \left(\boldsymbol{\varphi}_{\mathbb{P},\mathbb{O}}(x)_{13} \boldsymbol{\varphi}_{\mathbb{P},\mathbb{O}}(y)_{24} \right).$$

Sychronicity

Decomposition of finite quantum spaces

$$\mathrm{C}(\mathbb{P}) = \bigoplus_{l=1}^{N_{\mathbb{P}}} \mathrm{Mat}_{m_l}(\mathbb{C}), \quad \mathrm{C}(\mathbb{O}) = \bigoplus_{k=1}^{N_{\mathbb{O}}} \mathrm{Mat}_{n_k}(\mathbb{C}).$$

Let $\{{}^l f_{st} \ : l = 1, \ldots, N_{\mathbb{P}}, \ s, t = 1, \ldots m_l\}$, $\{{}^k e_{ij} \ : k = 1, \ldots, N_{\mathbb{O}}, \ i, j = 1, \ldots n_k\}$ be the corresponding systems of matrix units, and $\{{}^l f_s\}$, $\{{}^k e_i\}$ the corresponding standard bases of $\mathbb{C}^{m_l}$ and $\mathbb{C}^{n_k}$, resp.

Coefficients of quantum correlation

$$T\left({}^k e_{ij} \otimes {}^{k'} e_{i'j'}\right) = \sum_{s,t,s',t',l,l'} {}^{kk'}_{ll'} X^{(st),(s't')}_{(ij),(i'j')} \left({}^l f_{st} \otimes {}^{l'} f_{s't'}\right).$$

Synchronicity

Definition

A $(\mathbb{P}, \mathbb{O})$-correlation $T : \mathrm{C}(\mathbb{O}) \otimes \mathrm{C}(\mathbb{O}) \to \mathrm{C}(\mathbb{P}) \otimes \mathrm{C}(\mathbb{P})$ is called *synchronous* if

$$\sum_{s,t,i,j,k,l} \frac{1}{n_k m_l} {}^{kk}_{ll}X^{(st),(st)}_{(ij),(ij)} = N_{\mathbb{P}}.$$

But why...?

- For finite classical sets we have $n_k = m_l = 1$ for all k, l and there are no "interal indices" inside matrix blocks: ${}^{kk'}_{ll'}X^{(st),(s't')}_{(ij),(i'j')} \rightsquigarrow {}^{kk'}_{ll'}X$.
- We identify ${}^{kk'}_{ll'}X = p(k, k'|l, l')$.
- But then $\sum_{k,k'} p(k, k'|l, l) = 1$.
- Hence: $\sum_{k,l} p(k, k|l, l) = N_{\mathbb{P}} \Rightarrow \forall l \, p(k, k'|l, l) = 0$ whenever $k \neq k'$.

Synchronicity

Proposition: characterization of synchronicity

T is synchronous iff

$$\left\langle \phi \right| \, T \left(\sum_{i,j,k} \frac{1}{n_k} {}^k e_{ij} \otimes {}^k e_{ij} \right) \phi \Big\rangle = 1,$$

where

$$\phi = \frac{1}{\sqrt{N_{\mathbb{P}}}} \sum_{l} \frac{1}{\sqrt{m_l}} \sum_{s} \left({}^l f_s \otimes {}^l f_s \right) \in \left(\bigoplus_{l=1}^{N_{\mathbb{P}}} \mathbb{C}^{m_l} \right) \otimes \left(\bigoplus_{l=1}^{N_{\mathbb{P}}} \mathbb{C}^{m_l} \right).$$

Synchronous correlations from *realization*

- Realizable correlation is given by a state $\sigma \in \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_{\max} \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$:

$$T_\sigma(x \otimes y) = (\mathrm{id} \otimes \mathrm{id} \otimes \sigma)\left(\mathbf{\Phi}_{\mathbb{P},\mathbb{O}}(x)_{13}\mathbf{\Phi}_{\mathbb{P},\mathbb{O}}(y)_{24}\right).$$

- Write $\mathbf{\Phi}_{\mathbb{P},\mathbb{O}}({}^k e_{ij}) = \sum_{s,t,l} {}^l f_{st} \otimes {}^k_l V^{st}_{ij}$.

Lemma

Let τ be a trace on $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$. Then there exists a state σ_τ on $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \otimes_{\max} \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ such that

$$\sigma_\tau\left({}^k_l V^{st}_{ij} \otimes {}^{k'}_{l'} V^{s't'}_{i'j'}\right) = \tau\left({}^k_l V^{st}_{ij}\ {}^{k'}_{l'} V^{s't'}_{i'j'}\right).$$

Theorem

T_{σ_τ} is synchronous.

Synchronous realizable $(\mathbb{P}, \mathbb{O})$-correlations ...

... arise from tracial states on C^*-algebras generated by operators associated to the maps in the realization $(\Phi_1, \Phi_2, \omega_\xi)$ of the correlation. More precisely,

Theorem

Let T be a synchronous $(\mathbb{P}, \mathbb{O})$-correlation with realization $(\Phi_1, \Phi_2, \omega_\xi)$. Let $\{{}^k_l U^{st}_{ij}\}$ and $\{{}^k_l W^{st}_{ij}\}$ be elements of $\mathrm{B}(\mathrm{H})$ defined by

$$\Phi_1({}^k e_{ij}) = \sum_{l,s,t} {}^l f_{st} \otimes {}^k_l U^{st}_{ij}, \quad \Phi_2({}^k e_{ij}) = \sum_{l,s,t} {}^l f_{st} \otimes {}^k_l W^{st}_{ij}.$$

Then

(1) ${}^k_l W^{st}_{ij} \xi = \left({}^k_l U^{st}_{ij}\right) \xi$,

(2) ω_ξ restricted to $\mathrm{C}^* \left\langle \{{}^k_l W^{st}_{ij}\} \right\rangle$ is a trace,

(3) ω_ξ restricted to $\mathrm{C}^* \left\langle \{{}^k_l U^{st}_{ij}\} \right\rangle$ is a trace.

Moreover, let $\Lambda_1 : \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}}) \to \mathrm{B}(\mathrm{H})$ be the unique unital $*$-homomorphism s.t. $\Phi_1 = (\mathrm{id} \otimes \Lambda_1)\mathbf{\Phi}_{\mathbb{P},\mathbb{O}}$. Then $\tau = \omega_\xi \circ \Lambda_1$ is a trace on $\mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ and $T = T_{\sigma_\tau}$.

Conclusions

- We studied quantum space of maps between quantum spaces.
- The C^*-algebra of this quantum space has the lifting property and is RDF.
- We showed that the embedding $\mathbb{S}_{\mathbb{P},\mathbb{O}} \subset \mathrm{C}(\mathbb{M}_{\mathbb{P},\mathbb{O}})$ is hyperrigid, and we studied consequences of this fact.
- We discussed quantum correlations and synchronicity within this framework.