WHAT IS AN EQUIVARIANT INDEX?

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Γ discrete (countable) group

$M$ $C^\infty$-manifold, $\partial M = \emptyset$

$\Gamma \times M \to M$ smooth, proper, co-compact action of $\Gamma$ on $M$

"smooth" = each $\gamma \in \Gamma$ acts on $M$ by diffeomorphisms

"proper" = if $\Delta$ is any compact subset of $M$, then $\{\gamma \in \Gamma : \Delta \cap \gamma \Delta \neq \emptyset\}$ is finite

"co-compact" = the quotient space $M/\Gamma$ is compact
REMARKS

1. If \( p \in M \), then \( \{ \gamma \in \Gamma : \gamma p = p \} \) is a finite subgroup of \( \Gamma \).

2. \( M/\Gamma \) is a compact orbifold.

3. \( M \) is compact \( \iff \Gamma \) is finite.

\( \Gamma \times M \to M \) smooth, proper, co-compact action of \( \Gamma \) on \( M \).

Let \( D \) be a \( \Gamma \)-equivariant elliptic differential operator (or \( \psi \)DO) on \( M \).

What is the (equivariant) index of \( D \)?

\[ \text{Index}_\Gamma(D) = ? \]
EXAMPLE

\[ M = \mathbb{R} \]
\[ \Gamma = \mathbb{Z} \]
\[ \mathbb{Z} \times \mathbb{R} \to \mathbb{R} \]
\[ (n, t) \mapsto n + t \]
\[ D = -i \frac{d}{dx} \]
\[ C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R}) \]

\(-i\frac{d}{dx}\) is an unbounded operator on the Hilbert space \(L^2(\mathbb{R})\)

\[-i\frac{d}{dx} : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})\]

\(-i\frac{d}{dx}\) is essentially self-adjoint i.e.

\[-i\frac{d}{dx} : C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})\]

has a unique self-adjoint extension

What is the (equivariant) index of \(-i\frac{d}{dx}\)?

\[ \text{Index}_\mathbb{Z} \left( -i\frac{d}{dx} \right) = ? \]
**H** a Hilbert space

An **unbounded operator** on **H** is a pair \((\mathcal{D}, T)\) such that

1. \(\mathcal{D}\) is a vector subspace of **H**

2. \(\mathcal{D}\) is dense in **H**

3. \(T: \mathcal{D} \rightarrow \mathcal{H}\) is a \(\mathbb{C}\)-linear map

4. \((\mathcal{D}, T)\) is closable (i.e. the closure of graph \(T\) in \(\mathcal{H} \oplus \mathcal{H}\) is the graph of a \(\mathbb{C}\)-linear map

\[ P(\text{graph } T) \rightarrow \mathcal{H} \]

where \(P: \mathcal{H} \oplus \mathcal{H} \rightarrow \mathcal{H}, P(u, v) = u\)

\((\mathcal{D}, T)\) is **symmetric** \iff \(\langle T u, v \rangle = \langle u, T v \rangle \ \forall u, v \in \mathcal{D}\)

**CAUTION**

- symmetric \(\neq\) self-adjoint
- symmetric \(\iff\) self-adjoint
If \((\mathcal{D}, T)\) is an unbounded operator on \(H\), then

\[
\mathcal{D}(T^*) := \left\{ u \in H : \begin{array}{l}
v \mapsto \langle u, Tv \rangle \text{ extends from } \mathcal{D} \text{ to } H \\
to be bounded linear functional
\end{array} \right\}
\]

For \(u \in \mathcal{D}(T^*)\) and \(v \in H\), \(\langle u, Tv \rangle = \langle T^*u, v \rangle\)

\(T^* : \mathcal{D}(T^*) \to H\)

\((\mathcal{D}, T)\) is selfadjoint if \((\mathcal{D}, T) = (\mathcal{D}(T^*), T^*)\)

\(C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})\)

\((C_c^\infty(\mathbb{R}), -i \frac{d}{dx})\) has a unique self-adjoint extension \((\mathcal{D}, -i \frac{d}{dx})\)

\[
\mathcal{D} = \left\{ u \in L^2(\mathbb{R}) : -i \frac{d}{dx} \text{ in the distribution sense is in } L^2(\mathbb{R}) \right\}
\]

\[
= \left\{ u \in L^2(\mathbb{R}) : x \hat{u} \in L^2(\mathbb{R}) \right\}
\]

\(\hat{u} = \) the Fourier transform of \(u\)

\(x : \mathbb{R} \to \mathbb{R}, \quad x(t) = t \quad \forall t \in \mathbb{R}\)
$$\mathbb{Z} \times \mathbb{R} \to \mathbb{R}$$

$$(n, t) \mapsto n + t$$

Fundamental domain is [0, 1]

$$C^\infty_c((0, 1)) \subset L^2([0, 1])$$

$$(C^\infty_c((0, 1)), -i \frac{d}{dx})$$ has one self-adjoint extension for each $$\lambda \in \mathbb{C}$$ with $$|\lambda| = 1$$

Fix $$\theta \in [0, 1]$$

$$C^\infty_\theta([0, 1]) := \{ u \in C^\infty([0, 1]) : u(1) = e^{2\pi i \theta} u(0) \}$$

$$L^2([0, 1]) \supset C^\infty_\theta([0, 1]) \supset C^\infty((0, 1))$$

$$e^{2\pi i (n+\theta)x}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Is an orthonormal basis for $$L^2([0, 1])$$

$$e^{2\pi i (n+\theta)x} \in C^\infty_\theta([0, 1])$$

$$-i \frac{d}{dx} e^{2\pi i (n+\theta)x} = 2\pi (n + \theta) e^{2\pi i (n+\theta)x}$$
\[ e^{2\pi i(n+\theta)x}, \quad n = 0, \pm 1, \pm 2, \ldots \]
is an orthonormal basis for \(L^2([0, 1])\) consisting of eigen-vectors of the operator \(-i \frac{d}{dx}\).

The eigen-values are \(2\pi(n+\theta), n = 0, \pm 1, \pm 2, \ldots\)

Set
\[
\mathcal{D}_\theta := \left\{ \sum_{n=-\infty}^{\infty} \lambda_n e^{2\pi i(n+\theta)x} \in L^2([0, 1]) : \right. \\
\left. \sum_{n=-\infty}^{\infty} 2\pi(n + \theta)\lambda_n e^{2\pi i(n+\theta)x} \in L^2([0, 1]) \right\}
\]

\(L^2([0, 1]) \supset \mathcal{D}_\theta \supset C^\infty([0, 1]) \supset C^\infty((0, 1))\)

\((\mathcal{D}_\theta, -i \frac{d}{dx})\) is an unbounded self-adjoint operator on \(L^2([0, 1])\)

For \(\theta = 0\) and \(\theta = 1\) we have the same unbounded self-adjoint operator

Except for this, the unbounded self-adjoint operators \((\mathcal{D}_\theta, -i \frac{d}{dx})\) are all distinct

Spectrum of \((\mathcal{D}_\theta, -i \frac{d}{dx})\) is \(\{2\pi(n + \theta) : n = 0, \pm 1, \pm 2, \ldots\}\).
We shall now convert \((\mathcal{D}_\theta, -i \frac{d}{dx})\) to a bounded operator on \(L^2([0, 1])\)

**Functional calculus.** Apply the function \(\frac{x}{\sqrt{1+x^2}}\) to \((\mathcal{D}_\theta, -i \frac{d}{dx})\) to obtain \(T_\theta : L^2([0, 1]) \rightarrow L^2([0, 1])\).

Spectrum of \(T_\theta\) is

\[
\frac{x}{\sqrt{1+x^2}} \left( \text{sp} \left( \mathcal{D}_\theta, -i \frac{d}{dx} \right) \right) \cup \{-1, 1\} = \frac{2\pi(n+\theta)}{\sqrt{1 + (2\pi(n+\theta))^2}} \cup \{-1, 1\}
\]

\[
T_\theta(e^{2\pi i(n+\theta)x}) = \frac{2\pi(n+\theta)}{\sqrt{1 + (2\pi(n+\theta))^2}} e^{2\pi i(n+\theta)x}
\]

\(n = 0, \pm 1, \pm 2, \ldots\)

\(\theta \mapsto T_\theta\) is a loop of bounded self-adjoint operators on \(L^2([0, 1])\).

This loop \(\theta \mapsto T_\theta\) should be viewed as giving the index of

\((\mathcal{D}, -i \frac{d}{dx}) - i \frac{d}{dx} : \mathcal{D} \rightarrow L^2(\mathbb{R})\)

WHY ?
$H$ Hilbert space

$T: H \to H$ bounded operator on $H$

$$
\|T\| = \sup_{\langle v, v \rangle = 1} \langle Tv, Tv \rangle^{\frac{1}{2}}
$$

operator norm

A bounded operator $T: H \to H$ is **Fredholm** if

$$
\dim_{\mathbb{C}}(\ker T) < \infty \\
\text{and} \\
\dim_{\mathbb{C}}(\coker T) < \infty
$$

Let $T$ be a Fredholm operator on $H$

$$
\text{Index}(T) \coloneqq \dim_{\mathbb{C}}(\ker T) - \dim_{\mathbb{C}}(\coker T)
$$

$\mathcal{L}(H) \coloneqq \{\text{bounded operators } T: H \to H\}$

$\mathcal{F}(H) \coloneqq \{T \in \mathcal{L}(H) : T \text{ is Fredholm}\}$

$\mathcal{F}(H)$ is topologized by the operator norm topology

$$
\pi_0(\mathcal{F}(H)) = \mathbb{Z} \\
T \mapsto \text{Index}(T)
$$

$S, T \in \mathcal{F}(H)$ are in the same connected component of $\mathcal{F}(H)$ if and only if $\text{Index}(T) = \text{Index} S$
\( \mathcal{F}(H) \) is a classifying space for \( K^0 \)

\( K^0 = \text{Atiyah-Hirzerbruch K-theory} \)

Let \( X \) be any compact Hausdorff topological space

\[
K^0(X) := \text{Grothendieck group of } \mathbb{C}\text{-vector bundles on } X
\]

Notation: \( X, Y \) topological spaces

\[
[X, Y] = \left\{ \text{Homotopy classes of continuous maps } f : X \to Y \right\}
\]

\[
= \left\{ \text{Continuous maps } f : X \to Y \right\} / \sim
\]

\( \sim \) = homotopy

**Theorem 1** (Atiyah, Janich). Let \( X \) be any compact Hausdorff topological space. Then

\[
K^0(X) = [X, \mathcal{F}(H)]
\]
\( \mathcal{F}_{s.a.}(H) := \left\{ T \in \mathcal{L}(H) : \text{T is Fredholm and selfadjoint} \right\} \)

\( \mathcal{F}_{s.a.}(H) \) has three connected components

\( \mathcal{F}_{s.a.}(H) = \mathcal{F}_{s.a.}^- \cup \mathcal{F}_{s.a.}(H)^\# \cup \mathcal{F}_{s.a.}^+ \)

**Essential spectrum**

\( \mathcal{L}^{inv}(H) := \left\{ T \in \mathcal{L}(H) : \exists S \in \mathcal{L}(H) \text{ with } ST = TS = Id \right\} \)

\( Id(v) = v \quad \forall \, v \in H \)

\( \mathcal{L}(H) \supset \mathcal{F}(H) \supset \mathcal{L}^{inv}(H) \)

\( T \in \mathcal{L}(H) \)

\( \text{Spectrum}(T) := \{ \lambda \in \mathbb{C} : (\lambda Id - T) \notin \mathcal{L}^{inv}(H) \} \)

Essential spectrum \( (T) := \{ \lambda \in \mathbb{C} : (\lambda Id - T) \notin \mathcal{F}(H) \} \)

Essential spectrum \( (T) \subset \text{Spectrum}(T) \)

\( T \in \mathcal{F}(H) \iff 0 \notin \text{Essential spectrum}(T) \)

\( \mathcal{F}_{s.a.}^- = \left\{ T \in \mathcal{F}_{s.a.}(H) : \text{Essential spectrum}(T) \subset (-\infty, 0) \right\} \)

\( \mathcal{F}_{s.a.}^+ = \left\{ T \in \mathcal{F}_{s.a.}(H) : \text{Essential spectrum}(T) \subset (0, \infty) \right\} \)
\[ \mathcal{F}_{s.a.}^{\#} = \left\{ T \in \mathcal{F}_{s.a.}(H) : \right. \\
\quad \text{Essential spectrum}(T) \cap (-\infty, 0) \neq \emptyset \\
\quad \text{Essential spectrum}(T) \cap (0, \infty) \neq \emptyset \left. \right\} \]

\( \mathcal{F}_{s.a.}^{\#}(H) \) is a classifying space for \( K^1 \)

\( K^1 = \text{Atiyah-Hirzerbruch K-theory} \)

Let \( X \) be any compact Hausdorff topological space

\[ K^1(X) := \lim_{n \to \infty} [X, \text{GL}(n, \mathbb{C})] \]

**Theorem 2** (Atiyah, Singer). Let \( X \) be any compact Hausdorff topological space. Then

\[ K^1(X) = [X, \mathcal{F}_{s.a.}^{\#}(H)] \]

**Bott periodicity**

\[ \Omega \mathcal{F}(H) \sim \mathcal{F}_{s.a.}^{\#}(H) \]

\[ \Omega \mathcal{F}_{s.a.}^{\#}(H) \sim \mathcal{F}(H) \]

\[ \pi_j(\mathcal{F}(H)) = \begin{cases} \mathbb{Z} & \text{j even} \\ 0 & \text{j odd} \end{cases} \]

\[ \pi_j(\mathcal{F}_{s.a.}^{\#}(H)) = \begin{cases} 0 & \text{j even} \\ \mathbb{Z} & \text{j odd} \end{cases} \]
EXAMPLE

\[ -i \frac{d}{dx} : \mathcal{D} \rightarrow L^2(\mathbb{R}) \]

\[ \mathbb{Z} \times \mathbb{R} \rightarrow \mathbb{R} \]

\[ (n,t) \mapsto n + t \]

\[ \mathbb{S}^1 \rightarrow \mathcal{F}_{s,a}^\#(L^2([0,1])) \]

\[ e^{2\pi i \theta} \mapsto T_\theta \]

This loop is the generator of

\[ \pi_1(\mathcal{F}_{s,a}^\#(L^2([0,1]))) = \mathbb{Z} = K^1(\mathbb{S}^1) \]

\( \mathbb{S}^1 \) is the Pontrjagin dual of \( \mathbb{Z} \)

\( G \) abelian locally compact Hausdorff topological group

Pontrjagin dual \( \hat{G} \) is again an abelian locally compact Hausdorff topological group

\[ \hat{G} := \text{Hom}(G, \mathbb{S}^1) \]

\( \hat{G} \) is compact \( \iff \) \( G \) is discrete
\[ \Gamma \times M \to M \] smooth proper co-compact action of \( \Gamma \) on \( M \)

\[ D \] \( \Gamma \)-equivariant elliptic differential (or \( \psi \)DO) operator on \( M \)

Assume:

1. \( \Gamma \) is abelian

2. \( D \) is essentially self-adjoint

Let \( \Delta \) be a fundamental domain for the action of \( \Gamma \) on \( M \)

Each \( \varphi \in \hat{\Gamma} \) determines a boundary condition for \( D|_{\Delta} \)

Using this boundary condition, construct a bounded self-adjoint operator \( T\varphi \)

\[ \hat{\Gamma} \to \mathcal{H}_{s.a.}(H) \]

\[ \varphi \mapsto T\varphi \]

\( \text{Index}_{\Gamma}(D) \in K^1(\hat{\Gamma}) \)

Remark: \( \hat{\Gamma} \) is viewed here as a compact Hausdorff topological space. The group structure of \( \hat{\Gamma} \) is not being used.
\[ \Gamma \times M \to M \]

\[ D \]

Assume:

1. \( \Gamma \) is abelian

2. \( D \) has no self-adjoint property

\[ \hat{\Gamma} \to \mathcal{F}(H) \]

\[ \text{Index}_{\Gamma}(D) \in K^0(\hat{\Gamma}) \]

\[ \Gamma = \mathbb{Z} \oplus \mathbb{Z}, \quad M = \mathbb{R}^2 \]

\[ (\mathbb{Z} \oplus \mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}^2 \]

\[ ((n_1, n_2), (t_1, t_2)) \mapsto (n_1 + t_1, n_2 + t_2) \]

\[ D = \bar{\partial} = \frac{\partial}{\partial x_1} + i \frac{\partial}{\partial x_2} \]

\[ \mathbb{Z} \oplus \mathbb{Z} = S^1 \times S^1 \]

\[ K^0(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z} \]

\[ \begin{pmatrix} 1 \\ L \end{pmatrix} \]

\[ 1 = (S^1 \times S^1) \times \mathbb{C} \]

\[ L = \text{Hopf line bundle} \]

\[ \text{Index}_{\mathbb{Z} \oplus \mathbb{Z}}(\bar{\partial}) \in K^0(\mathbb{Z} \oplus \mathbb{Z}) = K^0(S^1 \times S^1) \]

\[ \text{Index}_{\mathbb{Z} \oplus \mathbb{Z}}(\bar{\partial}) = L - 1 \]
Application: Dirac operato formulation of Baum-Connes conjecture

$\Gamma$ is a (countable) discrete group

$C^*_r(\Gamma)$ denotes the reduced $C^*$-algebra of $\Gamma$

$C^*_r(\Gamma)$ is the completion of the purely algebraic group algebra $\mathbb{C}[\Gamma]$ via the (left) regular representation of $\Gamma$

$K^j(C^*_r(\Gamma))$ denotes the $j$-th $K$-theory group of $C^*_r(\Gamma)$, $j = 0, 1, 2, \ldots$

Bott periodicity: $K^j(C^*_r(\Gamma)) \cong K^{j+2}(C^*_r(\Gamma))$, $j = 0, 1, 2, \ldots$

If $\Gamma$ is abelian, then $K^j(C^*_r(\Gamma)) \cong K^j(\hat{\Gamma})$ where $K^j(\hat{\Gamma})$ is the Atiyah-Hirzerbruch $K$-theory of the Pontrjagin dual $\hat{\Gamma}$
Moral: If $\Gamma$ is not abelian, then $K_j(C^*_r(\Gamma))$ replaces $K^j(\hat{\Gamma})$.

We shall now define an abelian group $K^\text{top}_j(\Gamma)$, $j = 0, 1$.

Definition of $K^\text{top}_j$, $j = 0, 1$

Consider pairs $(M, E)$ such that

1. $M$ is a $C^\infty$-manifold, $\partial M = \emptyset$, with a given smooth, proper co-compact action of $\Gamma$

   $$\Gamma \times M \to M$$

2. $M$ has a given $\Gamma$-equivariant $\text{Spin}^C$-structure

3. $E$ is a $\Gamma$-equivariant vector bundle on $M$
\[ K^\text{top}_0(\Gamma) \oplus K^\text{top}_1(\Gamma) = \{(M, E)\}/ \sim \]

Addition will be disjoint union

\[ (M, E) + (M', E') = (M \cup M', E \cup E') \]

Each fiber of \( E \) is a finite dimensional vector space over \( \mathbb{C} \)

\[ \dim_{\mathbb{C}}(E_p) < \infty \quad p \in M \]

The equivalence relation

**Isomorphism** \((M, E)\) is isomorphic to \((M', E')\) iff \( \exists \) a \( \Gamma \)-equivariant diffeomorphism \( \psi : M \to M' \)

preserving the \( \Gamma \)-equivariant Spin\(^c\)-structures on \( M, M' \) and with

\[ \psi^*(E') \cong E \]

The equivalence relation \( \sim \) will be generated by three elementary steps

- **Bordism**
- **Direct sum - disjoint union**
- **Vector bundle modification**
Bordism \((M_0, E_0)\) is bordant to \((M_1, E_1)\) iff \(\exists (W, E)\) such that:

1. \(W\) is a \(C^\infty\) manifold with boundary, with a given smooth proper co-compact action of \(\Gamma\)

   \[\Gamma \times W \to W\]

2. \(W\) has a given equivariant Spin\(^c\)-structure

3. \(E\) is a \(\Gamma\)-equivariant vector bundle on \(W\)

4. \((\partial W, E|_{\partial W}) \cong (M_0, E_0) \cup (-M_1, E_1)\)

Direct sum - disjoint union

Let \(E, E'\) be two \(\Gamma\)-equivariant vector bundles on \(M\)

\[(M, E) \cup (M, E') \sim (M, E \oplus E')\]
Vector bundle modification

\((M, E)\)

Let \(F\) be \(\Gamma\)-equivariant Spin\(^c\) vector bundle on \(M\).

Assume that
\[
\dim_{\mathbb{R}}(F_p) \equiv 0 \mod 2 \quad p \in M
\]
for every fiber \(F_p\) of \(F\).

\[
1 = M \times \mathbb{R} \quad \gamma(p, t) = (\gamma p, t)
\]
\[
\gamma \in \Gamma \quad (p, t) \in 1
\]

\[
S(F \oplus 1) := \text{unit sphere bundle of } F \oplus 1
\]

\((M, E) \sim (S(F \oplus 1), \beta_+ \otimes \pi^*E)\)

\[
S(F \oplus 1) \xrightarrow{\pi} M
\]

This is a fibration with even-dimensional spheres as fibers.

\(F \oplus 1\) is a \(\Gamma\)-equivariant Spin\(^c\) vector bundle on \(M\) with odd dimensional fibers. Let \(\Sigma\) be the spinor bundle for \(F \oplus 1\).

\[
\text{Cliff}_{\mathbb{C}}(F_p \oplus \mathbb{R}) \otimes \Sigma_p \to \Sigma_p
\]

\[
\pi^* \Sigma = \beta_+ \oplus \beta_-
\]

\((M, E) \sim (S(F \oplus 1), \beta_+ \otimes \pi^*E)\)
\[ \{(M, E)\}/\sim = K_0^\text{top}(\Gamma) \oplus K_1^\text{top}(\Gamma) \]

\[ K_j^\text{top}(\Gamma) = \text{subgroup of } \{(M, E)\}/\sim \text{ consisting of all } (M, E) \text{ such that every connected component of } M \text{ has dimension } \equiv j \mod 2, \ j = 0, 1 \]

Notation: for \((M, E)\) \(D_E\) is the Dirac operator of \(M\) tensored with \(E\)

\[ F = \text{spinor bundle of } M \]

\[ D_E: C_c^\infty(M, F \otimes E) \rightarrow C_c^\infty(M, F \otimes E) \]
$$K_{j}^{\text{top}}(\Gamma) \rightarrow K_{j}(C_{r}^{*}(\Gamma)) \quad j = 0, 1$$

$$(M, E) \mapsto \text{Index}(D_{E})$$

**Conjecture (BC).** (P. Baum, A. Connes) For any (countable) discrete group

$$K_{j}^{\text{top}}(\Gamma) \rightarrow K_{j}(C_{r}^{*}(\Gamma)) \quad j = 0, 1$$

is an isomorphism

**Corollary.** If BC conjecture is true for $\Gamma$, then

1. Every element of $K_{j}(C_{r}^{*}(\Gamma))$ is of the form $\text{Index}(D_{E})$ for some $(M, E)$ (surjectivity)

2. $(M, E)$ and $(M', E')$ have

$$\text{Index}(D_{E}) = \text{Index}(D_{E'})$$

if and only if it is possible to pass from $(M, E)$ to $(M', E')$ by a finite sequence of the three elementary moves

- Bordism
- Direct sum - disjoint union
- Vector bundle modification

(injectivity)
Corollaries of BC

- Novikov conjecture

- Stable Gromov-Lawson-Rosenberg conjecture

- Idempotent conjecture

- Kadison-Kaplansky conjecture

- Mackey analogy

- Construction of the discrete series via Dirac induction (Parthasarathy, Atiyah-Schmid)

- Homotopy invariance of $\rho$-invariants (Keswani, Piazza-Schick)

Theorem. (N. Higson, G. Kasparov) If $\Gamma$ is a discrete group which is amenable (or a-t-menable), then BC is true for $\Gamma$.

Theorem. (I. Mineyev, G. Yu, V. Lafforgue) If $\Gamma$ is a discrete group which is hyperbolic (in Gromov’s sense), then BC is true for $\Gamma$. 