EQUIVARIANT CHERN CHARACTER

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Warsaw, 9 May 2006

A C* algebra (or a Banach algebra) with unit $1_A$

Grothendieck group of finitely generated (left) projective $A$-modules

$k_0(A) := \text{Grothendieck group of finitely generated (left) projective } A\text{-modules}$

$n = 1, 2, 3, \ldots$

$M_n(A) = \{n \times n \text{ matrices } [a_{ij}] : a_{ij} \in A\}$

$M_n(A)$ is a C* algebra (or a Banach algebra) with unit

$$
\begin{pmatrix}
1_A & 0 & \ldots & 0 \\
0 & 1_A & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1_A \\
\end{pmatrix}
$$
\[ \mathrm{GL}(1, A) \subset \mathrm{GL}(2, A) \subset \mathrm{GL}(3, A) \subset \ldots \]

\[ \mathrm{GL}(A) := \bigcup_{n=1}^{\infty} \mathrm{GL}(n, A) \]

\( \mathrm{GL}(A) \) is topologized by the direct limit topology i.e. \( U \subset \mathrm{GL}(A) \) is open iff \( U \cap \mathrm{GL}(n, A) \) is open in \( \mathrm{GL}(n, A) \) for all \( n = 1, 2, 3, \ldots \)

\[ j = 1, 2, 3, \ldots \]

\[ K_j(A) := \pi_{j-1}(\mathrm{GL}(A)) \]

Bott periodicity

\[ \Omega^2 \mathrm{GL}(A) \sim \mathrm{GL}(A) \]

\[ K_j(A) \simeq K_{j+2}(A), \quad j = 1, 2, 3, \ldots \]

\[ K_0(A), \quad K_1(A) \]
$X$ locally compact Hausdorff topological space

$X^+ = X \cup \{p_\infty\}$ one point compactification of $X$

$C_0(X) = \{\alpha : X^+ \to \mathbb{C} : \alpha \text{ is continuous } \alpha(p_\infty) = 0\}$

$C_0(X)$ is a C*-algebra

$x \in X^+, \alpha, \beta \in C_0(X), \lambda \in \mathbb{C}$

$(\alpha + \beta)x = \alpha x + \beta x$

$(\alpha\beta)x = (\alpha x)(\beta x)$

$(\lambda\alpha)x = \lambda(\alpha x)$

$\|\alpha\| = \sup_{x \in X^+} |\alpha(x)|$

$\alpha^* x = \overline{\alpha x}$

$X$ locally compact Hausdorff topological space

$X^+ = X \cup \{p_\infty\}$ one point compactification of $X$

$K_0(C_0(X)) = \ker\left( K^0(X^+) \to K^0(p_\infty) = \mathbb{Z} \right)$

$E \in \mathbb{C}$ vector bundle on $X^+$
$X$ locally compact Hausdorff topological space

$K_\ast(C_0(X))$ is Atiyah-Hirzerbruch K-theory

This is topological K-theory with compact supports

Atiyah-Hirzerbruch notation for this K-theory is $K^\ast(X)$

$$K_j(C_0(X)) = K^j(X)$$

$X$ compact Hausdorff $\implies$

$K_0(C_0(X)) = K^0(X) = \text{Grothendieck group of }$ 

$\mathbb{C}$ vector bundles on $X$

Chern character

$X$ locally compact Hausdorff topological space

$$\text{ch}: K_j(C_0(X)) \to \bigoplus_l H_c^{j+2l}(X; \mathbb{Q}), \quad j = 0, 1$$

$$\mathbb{Q} \otimes_{\mathbb{Z}} K_j(C_0(X)) \cong_{\sim} \bigoplus_l H_c^{j+2l}(X; \mathbb{Q})$$

- Čech cohomology

- Alexander Spanier cohomology

(with compact supports)
$X$ locally compact Hausdorff topological space

$\Gamma$ discrete (countable) group

$\Gamma \times X \to X$ continuous action of $\Gamma$ on $X$

$C^*_r(\Gamma, X) = C_0(X) \rtimes \Gamma$ is the reduced crossed-product $C^*$-algebra for the action of $\Gamma$ on $X$

Definition of $C^*_r(\Gamma, X)$

Extend the given action $\Gamma \times X \to X$ to $\Gamma \times X^+ \to X^+$ by

$\gamma p_\infty = p_\infty \quad \forall \gamma \in \Gamma$

$\Gamma$ then acts on $C_0(X)$ by $C^*$ algebra automorphisms $\Gamma \times C_0(X) \to C_0(X)$

$(\gamma f)x = f(\gamma^{-1}x) \quad f \in C_0(X), \gamma \in \Gamma, x \in X$

Form the purely algebraic crossed-product algebra $C_0(X) \rtimes_{\text{alg}} \Gamma$

$C_0(X) \rtimes_{\text{alg}} \Gamma = \left\{ \text{finite formal sums } \sum_{\gamma \in \Gamma} f_{\gamma}[\gamma] : f_{\gamma} \in C_0(X) \right\}$

$\left( \sum_{\gamma \in \Gamma} f_{\gamma}[\gamma] \right) + \left( \sum_{\gamma \in \Gamma} h_{\gamma}[\gamma] \right) = \sum_{\gamma \in \Gamma} (f_{\gamma} + h_{\gamma})[\gamma]$

$\lambda \left( \sum_{\gamma \in \Gamma} f_{\gamma}[\gamma] \right) = \sum_{\gamma \in \Gamma} (\lambda f_{\gamma})[\gamma] \quad \lambda \in \mathbb{C}$

$(f[\gamma])(h[g]) = (f)(\gamma h)[\gamma g] \quad \gamma, g \in \Gamma, f, h \in C_0(X)$
Complete $C_0(X) \rtimes_{\text{alg}} \Gamma$ to obtain $C_r^*(\Gamma, X)$

$$l^2(\Gamma) = \{ u : \Gamma \to \mathbb{C} : \sum_{\gamma \in \Gamma} \overline{u(\gamma)} u(\gamma) < \infty \}$$

$l^2(\Gamma)$ is a Hilbert space

$$(u + v)\gamma = u\gamma + v\gamma$$

$$(\lambda u)\gamma = \lambda(u\gamma)$$

$$\langle u, v \rangle = \sum_{\gamma \in G} \overline{u(\gamma)} v(\gamma)$$

$L^2(l^2(\Gamma))$ is the $C^*$-algebra of all bounded operators $T : l^2(\Gamma) \to l^2(\Gamma)$ with operator norm

$$\|T\| = \sup_{\langle u, u \rangle = 1} \left( \langle Tu, Tu \rangle \right)^{1/2}$$

Each $x \in X$ determines a homomorphism of algebras

$$\tau_x : C_0(X) \rtimes_{\text{alg}} \Gamma \to \mathcal{L}(l^2\Gamma)$$

$$(\tau_x(f[\gamma])u)(g) = f(gx)u(\gamma^{-1}g)$$

$u \in l^2(\Gamma), x \in X, \gamma, g \in \Gamma, f \in C_0(X)$

$C_r^*(\Gamma, X)$ is the completion of $C_0(X) \rtimes_{\text{alg}} \Gamma$ in the norm

$$\left\| \sum_{\gamma \in \Gamma} f_\gamma[\gamma] \right\| = \sup_{x \in X} \left\| \tau_x \left( \sum_{\gamma \in \Gamma} f_\gamma[\gamma] \right) \right\|$$
$\text{ch}: K_j(C^*_r(\Gamma, X)) \to ?$

If $\Gamma$ is not finite, then $K_j(C^*_r(\Gamma, X))$ can be viewed as the natural extension of Atiyah-Segal equivariant $K$-theory to the case when $\Gamma$ is not finite

$\Gamma$ finite $\implies$ $K_*(C^*_r(\Gamma, X))$ is Atiyah-Segal equivariant $K$-theory, denoted $K^*_\Gamma(X)$

**Theorem 1** (Slominska). $\Gamma$ finite

$\text{ch}: K^*_j(X) \to \bigoplus_l H^j+2l_c(\hat{X}/\Gamma; \mathbb{C})$

$K^*_j(X) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \bigoplus_l H^j+2l_c(\hat{X}/\Gamma; \mathbb{C})$
\( M \) \( C^\infty \) manifold, \( \partial M = \emptyset \)

\( \Gamma \) discrete (countable) group

\( \Gamma \times M \to M \) smooth action of \( \Gamma \) on \( M \)

"smooth" = each \( \gamma \in \Gamma \) acts by a diffeomorphism

\( C^*_r(\Gamma, M) = C_0(M) \rtimes_r \Gamma \) is the reduced crossed-product \( C^* \) algebra for the action of \( \Gamma \) on \( C_0(M) \)

\[
\text{ch} : K_j(C^*_r(\Gamma, M)) \to \bigoplus_l H_{j+2l}(\Gamma; \Omega^*_c(\hat{M})) , \; j = 0, 1
\]

and

\[
K_j(C^*_r(\Gamma, M)) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\cong} \bigoplus_l H_{j+2l}(\Gamma; \Omega^*_c(\hat{M}))
\]

**Theorem 2** (J. Raven). *If BC for \( \Gamma \) with coefficient algebra \( C_0(M) \) is true, then there is a Chern character*

\[
\text{ch} : K_j(C^*_r(\Gamma, M)) \to \bigoplus_l H_{j+2l}(\Gamma; \Omega^*_c(\hat{M})) , \; j = 0, 1
\]
$\gamma \in \Gamma, \ M^\gamma := \{ p \in M : \gamma p = p \}$

If $\text{order}(\gamma) = \infty$, then $M^\gamma$ can be anything e.g. $M^\gamma$ can be a Cantor set

**Lemma 3.** If $\text{order}(\gamma) < \infty$, then $M^\gamma$ is a $C^\infty$ submanifold of $M$.

**Proof.** $\text{order}(\gamma) < \infty \implies$ The subgroup $J$ of $\Gamma$ generated by $\gamma$ is a finite (cyclic) group

$\implies$

Can choose on $M$ a $C^\infty$ Riemannian metric which is $J$-invariant. Then $\gamma$ acts by an isometry and $M^\gamma$ is a totally geodesic $C^\infty$ submanifold of $M$. \hfill \square

$\widehat{M} := \coprod_{\gamma \in \Gamma, \ \text{order}(\gamma) < \infty} M^\gamma$

$= \{ (\gamma, p) \in \Gamma \times M : \gamma p = p \}$

$\widehat{M}$ is a $C^\infty$ manifold

$\Omega^*_c(\widehat{M}) = \text{the de Rham complex of } \mathbb{C}-\text{valued compactly supported } C^\infty \text{ differential forms on } \widehat{M}$

$0 \to \Omega^0_c(\widehat{M}) \overset{d}{\to} \Omega^1_c(\widehat{M}) \overset{d}{\to} \ldots \overset{d}{\to} \Omega^n_c(\widehat{M}) \to 0$
\( \Gamma \times \hat{M} \to \hat{M} \) smooth action of \( \Gamma \) on \( \hat{M} \)

\[ g(\gamma, p) = (g\gamma g^{-1}, gp), \ g \in \Gamma, \ (\gamma, p) \in \hat{M} \]

\[ \Gamma \times \Omega^*_c(\hat{M}) \to \Omega^*_c(\hat{M}) \]

\( \Omega^*_c(\hat{M}) \) is a complex of \( \Gamma \)-modules

\( H_l(\Gamma; \Omega^*_c(\hat{M})) \) is the \( l \)-th cohomology of \( \Gamma \) with coefficients \( \Omega^*_c(\hat{M}) \)

\( \mathbb{C}[\Gamma] \)

\( \mathbb{C}[\Gamma] \) is the purely algebraic group algebra

\[ \mathbb{C}[\Gamma] := \left\{ \text{Finite formal sums } \sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma] : \lambda_\gamma \in \mathbb{C} \right\} \]

Algebra homomorphism \( \varepsilon : \mathbb{C}[\Gamma] \to \mathbb{C} \)

\[ \varepsilon \left( \sum_{\gamma \in \Gamma} \lambda_\gamma [\gamma] \right) = \sum_{\gamma \in \Gamma} \lambda_\gamma \]

A (left) \( \Gamma \)-module is a \( \mathbb{C} \) vector space \( V \) with a given group homomorphism \( \Gamma \to \text{GL}_\mathbb{C}(V) \)

Equivalently a (left) \( \Gamma \)-module is a unital (left) \( \mathbb{C}[\Gamma] \)-module
$V$ $\Gamma$-module

$H_l(\Gamma; V) = l$-th cohomology of $\Gamma$ with coefficients $V$, $l = 0, 1, 2, 3, \ldots$

$H_l(\Gamma; V) := \text{Tor}^l_{\mathbb{C}[\Gamma]}(\mathbb{C}, V)$

$0 \leftarrow V \overset{\partial}{\leftarrow} C[\Gamma] \otimes_{\mathbb{C}} V \overset{\partial}{\leftarrow} C[\Gamma] \otimes_{\mathbb{C}} C[\Gamma] \otimes_{\mathbb{C}} C[\Gamma] \otimes_{\mathbb{C}} C[\Gamma] \otimes_{\mathbb{C}} \partial$

$\partial(a_0 \otimes a_1 \otimes \ldots \otimes a_n \otimes v) =$

$\varepsilon(a_0) a_1 \otimes \cdots \otimes a_n \otimes v$

$+ \sum_{j=0}^{n-1} (-1)^{j+1} a_0 \otimes \cdots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes a_{j+2} \otimes \cdots \otimes a_n \otimes v$

$+ (-1)^{n+1} a_0 \otimes \cdots \otimes a_{n-1} \otimes a_n v$

$a_0, a_1, \ldots, a_n \in C[\Gamma], \ v \in V$

$V_{\Gamma}$ is the $\Gamma$-coinvariants

$V^\Gamma := \{v \in V : \gamma v = v \ \forall \ \gamma \in V\}$

$V^\Gamma$ is the $\Gamma$-invariants

$H_l(\Gamma; V)$ is the $l$-th derived functor of $V \mapsto V_{\Gamma}$
Let
\[ \Psi = \left\{ 0 \xrightarrow{d} V^0 \xrightarrow{d} V^1 \xrightarrow{d} \ldots \right\} \]
be a complex of (left) \( \Gamma \)-modules

To define \( H_l(\Gamma; \Psi) \), \( l \in \mathbb{Z} \) form (first quadrant bicomplex) \( \{A^{i,j}\} \)
\[ A^{i,j} := \mathbb{C}[\Gamma] \otimes \mathbb{C}[\Gamma] \otimes \ldots \otimes \mathbb{C}[\Gamma] \otimes V^j \]

Totalize this bicomplex \( \{A^{i,j}\} \) by setting
\[ D_l := \bigoplus_{i-j=l} A^{i,j} \quad l \in \mathbb{Z} \]
\[ D_* = \{ \ldots D_{-1} \leftarrow D_0 \leftarrow D_1 \leftarrow \ldots \} \]

\( D_* \) is a complex of \( \mathbb{C} \) vector spaces

\[ H_l(\Gamma; \Psi) := H_l(D_*) \]
\[ H_l(\Gamma; \Omega^*_c(\hat{M})), \quad l \in \mathbb{Z} \]
Two extreme cases

1. The action of $\Gamma$ on $M$ is proper

2. $M = \ast$

Ad. 1.

"proper" = The map $\Gamma \times M \to M \times M$, $(\gamma, p) \mapsto (\gamma p, p)$ is proper (i.e. the preimage of any compact set in $M \times M$ is compact)

Equivalently, if $\Delta \subset M$ is any compact subset of $M$, then $\{\gamma \in \Gamma : \Delta \cap \gamma \Delta \neq \emptyset\}$ is finite

action of $\Gamma$ on $M$ is smooth and proper $\implies$ $M/\Gamma$ is an orbifold

action of $\Gamma$ on $\hat{M}$ is smooth and proper $\implies$ $\hat{M}/\Gamma$ is an orbifold
When the action of $\Gamma$ is smooth and proper, $H_*(\Gamma; \Omega_c^*(\widehat{M})) = ?$

**Answer:** When the action of $\Gamma$ on $M$ is smooth and proper

$$H_*(\Gamma; \Omega_c^*(\widehat{M})) = H_*(\widehat{M}/\Gamma; \mathbb{C})$$

**Why?** action of $\Gamma$ on $M$ is smooth and proper $\implies$

$$H_j(\Gamma; \Omega_c^*(\widehat{M})) = 0 \text{ for } j > 0, \ r = 0, 1, 2 \implies$$

$$H_j(\Gamma; \Omega_c^*(\widehat{M})) \text{ is the homology of }$$

$$0 \to (\Omega_c^0(\widehat{M}))_\Gamma \xrightarrow{d} (\Omega_c^1(\widehat{M}))_\Gamma \xrightarrow{d} (\Omega_c^2(\widehat{M}))_\Gamma \xrightarrow{d} \ldots$$

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Moriyoshi’s lemma

$\Gamma$ (countable) discrete group

$W$ $C^\infty$ manifold

$\Gamma \times W \to W$ smooth proper action of $\Gamma$ on $W$

$\omega \in \Omega_c^*(W)$ smooth compactly supported $\mathbb{C}$-valued differential form on $W$

**Notation:** For $\gamma \in \Gamma$, $\gamma_* \omega$ denotes $\omega$ translated by $\gamma$

**Terminology:** A closed set $\Delta \subset W$ is $\Gamma$-compact if $\gamma \Delta = \Delta$ for all $\gamma \in \Gamma$ and $\Delta/\Gamma$ is compact
Then $\sum_{\gamma \in \Gamma} \gamma^* \omega$ makes sense and $\sum_{\gamma \in \Gamma} \gamma^* \omega$ is a $\Gamma$-invariant differential form with $\Gamma$-compact support

$$\sum_{\gamma \in \Gamma} \gamma^* \omega \in \left[ \Omega^*_c(W) \right]_{\Gamma}$$

Consider the map

$$\eta: \Omega_c^*(W) \to \left[ \Omega^*_c(W) \right]_{\Gamma}$$

$$\eta(\omega) = \sum_{\gamma \in \Gamma} \gamma^* \omega$$

For all $g \in \Gamma$ $\eta(g^* \omega) = \eta(\omega)$, so $\eta$ factors through

$$\Omega_c^*(W) \to \left[ \Omega^*_c(W) \right]_{\Gamma} \to \left[ \Omega^*_c(W) \right]_{\Gamma}$$
Lemma 4.

\[ [\Omega^*_c(W)]_{\Gamma} \overset{\cong}{\rightarrow} [\Omega^*_{\Gamma-\text{compact}}(W)]_{\Gamma} \]

Lemma 5.

\[ H_\ast\left([\Omega^*_{\Gamma-\text{compact}}(W)]_{\Gamma}\right) = H^\ast_c(W/\Gamma; \mathbb{C}) \]

Proof. this is a slight extension of the de Rham theorem i.e.

\[ 0 \rightarrow [\Omega^0_{\Gamma-\text{compact}}(W)]_{\Gamma} \rightarrow [\Omega^1_{\Gamma-\text{compact}}(W)]_{\Gamma} \]

\[ \rightarrow [\Omega^2_{\Gamma-\text{compact}}(W)]_{\Gamma} \rightarrow \cdots \]

is a resolution of the constant sheaf on \( W/\Gamma \).

\[ \square \]

Corollary 6. If the action of \( \Gamma \) on \( M \) is smooth and proper then

\[ H_\ast(\Gamma; \Omega^*_c(\hat{M})) = H^\ast_c(\hat{M}/\Gamma; \mathbb{C}) \]

Corollary 7. (Since BC for \( \Gamma \) with coefficient algebra \( C^0_0(M) \) is true when the action of \( \Gamma \) is proper) The Chern character

\[ \text{ch}: K_j(C^*_r(\Gamma, M)) \rightarrow \bigoplus_l H^{j+2l}_c(\hat{M}/\Gamma; \mathbb{C}) \]

exists.
$M = \ast$

Let $S\Gamma = \{\gamma \in \Gamma : \text{order}(\gamma) < \infty\}$

$\ast = S\Gamma$

$\Omega_c^r(\ast) = 0$ for $r > 0$

$\Omega_c^0(\ast) = F\Gamma$

$F\Gamma = \{\text{Finite formal sums } \sum_{\gamma \in S\Gamma} \lambda_\gamma [\gamma] : \lambda_\gamma \in \mathbb{C}\}$

$F\Gamma$ is a $\Gamma$-module

\[
\left( \sum_{\gamma \in S\Gamma} \lambda_\gamma [\gamma] \right) + \left( \sum_{\gamma \in S\Gamma} \mu_\gamma [\gamma] \right) = \sum_{\gamma \in S\Gamma} (\lambda_\gamma + \mu_\gamma) [\gamma]
\]

$\lambda \left( \sum_{\gamma \in S\Gamma} \lambda_\gamma [\gamma] \right) = \sum_{\gamma \in S\Gamma} \lambda \lambda_\gamma [\gamma], \lambda \in \mathbb{C}$

$g \left( \sum_{\gamma \in S\Gamma} \lambda_\gamma [\gamma] \right) = \sum_{\gamma \in S\Gamma} \lambda_\gamma [g\gamma g^{-1}], g \in \Gamma$

Problem: Forget about BC and give a direct construction of the Chern character

$\text{ch}: K_j(C^*_r(\Gamma, M)) \to \bigoplus_l H_{c+2l}^j(\widetilde{M}/\Gamma; \mathbb{C})$

(Assuming action of $\Gamma$ on $M$ is smooth and proper)
The direct construction is done as follows

For simplicity, shall assume \( j = 0 \) and \( M/\Gamma \) compact

Remark: \( M/\Gamma \) compact \( \Rightarrow \) \( \hat{M}/\Gamma \) compact

**Theorem 8** (W. lück and R. Oliver). Let \( W \) be a \( C^\infty \) manifold with a given smooth proper and co-compact action of \( \Gamma \). Then

\[ K_0(C^*_r(\Gamma, W)) = \text{Grothendieck group of } \Gamma\text{-equivariant } \mathbb{C} \text{ vector bundles on } W \]

Localized Chern character

\( W \) as in the Lück-Oliver theorem

\[ \text{ch}_{\text{local}}: K_0(C^*_r(\Gamma, W)) \to \bigoplus H^{2l}(W/\Gamma; \mathbb{C}) \]

is constructed as follows

Let \( F \) be a \( \Gamma \)-equivariant \( C^\infty \) \( \mathbb{C} \) vector bundle on \( W \)

Choose a \( \Gamma \)-equivariant connection \( D \) for \( F \)

Consider the \( \Gamma \)-equivariant differential form

\[ \text{ch}(K) = \text{Tr} \left( \exp \left( \frac{K}{2\pi i} \right) \right) \]

\( K = \text{curvature}(D) \)
\( \operatorname{ch}(K) \in [\Omega^*(W)]^\Gamma \implies \operatorname{ch}(K) \) determines an element in \( \bigoplus H^{2l}(W/\Gamma; \mathbb{C}) \).

This is \( \operatorname{ch}_{\text{(local)}}(F) \)

Local Chern character

Alternate (more topological) construction of \( \operatorname{ch}_{\text{(local)}}: K_0(C^*_\Gamma(\Gamma,W)) \to \bigoplus H^{2l}(W/\Gamma; \mathbb{C}) \)

\( W \) as in the L"ueck-Oliver theorem

Let \( F \) be a \( \Gamma \)-equivariant \( \mathbb{C} \) vector bundle on \( W \)

\[
\begin{array}{ccc}
E\Gamma \times_\Gamma F & \xrightarrow{\varphi} & BU(r) \\
\downarrow & & \\
E\Gamma \times_\Gamma W & & \\
\end{array}
\]

\( \varphi = \text{classifying map for } E\Gamma \times_\Gamma F \)

\( r = \text{fiber dimension of } E\Gamma \times_\Gamma F \)
\[ \text{ch}_{\text{universal}} \in \prod_{l=0}^{\infty} H^{2l}(BU(r); \mathbb{Q}) \]

\[ \varphi^*(\text{ch}_{\text{universal}}) \in \prod_{l=0}^{\infty} H^{2l}(E\Gamma \times_{\Gamma} W; \mathbb{Q}) \]

\[ E\Gamma \times_{\Gamma} W \]

\[ B\Gamma \xrightarrow{\sim} W/\Gamma \]

\[ H^*(W/\Gamma; \mathbb{Q}) \cong H^*(E\Gamma \times_{\Gamma} W; \mathbb{Q}) \]

\[ \varphi^*(\text{ch}_{\text{universal}}) \in \prod_{l=0}^{\infty} H^{2l}(E\Gamma \times_{\Gamma} W; \mathbb{Q}) \]

This is \( \text{ch}_{\text{local}}(F) \)

\[ \text{ch}: K_0(C^*_r(\Gamma, M)) \to \bigoplus_{l} H^{2l}(\widehat{M}/\Gamma; \mathbb{C}) \]

Action of \( \Gamma \) on \( M \) assumed to be smooth, proper, and co-compact

\[ \widehat{M} = \{ (\gamma, p) \in \Gamma \times M : \gamma p = p \} \]

\[ \Gamma \times \widehat{M} \to \widehat{M} \]

\[ g(\gamma, p) = (g\gamma g^{-1}, gp), \quad g \in \Gamma, (\gamma, p) \in \widehat{M} \]

Let \( F \) be a \( \Gamma \)-equivariant vector bundle on \( \widehat{M} \)

Define \( \theta: F \to F \) for \( v \in F(\gamma; p) \)

\[ \theta(v) = \gamma v \in F(\gamma; p) \]
$F$ is any $\Gamma$-equivariant $\mathbb{C}$ vector bundle on $\hat{M}$

$\theta$ is an automorphism of $F$

$\theta$ is of finite order, $\theta^m = Id$ for some positive integer $m$

$$F = \frac{F_1}{\zeta_1} \oplus \frac{F_2}{\zeta_2} \oplus \ldots \oplus \frac{F_t}{\zeta_t}$$

Each $\zeta_j \in \mathbb{C}$ is a root of unity

$$K_0(C^*_r(\Gamma, \hat{M})) \to \bigoplus_l H^{2l}(\hat{M}/\Gamma; \mathbb{C})$$

$$F \mapsto \sum_{\nu=1}^t \zeta_\nu \text{ch}_{\text{(local)}}(F_\nu)$$
\[ H_l(\Gamma; \Omega^*_c(\hat{\ast})) = H_l(\Gamma; F\Gamma) \quad l = 0, 1, 2, \ldots \]

\[ \mathbb{K}^\Gamma_j(E\Gamma) \longrightarrow \mathbb{K}_j(C^*_r(\Gamma)) \]

\[ \bigoplus_l H_l(\Gamma; F\Gamma) \]

\[ \mathbb{K}^\Gamma_j(E\Gamma) \otimes \mathbb{C} \xrightarrow{\cong} \bigoplus_l H_{j+2l}(\Gamma; F\Gamma) \]

\[ \Gamma \times M \rightarrow M \text{ smooth action} \]

\[ \text{ch: } \mathbb{K}_j(C^*_r(\Gamma, M)) \xrightarrow{\cong} \bigoplus_l H_{j+2l}(\Gamma; \Omega^*_c(\hat{M})) \]

\[ X \text{ locally compact Hausdorff topological space} \]

\[ \Gamma \times X \rightarrow X \text{ continuous action} \]

\[ \text{ch: } \mathbb{K}_j(C^*_r(\Gamma, X)) \xrightarrow{\cong} \bigoplus_l H_{j+2l}(\Gamma; C^*_c(\hat{X})) \]

\[ C^*_c(\hat{X}) = \text{ Alexander-Spanier cochains with compact supports on } \hat{X} \]

\[ \hat{X} = \{ (\gamma, x) \in \Gamma \times X : \text{order}(\gamma) < \infty, \gamma x = x \} \]