REPRESENTATIONS OF GROUPOIDS AND IMPRIMITIVITY SYSTEMS

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Basic publications

- L. Pysiak, *Imprimitivity theorem for groupoid representations*, Demonstratio Mathematica 44 (2011), 29-48.
- L. Pysiak, *Groupoids, their representations and imprimitivity systems*, Demonstratio Mathematica 37 (2004), 661-670.
- M. Heller, Z. Odrzygóźdź , L. Pysiak, W. Sasin, *Structure of Malicious Singularities*, Int. J. Theor. Phys., 42, (2003), 427 - 441.
- M. Heller, L. Pysiak, W. Sasin, *Conceptual unification of gravity and quanta*, Int. J. Theor. Phys., 46 (2007), 2494- 2512.

Basic references

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Concept of groupoid

We recall that a *groupoid* $\mathcal{G}$ over X, or a *groupoid with base* X, is a set with a partially defined multiplication "$\circ$" on a subset $\mathcal{G}^2$ of $\mathcal{G} \times \mathcal{G}$, and an inverse map $g \to g^{-1}$ defined for every $g \in \mathcal{G}$. The multiplication is associative when defined. One has an injection $\epsilon : X \to \mathcal{G}$ called the identity section (and $\epsilon(x)$ being an unit at $x \in X$) and two structure maps $d, r : \mathcal{G} \to X$ called the source map and the target map respectively, such that

$$\epsilon(d(g)) = g^{-1} \circ g$$

$$\epsilon(r(g)) = g \circ g^{-1}$$

for $g \in \mathcal{G}$.

Let us introduce the following fibrations in the set $\mathcal{G}$:

$$\mathcal{G}_x = \{g \in \mathcal{G} : d(g) = x\}$$

$$\mathcal{G}^x = \{g \in \mathcal{G} : r(g) = x\}$$

for $x \in X$. Let us also denote $\mathcal{G}_x^y = \mathcal{G}^x \bigcap \mathcal{G}_y$, and consider the set $\mathcal{G}_x^x = \mathcal{G}^x \bigcap \mathcal{G}_x$ for $x \in X$. It has the group structure and is called *the isotropy group of the point* x. It is clear that the set $\Gamma = \bigcup_{x \in X} \mathcal{G}_x^x$ has the structure of a subgroupoid of $\mathcal{G}$ over the base X (all the structure maps are the restrictions of the structure maps of $\mathcal{G}$ to Γ).
We call $\mathcal{G}$ *a transitive groupoid*, if for each pair of elements $x_1, x_2 \in X$ there exists $g \in \mathcal{G}$ such that $d(g) = x_1$ and $r(g) = x_2$.
A groupoid $\mathcal{G}$ is *a topological groupoid* if $\mathcal{G}$ and X are topological spaces and all structure maps are continuous (in particular, the embedding ϵ is a homeomorphism of X onto its image).
In the following we assume that $\mathcal{G}$ (and thus X) is a locally compact Hausdorff space.

Pair groupoid

Example

A pair groupoid. Let X be a locally compact Hausdorff space. Take $\mathcal{G} = X \times X$. We define the set $\mathcal{G}^2$ of composable elements as $\mathcal{G}^2 = \{((x,y),(y,z)) : x,y,z \in X\} \subset \mathcal{G} \times \mathcal{G}$ and a multiplication, for $((x,y),(y,z)) \in \mathcal{G}^2$, by

$$(x,y) \circ (y,z) = (x,z).$$

Moreover, we have: $(x,y)^{-1} = (y,x)$, $d(x,y) = y$, $r(x,y) = x$, $\epsilon(x) = (x,x)$. With such defined structure maps $\mathcal{G}$ is a groupoid, called *pair groupoid*.

Transformation groupoid

Example

A transformation groupoid. Let X be a locally compact Hausdorff space, and G a locally compact group. Let G act continuously on X to the right, $X \times G \to X$. Denote $(x, g) \mapsto xg$. We introduce the groupoid structure on the set $\mathcal{G} = X \times G$ by defining the following structure maps. The set of composable elements
$\mathcal{G}^2 = \{((xg, h), (x, g) : x \in X, g, h \in G\} \subset \mathcal{G} \times \mathcal{G}$, and the multiplication for $((xg, h), (x, g)) \in \mathcal{G}^2$ is given by

$$(xg, h) \circ (x, g) = (x, gh).$$

And also $(x, g)^{-1} = (xg, g^{-1})$, $d(x, g) = x$, $r(x, g) = xg$, $\epsilon(x) = (x, e_G)$. This groupoid is called *the transformation groupoid.*

Right Haar System

Definition

A right Haar system for the groupoid $\mathcal{G}$ is a family $\{\lambda_x\}_{x\in X}$ of regular Borel measures defined on the sets $\mathcal{G}_x$ (which are locally compact Hausdorff spaces) such that the following three conditions are satisfied:

1. the support of each λ_x is the set $\mathcal{G}_x$,
2. (continuity) for any $f \in C_c(\mathcal{G})$ the function f^0, where

$$f^0(x) = \int_{\mathcal{G}_x} f d\lambda_x,$$

   belongs to $C_c(X)$,
3. (right invariance) for any $g \in \mathcal{G}$ and any $f \in C_c(\mathcal{G})$,

$$\int_{\mathcal{G}_{r(g)}} f(h \circ g) d\lambda_{r(g)}(h) = \int_{\mathcal{G}_{d(g)}} f(u) d\lambda_{d(g)}(u).$$

One can also consider the family $\{\lambda^x\}_{x\in X}$ of left-invariant measures, each λ^x being defined on the set $\mathcal{G}^x$ by the formula $\lambda^x(E) = \lambda_x(E^{-1})$ for any Borel subset E of $\mathcal{G}^x$ (where $E^{-1} = \{g \in \mathcal{G} : g^{-1} \in E\}$). Then the invariance condition assumes the form:

$$\int_{\mathcal{G}^{d(g)}} f(g \circ h) d\lambda^{d(g)}(h) = \int_{\mathcal{G}^{r(g)}} f(u) d\lambda^{r(g)}(u).$$

Now, let μ be a regular Borel measure on X. We can consider the following measures which will be called *measures associated with* μ: $\nu = \int \lambda_x d\mu(x)$ on $\mathcal{G}$, $\nu^{-1} = \int \lambda^x d\mu(x)$ and $\nu^2 = \int \lambda_x \times \lambda^x d\mu(x)$ on $\mathcal{G}^2$. If $\nu = \nu^{-1}$ we say that the measure μ is *a $\mathcal{G}$-invariant measure on X.*

Locally trivial groupoids

Definition

A topological groupoid $\mathcal{G}$ on X is called *locally trivial* if there exist a point $x \in X$, an open cover $\{U_i\}$ of X and continuous maps $s_{x,i} : U_i \to \mathcal{G}_x$ such that $r \circ s_i = id_{U_i}$ for all i.

Proposition

Assume that $\mathcal{G}$ is a locally trivial groupoid on X and X is second countable space. Let μ be a regular Borel measure on X. Then

1. *$\mathcal{G}$ is transitive,*
2. *all isotropy groups of $\mathcal{G}$ are isomorphic with each other,*
3. *for every $y \in X$ there exist an open cover $\{V_j\}$ of X and continuous maps $s_{y,j} : V_j \to \mathcal{G}_y$ such that $r \circ s_j = id_{v_j}$,*
4. *for every $x \in X$ there exists a section $s_x : X \to \mathcal{G}_x$ which is μ-measurable, i.e., for every Borel set B in $\mathcal{G}_x$, $s_x^{-1}(B)$ is μ-measurable subset of X,*
5. *the section s_x is $\mu -$ a.e. continuous on X.*

Groupoid representation

Definition

A unitary representation of a groupoid $\mathcal{G}$ is the pair $(\mathcal{U}, \mathbf{H})$ where $\mathbf{H}$ is a Hilbert bundle over X and $\mathcal{U} = \{U(g)\}_{g\in\mathcal{G}}$ is a family of unitary maps $U(g) : H_{d(g)} \to H_{r(g)}$ such that:

1. $U(\epsilon(x)) = id_{H_x}$ for all $x \in X$,
2. $U(g) \circ U(h) = U(g \circ h)$ for $\nu^2 - a.e.$ $(g, h) \in \mathcal{G}^2$,
3. $U(g^{-1}) = U(g)^{-1}$ *for* $\nu - a.e.$ $g \in \mathcal{G}$,
4. For every $\phi, \psi \in L^2(X, \mathbf{H}, \mu)$,

$$\mathcal{G} \ni g \to (U(g)\phi(d(g)), \psi(r(g)))_{r(g)} \in \mathcal{C}$$

is ν-measurable on $\mathcal{G}$. (Here $L^2(X, \mathbf{H}, \mu)$ denotes the space of square-integrable sections of the bundle $\mathbf{H}$, and $(\cdot, \cdot)_x$ denotes the scalar product in the Hilbert space H_x.)

Properties of representations

Definition

Unitary representations $(\mathcal{U}_1, \mathbf{H}_1)$ and $(\mathcal{U}_2, \mathbf{H}_2)$ of a groupoid $\mathcal{G}$ are said to be *unitarily equivalent* if there exists a family $\{A_x\}_{x \in X}$ of isomorphisms of Hilbert spaces $A_x : H_{1x} \to H_{2x}, x \in X$ such that for every $x, y \in X$ and for $\nu-$a.e. $g \in \mathcal{G}_x^y$ the following diagram commutes

$$\begin{array}{ccc} H_{1x} & \xrightarrow{U_1(g)} & H_{1y} \\ {\scriptstyle A_x}\downarrow & & \downarrow{\scriptstyle A_y} \\ H_{2x} & \xrightarrow[U_2(g)]{} & H_{2y} \end{array}$$

Definition

A unitary representation $(\mathcal{U}, \mathbf{H})$ is called *irreducible* if it has no proper subrepresentations.

Examples of representations

Example

Let $H_x = L^2(\mathcal{G}_x, d\lambda_x)$, for $x \in X$, be a Hilbert space of square λ_x-integrable functions on $\mathcal{G}_x$, and for $g \in \mathcal{G}_x^y, x, y \in X$ and $f \in H_x$ define $U(g) : H_x \to H_y$ by

$$(U(g)f)(g_1) = f(g_1 \circ g),$$

for $g_1 \in \mathcal{G}_y$.
A representation $(\mathcal{U}, \mathbf{H})$ is called *regular representation of the groupoid $\mathcal{G}$*
.

Example

Now let us consider the regular representation of a pair groupoid $\mathcal{G}_0 = X \times X$. Let μ be a regular Borel measure on X. Now we can identify $H_x = L^2(X.\mu)$. Then

$$\mathcal{U}(x, y) = id|_{H_x}, \ for(x, y) \in X$$

Generalized regular representation of a groupoid algebra

Consider the noncommutative algebra $\mathcal{A} = C_c(\mathcal{G}_0)$ of continuous compactly supported functions on the pair groupoid $\mathcal{G}_0$ with multiplication given by the following convolution:

$$(a * b)(x, y) = \int a(x, z)b(z, y)d\mu(z).$$

Such defined algebra will be called the groupoid algebra of $\mathcal{G}_0$. We shall consider a representation $\widetilde{\pi}$ of $\mathcal{A}$ in the space $L^2(X, H, \mu)$ of square-integrable functions on X with values in a Hilbert space H.

$$\widetilde{\pi} : \mathcal{A} \to B(L^2(X, H, \mu))$$

given by the formula

$$[\widetilde{\pi}(a)\psi](x) = \int a(x, y)\psi(y)d\mu(y),$$

where $a \in \mathcal{A}$ and $\psi \in L^2(X, H, \mu)$. This representation will be called also generalized regular representation.

$\mathcal{G}_0$-consistent representation

Let $\mathbf{W} = \{W_x\}_{x\in X}$ be a Hilbert bundle over X and let us consider a new Hilbert bundle $\mathbf{H} = \{H_x\}_{x\in X}$ of the form $H_x = L^2(X, W_x)$. Take a generalized regular representation $\widetilde{\pi}_x$ of the groupoid algebra $\mathcal{A}$ in the spaces H_x:

$$\widetilde{\pi}_x : \mathcal{A} \to B(L^2(X, H_x, \mu)), \quad x \in X.$$

Definition

Let $(\mathcal{U}, \mathbf{H})$ be an unitary representation of the groupoid $\mathcal{G}$. We call it a $\mathcal{G}_0$*-consistent representation*, if the following condition holds:

$$U(g)\widetilde{\pi}_x(a)U(g^{-1}) = \widetilde{\pi}_y(a)$$

for $g \in \mathcal{G}_x^y$, $a \in \mathcal{A}$, and $x, y \in X$.

Group induced representation

Let G be a Lie group and K its closed subgroup. We assume, for simplicity, that $X = K\backslash G$ has a G-invariant measure μ. We consider $\mathcal{H}_L$, a Hilbert space consisting of measurable functions ϕ on G with values in V, such that

$$\phi(hg) = L(h)\phi(g), h \in K,$$

and

$$\int_X ||\phi([g])||^2_V d\mu([g]) < \infty$$

where $[g]$ denotes the image of g in X under the projection $G \to K\backslash G = X$. We introduce the inner product

$$(\phi_1, \phi_2)_{\mathcal{H}_L} = \int_X (\phi_1(x), \phi_2(x))_V d\mu(x).$$

Then we define the representation U^L of G on $\mathcal{H}_L$ given by the formula

$$(U^L(g)f)(g_0) = f(g_0 g), \; g_0, g \in G, \; f \in \mathcal{H}_L.$$

$(U^L, \mathcal{H}_L)$ is called *induced by the representation L of K*.

Imprimitivity system of group G

Definition

Let (U, H) be a unitary representation of the group G, X a $G-$space and P a projection valued measure on the Borel sets of X, $P(B)$ being orthogonal projection on H, and $P(X) = id_H$. The pair (U, P) is called *a system of imprimitivity (S.I. for short) of the group G for the representation U* , if

$$U(g)P(B)U(g^{-1}) = P(Bg^{-1}),$$

where $Bg^{-1} = \{xg^{-1}, x \in B, g \in G\}$, and B a Borel set in X.

Next I present an equivalent definition of S.I.

Imprimitivity system of group G

Definition

Let (U, H) be a unitary representation of the group G, and π be a nondegenerate representation of $* -$ *algebra* $C_0(X)$ of continuous functions on X, vanishing at infinity. The pair of representations (U, π) is called *a system of imprimitivity (S.I. for short) of the group G for the representation U* , if the representations π, U satisfy the following covariance condition:

$$U(g)\pi(f)U(g^{-1}) = \pi(R_g f),$$

where $R_g f(x) = f(xg), \; x \in X, \; g \in G, \; f \in C_0(X)$.

The classical Mackey's imprimitivity theorem states, that every unitary representation of the group G for which there exists a transitive imprimitivity system is equivalent to representation induced by some representation of subgroup K. (The transitivity of S.I. means that $X = K\backslash G$).

Representations of a transformation groupoid

Let G be a Lie group and K its closed subgroup. Consider representations of the transformation groupoid of the form $\mathcal{G} = X \times G$, where $X = K\backslash G$.

Theorem

There exists a one-to-one correspondence between unitary representations of the transformation groupoid $\mathcal{G}$ and the systems of imprimitivity of the group G.

Proof of theorem

Proof. Let $(\mathcal{U}, \mathcal{H})$ be a u.r. of $\mathcal{G}$ in a Hilbert bundle $\mathcal{H}$ over X. Denote $\mathbf{H} = \int_{\oplus} H_x d\mu(x)$ and define $U(g) : \mathbf{H} \to \mathbf{H}$ as

$$U(g) = \int_{\oplus} U(x, g) d\mu$$

Then $(U, \mathbf{H})$ is a u.r. of the group G in the Hilbert space $\mathbf{H}$. Moreover for $f \in C_0(X)$

$$U(g)\pi(f) = \pi(R_g f)U(g)$$

Thus we obtain a S.I. (U, π) of the group G.

For simplicity I present another part of proof in the finite case. Now choose a S.I. (U, P). Denote $H_x = P_x H$ and define:

$$\mathcal{U}(x, g) : H_x \to H_{gx}$$

by the formula:

$$\mathcal{U}(x,g)h = U(g^{-1})|_{H_x}h \qquad \text{for } h \in H_x,$$

Observe that $\mathcal{U}(x,g)h = P_{xg}U(g^{-1})h$, by the property of S.I. But it means that $\mathcal{U}(x,g)h \in H_{gx}$. Let us check the conditions of groupoid representation. Indeed, one has $\mathcal{U}(x,e)h = U(e)|_{H_x}h = h$, for $h \in H_x$. Further $\mathcal{U}(xg_2,g_1) \circ \mathcal{U}(x,g_2) = U(g_1^{-1})|_{H_{xg_2}} \circ U(g_2^{-1})|_{H_x} =$ $U((g_2g_1)^{-1})|_{H_x} = \mathcal{U}(x,g_2g_1)$. And finally $\mathcal{U}(xg,g^{-1}) = U(g)|_{H_{xg}} = (\mathcal{U}(x,g))^{-1}$. Thus we have constructed the representation $(\mathcal{U},\overline{\mathcal{H}})$ of $\mathcal{G}$, corresponding to the S.I. given.

The space of induced representation

Assume that there is given a unitary representation $(\tau, \mathbf{W})$ of the subgroupoid Γ. Here $\mathbf{W}$ is a Hilbert bundle over X. Let W_x denote a fiber over $x \in X$ which is a Hilbert space with the scalar product $\langle \cdot, \cdot \rangle_x$, and let $W = \cup_{x \in X} W_x$ denote the total space of the bundle $\mathbf{W}$.
Let us define, for every $x \in X$, the space $\mathcal{W}_x$ of W-valued functions F defined on the set $\mathcal{G}_x$ satisfying the following four conditions:

1. $F(g) \in W_{r(g)}$ for every $g \in \mathcal{G}_x$,
2. for every μ-Borel measurable r-section $s_x : X \to \mathcal{G}_x$ (see Proposition) the composition $F \circ s_x$ is a μ-measurable section of the bundle $\mathbf{W}$,
3. $F(\gamma \circ g) = \tau(\gamma) F(g)$ for $g \in \mathcal{G}_x, \ \gamma \in \Gamma_{r(g)}$,
4. $\int \langle F(s_x(y)), F(s_x(y)) \rangle_y d\mu(y) < \infty$.

We identify two functions $F, F' \in \mathcal{W}_x$ which differ on the zero-measure sets, and introduce the scalar product $(\cdot,\cdot)_x$ in the space $\mathcal{W}_x$

$$(F_1, F_2)_x = \int \langle F_1(s_x(y)), F_2(s_x(y))\rangle_y d\mu(y)$$

where s_x is a fixed section determined by Proposition.
The spaces $\mathcal{W}_x$, $x \in X$, with these scalar products are Hilbert spaces. It is easily seen that they are isomorphic to the Hilbert space $L^2(X, \mathbf{W})$ of square-integrables sections of the bundle $\mathbf{W}$. Now, let us denote $\mathcal{W} = \{\mathcal{W}_x\}_{x \in X}$. It is a Hilbert bundle over X.

Induced representation of groupoid

Definition

The representation of the groupoid $\mathcal{G}$ *induced by the representation* $(\tau, \mathbf{W})$ of the subgroupoid Γ is the pair $(U^\tau, \mathcal{W})$ where, for $g \in \mathcal{G}_x^y$, we define $U^\tau(g) : \mathcal{W}_x \to \mathcal{W}_y$ by

$$(U^\tau(g_0)F)(g) = F(g \circ g_0).$$

It is clear that $(U^\tau, \mathcal{W})$ is a unitary groupoid representation.

The structure of transformation groupoid

Let us denote

$$\mathcal{G}_x = \{(x, g) \in \mathcal{G} : g \in G\},$$

$$\mathcal{G}^y = \{(yg^{-1}, g) \in \mathcal{G} : g \in G\}.$$

Let us also denote the isotropy group $\mathcal{G}_x^x$ by Γ_x, $\Gamma_x = \{(x, k) : k \in K_x\}$, where K_x is a subgroup of G of the form $K_x = g_0^{-1} K g_0$ where $g_0 \in G$ is an element of the coset x ($x = [g_0]$). Indeed, for $k_x \in K_x$ we have $xk_x = [g_0]g_0^{-1}kg_0 = [kg_0] = x$. Denote by s_0 a Borel section of the principal bundle $G \to K\backslash G = X$, i. e., $[s_0(x)] = Ks_0(x) = x$.

Now, for a function $f \in C_c(\mathcal{G}_x)$, let us define $f_x(y,k) = f(x, s_0(x)^{-1} k s_0(y))$, and

$$\int_{\mathcal{G}_x} f(\mathbf{g}) d\lambda_x(\mathbf{g}) = \int_X \int_K f_x(y,k) dk d\mu(y).$$

Proposition

The collection $\{\lambda_x\}_{x \in X}$ is a right Haar system on the groupoid $\mathcal{G}$.

Now, we shall consider representations of the isotropy subgroupoid Γ. As we have seen, $\Gamma = \bigcup_{x \in X}\{x\} \times K_x$ with $K_x = g^{-1}Kg$ and $g \in G$ such that its coset in X is equal to x ($[g] = x$). We can use $g = s_0(x)$.
Let $(\tau, \mathbf{W})$ be a unitary representation of the groupoid Γ in a Hilbert bundle $\mathbf{W} = \{W_x\}_{x \in X}$.

Definition

A representation $(\tau, \mathbf{W})$ is called X-consistent if there exist a unitary representation (τ_0, W_0) of the group K and a family of Hilbert space isomorphisms

$$A_x : W_0 \to W_x, \;\; x \in X$$

such that, for $\gamma \in \Gamma_x$ of the form $\gamma = (x, s_0(x)^{-1}ks_0(x))$,

$$\tau(\gamma) = A_x \tau_0(k) A_x^{-1}.$$

Induced representations of $\mathcal{G} = X \times G$

In the sequel we shall consider the representation of the groupoid $\mathcal{G} = X \times G$ induced by X - consistent representation $(\tau, \mathbf{W})$ of the subgroupoid Γ, and we shall establish its connection with the induced representation in the Mackey sense of the group G. Now condition 3 of the definition of the space $\mathcal{W}_x$ assumes the form

$$F(\gamma \circ (x, g)) = \tau(\gamma) F(x, g)$$

where $x, y \in X, \; y = xg, \; g \in G, \; \gamma \in \Gamma_y = \{y\} \times K_y$. Thus we have $\gamma = (y, s_0(y)^{-1} k s_0(y))$ for an element $k \in K$. Then, by the definition of X-consistent representation, we can write

$$F(\gamma \circ (x, g)) = (A_y \tau_0(k) A_y^{-1}) F(x, g).$$

Let introduce a function $\phi : G \to W_0$ defined by the formula $\phi(k s_0(y)) = A_y^{-1}(F(x, s_0(x)^{-1} k s_0(y)))$. Then the function ϕ has the property $\phi(kg) = \tau_0(k)\phi(g)$.

We shall use the notation (L, W_0) for the unitary representation of the group K in the space W_0, $L = \tau_0$. Thus we have $\phi(kg) = L(k)\phi(g)$ and we can consider the Hilbert space $\mathcal{H}_L$ introduced above as well as the representation $(U^L, \mathcal{H}_L)$ of the group G induced in the sense of Mackey by L from the subgroup K.

The following theorem establishes a connection of the induced representation $(\mathcal{U}^\tau, \mathcal{W})$ of the groupoid $\mathcal{G}$ with the representation $(U^L, \mathcal{H}_L)$ of the group G.

Denote by R_g, $g \in G$, the following operator acting in the space $\mathcal{W}_x, x \in X$, $y = xg$,

$$(R_g F)(x, h) = (A_{xh} A_{xhg}^{-1})(F(x, hg)).$$

Then we have the family of unitary G-representations $(R, \mathcal{W}_x), x \in X$. (The unitarity follows from the fact that the measure μ is G-invariant and the operators A_{xh}, A_{xhg} are Hilbert space isomorphisms.)

Relation with group induced representation

Theorem

1. *For every $x \in X$ the G-representation $(R, \mathcal{W}_x)$ is unitarily equivalent to the induced representation $(U^L, \mathcal{H}_L)$.*
2. *All representations $(R, \mathcal{W}_x)$, $x \in X$, are unitarily equivalent to each other. The equivalence is given by the operators $I_x^y : \mathcal{W}_x \to \mathcal{W}_y$,*

$$(I_x^y F)(y, s_0(y)^{-1} k s_0(z)) = (A_y A_z^{-1})(F(x, s_0(x)^{-1} k s_0(z))),$$

$x, y \in X$.

Proof of theorem

Proof.
We define the linear map $J_x : \mathcal{W}_x \to \mathcal{H}_L$ by $(J_x F)(g) = \phi(ks_0(y)) = A_y^{-1}(F(x, s_0(x)^{-1}ks_0(y)))$ where $g = ks_0(y)$. J_x is a linear isomorphism since A_y is an isomorphism and it is easily seen that J_x preserves scalar products of $\mathcal{W}_x$ and $\mathcal{H}_L$ and so it is a Hilbert space isomorphism. To see that it defines an equivalence of representations, we have to show that, for $g \in G$, the following diagram is commutative

$$\begin{array}{ccc} \mathcal{W}_x & \xrightarrow{R_g} & \mathcal{W}_x \\ {\scriptstyle J_x}\downarrow & & \downarrow{\scriptstyle J_x} \\ \mathcal{H}_L & \xrightarrow[U^L(g)]{} & \mathcal{H}_L \end{array}$$

Let us compute $(U^L(g)J_x)(F)(h)$. It is sufficient to take $h = s_0(y)$ and to notice that each $g \in G$ can be written in the form $g = s_0(y)^{-1}ks_0(z)$, for $z \in X, z = yg$ and an element $k \in K$.

$$(U^L(s_0(y)^{-1}ks_0(z))J_x)(F)(s_0(y)) = (J_xF)(ks_0(z)) =$$
$$= L(k)A_z^{-1}(F(x, s_0(x)^{-1}s_0(z))).$$

On the other hand

$$(J_xR_g)(F)(s_0(y)) = A_y^{-1}((R_gF)(x, s_0(x)^{-1}s_0(y))) =$$
$$= A_y^{-1}(A_yA_z^{-1})(F(x, s_0(x)^{-1}ks_0(z))) = A_z^{-1}\tau(\gamma)(F(x, s_0(x)^{-1}s_0(z))) =$$
$$= A_z^{-1}A_z\tau_0(k)A_z^{-1}(F(x, s_0(x)^{-1}s_0(z))) = L(k)A_z^{-1}(F(x, s_0(x)^{-1}s_0(z))).$$

Now it is a simple observation that $I_x^y = J_y^{-1}J_x$. $\diamond$

Imprimitivity system of groupoid

Definition

Let $(\mathcal{U}, \mathbf{H})$ be an unitary $\mathcal{G}_0$-consistent representation of the groupoid $\mathcal{G}$. Consider the commutative algebra $L^\infty(X)$ and a family $\pi = (\pi_x)_{x\in X}$ of its representations in the Hilbert spaces $L^2(X, W_x)$ respectively, given by the operators of multiplication by a function:
$L^\infty(X) \ni f \to \pi_x(f) \in B(L^2(X, W_x))$ where, for $z \in X, \psi \in L^2(X, W_x)$

$$[\pi_x(f)\psi](z) = f(z)\psi(z).$$

We say that the representation $\mathcal{U}$ has *a system of imprimitivity* $(\mathcal{U}, \pi)$ if for every $f \in L^\infty(X)$, and for μ - a.e. $x, y \in X$, and ν - a.e. $g \in \mathcal{G}_x^y$ the following condition holds:

$$U(g)\pi_x(f)U(g^{-1}) = \pi_y(f).$$

Imprimitivity theorem for groupoid

Theorem

If, for a representation $(\mathcal{U}, \mathbf{H})$, there exists a system of imprimitivity $(\mathcal{U}, \pi)$ then the representation $\mathcal{U}$ is equivalent to the representation $\mathcal{U}^{\tau}$ induced by some representation $(\tau, \mathbf{W})$ of the subgroupoid Γ.

Let us observe that, for $\gamma \in \Gamma_x = \mathcal{G}_x^x$, the covariance condition of the imprimitivity system reduces to the following one

$$U(\gamma)\pi_x(f)U(\gamma^{-1}) = \pi_x(f).$$

It follows that $U(\gamma)$ are decomposable, i.e., for $\mu - a.e.\ y \in X$, there exists an operator $U(\gamma)_y \in B(W_x)$ such that, for $\psi \in L^2(X, W_x)$, $(U(\gamma)\psi)(y) = U(\gamma)_y(\psi(y))$.

Morover,notice that the Hilbert space $L^2(X, W_x)$ is isomorphic to the tensor product of Hilbert spaces $L^2(X) \bigotimes W_x$.

And more

Lemma 1

Lemma

If for a representation $(U, \mathbf{H})$ there exists a system of imprimitivity, then

1. *there exists a unitary representation (τ_x, W_x) of the group Γ_x such that $U(\gamma) = id_{L^2} \otimes \tau_x(\gamma)$ for every $\gamma \in \Gamma_x$ and μ - a.e. $x \in X$. (In particular it means that the function $X \ni y \to U(\gamma)_y \in B(H_x)$ is a constant field of operators),*
2. *we can define a representation $(\tau, \mathbf{W})$ of the subgroupoid Γ such that, for $\gamma \in \Gamma_x$, $\tau(\gamma) = \tau_x(\gamma)$,*

Proof of Lemma 1

Proof: A decomposable operator $U(\gamma)$ in the space $L^2(X) \bigotimes W_x$ has the form $[U(\gamma)(\psi \otimes h)](y) = \psi(y) \otimes U(\gamma)_y h$. We have to show that it is of the form $id_{L^2} \otimes \tau_x(\gamma)$, where $\tau_x(\gamma) \in B(W_x)$. Since $(\mathcal{U}, \mathbf{H})$ is a $\mathcal{G}_0$-consistent representation, the following commutation relation holds:

$$U(\gamma^{-1})\widetilde{\pi}_x(a)U(\gamma) = \widetilde{\pi}_x(a)$$

for $a \in \mathcal{A}, \; \gamma \in \Gamma_x$, and $x \in X$. But this implies that

$$U(\gamma^{-1})AU(\gamma) = A$$

for every A of the form $A = A_0 \otimes id_{W_x}$, $A_0 \in B(L^2(X))$. Then it follows that $U(\gamma) = id_{L^2} \otimes \tau_x(\gamma)$. It is clear that all $\tau_x(\gamma)$ are unitary in W_x. Thus τ_x is a unitary representation of the group Γ_x in the Hilbert space W_x. This ends the proof of Lemma.

Lemma 2

Lemma

1. *The representations* $\tau_x, x \in X$ *are equivalent to each other, as representations of isomorphic groups* Γ_x*.*
2. *The operators* $U(g) : H_x \to H_y$*, where* $H_x = L^2(X, W_x)$*,* $H_y = L^2(Y, W_y)$ *for* $g \in \mathcal{G}_x^y$*, are decomposable, i.e., there exist unitary operators* $U^0(g) : W_x \to W_y$ *such that for* $\psi \in L^2(X, W_x)$ *and, for* $z \in X$*,*

$$(U(g)\psi)(z) = (U^0(g))(\psi(z)).$$

Moreover, the operator $U^0(g) : W_x \to W_y$ *does not depend of* $z \in X$*.*

Proof of Lemma 2

Proof: First we shall prove part 2. Denote by $i_x^y : W_x \to W_y$ an isomorphism of Hilbert spaces and define the unitary map $R_x^y : L^2(X, W_x) \to L^2(X, W_y)$ by $(R_x^y\psi)(z) = i_x^y(\psi(z))$, $\psi \in L^2(X, W_x), z \in X$. Consider the composition of unitary maps $U(g) \circ (R_x^y)^{-1} : L^2(X, W_y) \to L^2(X, W_y)$ where $g \in \mathcal{G}_x^y$. By using the property of the imprimitivity system for $U(g)$, we obtain

$$U(g) \circ (R_x^y)^{-1} \circ \pi_y(f) = \pi_y(f) \circ U(g) \circ (R_x^y)^{-1}$$

for $f \in L^\infty(X)$.
This means that the operator $U(g) \circ (R_x^y)^{-1}$ is decomposable in $L^2(X, W_y)$. But (R_x^y) is a decomposable map by definition, therefore $U(g)$ is decomposable as the composition of decomposable maps. As in the proof of Lemma 1 we conclude that $U^0(g)$ does not depend of $z \in X$ and is unitary.

To prove part 1 let us first observe that the isotropy groups Γ_x are isomorphic to each other $x \in X$. Indeed, taking an element $g \in \mathcal{G}_x^y$ we define the isomorphism $i : \Gamma_x \to \Gamma_y$ by the formula $i(\gamma) = g \circ \gamma \circ g^{-1}$ for $\gamma \in \Gamma_x$. Now, we have $U(i(\gamma)) = id_{L^2} \otimes \tau_y(i(\gamma))$ as in the proof of Lemma 1. On the other hand, $U(i(\gamma)) = U(g) \circ U(\gamma) \circ U(g^{-1}) = (id_{L^2} \otimes U^0(g)) \circ (id_{L^2} \otimes \tau_x(\gamma)) \circ (id_{L^2} \otimes U^0(g)^{-1}) = id_{L^2} \otimes (U^0(g) \circ \tau_x(\gamma) \circ U^0(g)^{-1})$. Therefore, we have $\tau_y(i(\gamma)) = U^0(g) \circ \tau_x(\gamma) \circ U^0(g)^{-1}$, but this means that the representations τ_y and τ_x are equivalent.

Idea of proof of the theorem

- Define a family of linear maps of Hilbert spaces

$$J_x : H_x \to \mathcal{W}_x, \;\; x \in X$$

- The maps J_x are unitary isomorphisms.
- J_x are intertwining maps, i.e., the diagram commutes:

$$\begin{array}{ccc} H_x & \xrightarrow{U(g)} & H_z \\ {\scriptstyle J_x}\downarrow & & \downarrow{\scriptstyle J_z} \\ \mathcal{W}_x & \xrightarrow[U^{\tau}(g)]{} & \mathcal{W}_z \end{array}$$

Proof of the theorem

Proof. Let us consider the spaces $\{\mathcal{W}_x\}_{x \in X}$, connected to the representation τ of Lemma 1 and the corresponding induced representation U^τ. We shall show that the representation $(U, \mathbf{H})$ is equivalent to $(U^\tau, \mathcal{W})$. We define a family of isomorphisms of Hilbert spaces $J_x : H_x \to \mathcal{W}_x$ for μ - a.e. $x \in X$. Since $H_x = L^2(X, W_x)$, for $\psi \in H_x$, $g \in \mathcal{G}_x$, and $r(g) = y$, we put $F(g) = (J_x\psi)(g) = (U(g)(\psi))(y)$. The definition is correct since by Lemma 2 we have $(U(g)\psi)(y) = U^0(g)(\psi(y))$, and $U^0(g)$ does not depend of $y \in X$. Since $U(g)\psi \in L^2(X, W_y)$, therefore $[U(g)(\psi)](y) \in W_y$. Also it is clear that $F(\gamma \circ g) = \tau(\gamma)(F(g))$ for $\gamma \in \Gamma_y$.

To see the square-integrability let us write

$$\int \langle F(s_x(y)), F(s_x(y)) \rangle_y d\mu(y) =$$

$$= \int \langle U^0(s_x(y))(\psi)(y), U^0(s_x(y))(\psi)(y) \rangle_y d\mu(y) = \int \langle \psi(y), \psi(y) \rangle_y d\mu(y) =$$

$$= \| \psi \|_{H_x} < \infty.$$

This also shows that J_x are unitary maps and are injective.

To see that J_x map onto $\mathcal{W}_x$, we can give the formula for J_x^{-1}: $(J_x^{-1}F)(y) = (U^0(g))^{-1}(F(g))$ where $F \in \mathcal{W}_x$ and $g \in \mathcal{G}_x^y$. Then the right-hand side does not change if we take other element $g_1 \in \mathcal{G}_x^y$. Indeed, since $g_1 = \gamma \circ g$, for an element $\gamma \in \Gamma_y$, therefore we have $(U^0(\gamma \circ g))^{-1}(F(\gamma \circ g)) = ((U^0(g))^{-1}(\tau(\gamma))^{-1}(\tau(\gamma))(F(g)) = (U^0(g))^{-1}(F(g))$. This shows that J_x, $x \in X$, are isomorphisms of Hilbert spaces.

Now we can see that J_x are intertwining maps for the representations U and U^τ, i.e., that the following diagram commutes

$$\begin{array}{ccc} H_x & \xrightarrow{U(g)} & H_z \\ {\scriptstyle J_x}\downarrow & & \downarrow{\scriptstyle J_z} \\ \mathcal{W}_x & \xrightarrow[U^\tau(g)]{} & \mathcal{W}_z \end{array}$$

for μ-a.e. $x, z \in X$ and ν - a.e. $g \in \mathcal{G}_x^z$. Let $\psi \in H_x$. Then, for $h \in \mathcal{G}_z^y$, we have $[(J_z U(g))(\psi)](h) = [(U(h)(U(g))(\psi)](y) = U(h \circ g)(\psi(y)) = U^0(h \circ g)(\psi(y))$. On the other hand,
$U^\tau(g) J_x(\psi)(h) = [J_x(\psi)](h \circ g) = [U(h \circ g)(\psi)](y)$. This ends the proof of Theorem.

Energy-momentum space of a particle

Consider the energy-momentum space H of a particle, $H = \{(p_0, p_1, p_2, p_3) \in \mathbf{R}^4 : p_0^2 - p_1^2 - p_2^2 - p_3^2 = m\}$. We have an action of the group $G = SL_2(\mathbf{C})$ on the hyperboloid H.
To describe the action we identify H with the set $\overline{H}$ of hermitian 2×2-matrices with determinant equal to m,

$$(p_0, p_1, p_2, p_3) \mapsto \begin{pmatrix} p_0 - p_3 & p_2 - ip_1 \\ p_2 + ip_1 & p_0 + p_3 \end{pmatrix}$$

and we let to act $g \in G$ on $\overline{H}$ to the right in the following way, $\overline{H} \ni A \mapsto g^* A g \in \overline{H}$. (It is clear that $det(g^* A g) = det A = m$).

Next, we see that the isotropy group of the element $(p_0, 0, 0, 0)$, $p_0 = \sqrt{m}$ is equal to $K = SU(2)$. Thus we deduce that the homogoneus space $K\backslash G$ is diffeomorphic to H. We can take the phase space of a particle of the mass m as the space $\mathcal{G} = K\backslash G \times G = H \times G$ and consider the algebraic structure of transformation groupoid on it.
Let $(\mathcal{U}, \mathcal{W})$ be a unitary representation of the groupoid $\mathcal{G}$ in a Hilbert bundle $\mathcal{W}$.

An imprimitivity system and a particle

Assume that there exists an imprimitivity system $(\mathcal{U}, \pi)$ for $(\mathcal{U}, \mathcal{W})$. We say that a particle of mass m is represented by the pair $(\mathcal{U}, \pi)$. We say that it is an elementary particle if the imprimitivity system $(\mathcal{U}, \pi)$ is irreducible [13], [14]. Equivalently (on the strength of the Imprimitivity Theorem), we can say that the particle is an induced representation $(\mathcal{U}^\tau, \mathcal{W})$ where τ is a unitary representation of the isotropy subgroupoid Γ. In the same way, we can say that the particle is elementary if the inducing representation τ is irreducible and, in turn, this means that the representation (L, W_0), $L = \tau_0$, of the group $K = SU(2)$ is irreducible. Then the representation (L, W_0) is called the spin of the particle.

Thank you for your attention

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