Jacobi's formula

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In matrix calculus, **Jacobi's formula** expresses the derivative of the determinant of a matrix A in terms of the adjugate of A and the derivative of A.[1] If A is a differentiable map from the real numbers to $n \times n$ matrices,

$$\frac{d}{dt} \det A(t) = \mathrm{tr}(\mathrm{adj}(A(t)) \, \frac{dA(t)}{dt}).$$

Equivalently, if dA stands for the differential of A, the formula is

$$d \det(A) = \mathrm{tr}(\mathrm{adj}(A) \, dA).$$

It is named after the mathematician C.G.J. Jacobi.

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Derivation

We first prove a preliminary lemma:

Lemma. Let A and B be a pair of square matrices of the same dimension n. Then

$$\sum_i \sum_j A_{ij} B_{ij} = \mathrm{tr}(A^{\mathrm{T}} B).$$

Proof. The product AB of the pair of matrices has components

$$(AB)_{jk} = \sum_i A_{ji} B_{ik}.$$

Replacing the matrix A by its transpose A^{T} is equivalent to permuting the indices of its components:

$$(A^{\mathrm{T}} B)_{jk} = \sum_i A_{ij} B_{ik}.$$

The result follows by taking the trace of both sides:

$$\mathrm{tr}(A^{\mathrm{T}} B) = \sum_j (A^{\mathrm{T}} B)_{jj} = \sum_j \sum_i A_{ij} B_{ij} = \sum_i \sum_j A_{ij} B_{ij}. \ \square$$

Theorem. (Jacobi's formula) For any differentiable map A from the real numbers to $n \times n$ matrices,

$$d\det(A) = \operatorname{tr}(\operatorname{adj}(A)\,dA).$$

Proof. Laplace's formula for the determinant of a matrix *A* can be stated as

$$\det(A) = \sum_j A_{ij} \operatorname{adj}^{\mathrm{T}}(A)_{ij}.$$

Notice that the summation is performed over some arbitrary row *i* of the matrix.

The determinant of *A* can be considered to be a function of the elements of *A*:

$$\det(A) = F\left(A_{11}, A_{12}, \ldots, A_{21}, A_{22}, \ldots, A_{nn}\right)$$

so that, by the chain rule, its differential is

$$d\det(A) = \sum_i \sum_j \frac{\partial F}{\partial A_{ij}}\,dA_{ij}.$$

This summation is performed over all $n \times n$ elements of the matrix.

To find $\partial F/\partial A_{ij}$ consider that on the right hand side of Laplace's formula, the index *i* can be chosen at will. (In order to optimize calculations: Any other choice would eventually yield the same result, but it could be much harder). In particular, it can be chosen to match the first index of $\partial / \partial A_{ij}$:

$$\frac{\partial \det(A)}{\partial A_{ij}} = \frac{\partial \sum_k A_{ik} \operatorname{adj}^{\mathrm{T}}(A)_{ik}}{\partial A_{ij}} = \sum_k \frac{\partial (A_{ik} \operatorname{adj}^{\mathrm{T}}(A)_{ik})}{\partial A_{ij}}$$

Thus, by the product rule,

$$\frac{\partial \det(A)}{\partial A_{ij}} = \sum_k \frac{\partial A_{ik}}{\partial A_{ij}} \operatorname{adj}^{\mathrm{T}}(A)_{ik} + \sum_k A_{ik} \frac{\partial \operatorname{adj}^{\mathrm{T}}(A)_{ik}}{\partial A_{ij}}.$$

Now, if an element of a matrix A_{ij} and a cofactor $\operatorname{adj}^{\mathrm{T}}(A)_{ik}$ of element A_{ik} lie on the same row (or column), then the cofactor will not be a function of A_{ij}, because the cofactor of A_{ik} is expressed in terms of elements not in its own row (nor column). Thus,

$$\frac{\partial \operatorname{adj}^{\mathrm{T}}(A)_{ik}}{\partial A_{ij}} = 0,$$

so

$$\frac{\partial \det(A)}{\partial A_{ij}} = \sum_k \operatorname{adj}^{\mathrm{T}}(A)_{ik} \frac{\partial A_{ik}}{\partial A_{ij}}.$$

All the elements of *A* are independent of each other, i.e.

$$\frac{\partial A_{ik}}{\partial A_{ij}} = \delta_{jk},$$

where δ is the Kronecker delta, so

$$\frac{\partial \det(A)}{\partial A_{ij}} = \sum_k \operatorname{adj}^{\mathrm{T}}(A)_{ik}\delta_{jk} = \operatorname{adj}^{\mathrm{T}}(A)_{ij}.$$

Therefore,

$$d(\det(A)) = \sum_i \sum_j \operatorname{adj}^{\mathrm{T}}(A)_{ij}\, dA_{ij},$$

and applying the Lemma yields

$$d(\det(A)) = \operatorname{tr}(\operatorname{adj}(A)\, dA).\ \square$$

Corollary

For any invertible matrix A, the inverse A^{-1} is related to the adjugate by $A^{-1} = (\det A)^{-1} \operatorname{adj} A$. It follows that if $A(t)$ is invertible for all t, then

$$\frac{d}{dt}\det A(t) = (\det A(t))\operatorname{tr}\left(A(t)^{-1}\frac{d}{dt}A(t)\right).$$

Furthermore, taking $A(t) = \exp(tB)$, this reduces to

$$\frac{d}{dt}\det e^{tB} = (\operatorname{Tr}B)\ \det e^{tB},$$

solved by

$$\boxed{\det e^{tB} = e^{\operatorname{Tr}\ tB}\ ,}$$

a useful relation connecting the trace to the determinant of the associated matrix exponential.

Notes

1. ^ Magnus & Neudecker (1999), Part Three, Section 8.3

References

- Magnus, Jan R.; Neudecker, Heinz (1999), *Matrix Differential Calculus with Applications in Statistics and Econometrics*, Wiley, ISBN 0-471-98633-X
- Bellmann, Richard (1987), *Introduction to Matrix Analysis*, SIAM, ISBN 0898713994