

Field of Rational Functions

On page 4 of the textbook, we read, “there are ordered fields which do not have the Archimedean property.” The purpose of these notes is to describe one.

The members of the field F will be *rational functions*. These are the functions of one variable that are expressible as a quotient of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomial functions (with real coefficients) having no common linear factors and q is not identically zero. For example,

$$\frac{5x^2 + \pi}{x^3 - x^2 + x - 1}, \quad 2, \quad (\sqrt{17})x$$

are three rational functions. Here by “2” is meant the function that is constantly 2. More generally, each rational function is expressible in the form

$$f(x) = \frac{a_mx^m + a_{m-1}x^{m-1} + \cdots + a_0}{b_nx^n + b_{n-1}x^{n-1} + \cdots + b_0}$$

where $b_n \neq 0$. In fact without loss of generality, we can always take $b_n = 1$.

The domain of a rational function consists of all but the finitely many real numbers where the denominator is 0. And at those points, the numerator is nonzero, lest there be a common linear factor. So when we compare two rational functions f and g , we can be sure that the “common domain” $\text{dom } f \cap \text{dom } g$ consists of all but finitely many real numbers. (Such sets are called *cofinite*.)

To complete the description of the field, we need specify the addition (+) and multiplication (\cdot) operations on the set F of rational functions, and we need to single out two members of F as the 0 and 1 elements. The latter is easy; the elements 0 and 1 are the constant functions having the values 0 and 1, respectively.

We define addition (+) and multiplication (\cdot) operations on the set F of rational functions as on page 12, but with one additional wrinkle. Page 12 defines $f + g$ to be the function with domain $\text{dom } f \cap \text{dom } g$ whose value at x is $f(x) + g(x)$, and defines multiplication similarly. The extra wrinkle is that we also need to remove any removable singularities. Algebraically, this amounts to canceling any common linear factors in the numerator and denominator.

Here are two examples. Let

$$f(x) = \frac{1}{x^2 - 1}, \quad g(x) = \frac{x}{x^2 - 1}, \quad h(x) = x + 1.$$

Thus $\text{dom } f$ and $\text{dom } g$ both lack -1 and 1 . For $f + g$ we have the function whose value at x is

$$\frac{1}{x^2 - 1} + \frac{x}{x^2 - 1} = \frac{1 + x}{x^2 - 1} = \frac{1}{x - 1}.$$

In the last step, we have canceled the common factor of $x + 1$, thereby removing the singularity at -1 . The sum, $1/(x - 1)$, lacks only 1 from its domain. Similarly for $f \cdot h$ we have the function whose value at x is

$$\frac{1}{x^2 - 1} \cdot (x + 1) = \frac{x + 1}{x^2 - 1} = \frac{1}{x - 1}.$$

Once again, in the last step we have canceled the common factor of $x + 1$, thereby removing the singularity at -1 . The product, $1/(x - 1)$, lacks only 1 from its domain.

These two examples illustrate the general procedure: For addition, we first add the functions as on page 12, and then we remove any removable singularities. And the same applies to multiplication. To say this in more algebraic terms:

$$\left(\frac{p}{q} + \frac{r}{s}\right)(x) = \frac{p(x)s(x) + q(x)r(x)}{q(x)s(x)}$$

but any common linear factors in the numerator and denominator must be canceled.

$$\left(\frac{p}{q} \cdot \frac{r}{s}\right)(x) = \frac{p(x)r(x)}{q(x)s(x)}$$

but again any common linear factors in the numerator and denominator must be canceled.

Theorem 1. F , together with $+$, \cdot , 0 , and 1 as defined above, forms a field.

Proof. We need to verify the nine field axioms listed on page 1. The commutative, associative, and distributive laws (i.e., P1, P2, P5, P6, and P9) all follow from the corresponding laws for the reals. For example, in P1 we want to verify that $f + g = g + f$ for any f and g in F . Two functions are equal iff they have the same domain and the same values at each point of that domain. Here $f + g$ has at x the value $f(x) + g(x)$. And $g + f$ has at x the value $g(x) + f(x)$. It is a fact about the reals that addition of numbers is commutative. The extra wrinkle of removing singularities complicates the picture a little, but not too much.

P3 (0 is an additive identity) and P7 (1 is a multiplicative identity) are “trivial” (meaning that they are trivial once one stops and looks at them). And for P7, we note that 0 and 1 are different functions.

For P4, existence of additive inverses, we take $-f$ to be the function whose domain is exactly the same as $\text{dom } f$ and whose value at x is $-f(x)$. Then $f(x) + (-f(x)) = 0$ everywhere except the points missing from $\text{dom } f$. After removing those singularities, $(f + (-f))(x) = 0$ everywhere.

For P8, we need, for each nonzero rational function f , a multiplicative inverse f^{-1} . (The notation here is potentially misleading. We write “ f^{-1} ” as in the statement of P8. But what is meant is the *reciprocal* of f , not the *inverse function*.) We take the obvious candidate:

$$\text{If } f = p/q \text{ then } f^{-1} = q/p.$$

The domain of f^{-1} will lack the finitely values of x for which $p(x) = 0$. But it will lack nothing else; for values of x not in $\text{dom } f$, the above shows that $q(x) = 0$ and $f^{-1}(x) = 0$.

Then $f(x) \cdot f^{-1}(x) = 1$ everywhere except the points missing from $\text{dom } f$ or from $\text{dom } f^{-1}$. After removing those singularities, $(f \cdot f^{-1})(x) = 1$ everywhere. Here is an example:

$$f(x) = \frac{x-2}{x-3} \quad \text{and} \quad (f^{-1})(x) = \frac{x-3}{x-2}$$

where $\text{dom } f$ lacks 3 and $\text{dom } f^{-1}$ lacks 2 . \dashv

Next we need to augment this field with an ordering relation \leq making it into an ordered field. By Problem 12 on page 6, it suffices to specify the set P of positive elements. Here is the rule:

$$f(x) = \frac{a_m x^m + a_{m-1} x^{m-1} + \cdots + a_0}{x^n + b_{n-1} x^{n-1} + \cdots + b_0} \in P \quad \text{iff} \quad a_m > 0.$$

That is, when the leading coefficient (the coefficient of the highest power of x) in the denominator is 1 , the leading coefficient in the numerator determines the sign.

Clearly we have property (a): Exactly one of these alternatives

$$f \in P, \quad f = 0, \quad (-f) \in P$$

holds. Next we need (b): The set P is closed under addition and multiplication. Closure under multiplication is relatively easy to see. Look at addition:

$$\frac{p(x)}{q(x)} + \frac{r(x)}{s(x)} = \frac{p(x)s(x) + q(x)r(x)}{q(x)s(x)}.$$

where in both q and s the leading coefficients are 1 . For the sum, the denominator qs still has leading coefficient 1 . What is the leading coefficient in the numerator?

Case I: Suppose $\deg p + \deg s > \deg q + \deg r$. Then the leading coefficient in the sum is the leading coefficient of p .

Case II: Suppose $\deg p + \deg s < \deg q + \deg r$. Then the leading coefficient in the sum is the leading coefficient of r .

Case III: Suppose $\deg p + \deg s = \deg q + \deg r$. Then the leading coefficient in the sum is the sum of the leading coefficient in p and the leading coefficient in q .

The point is that in all three cases, the sum of functions in P is still in P .

As in Problem 12, we now define $f \leq g$ to hold iff either $f - g$ (which is by definition $f + (-g)$) is in P , or else $f = g$.

Theorem 2. F , together with $+$, \cdot , 0 , 1 and \leq as defined above, forms an ordered field.

For example,

$$\frac{5x^2 + \pi}{x^3 - x^2 + x - 1} < 2 < (\sqrt{17})x$$

in this field.

Observe that the identity function $f(x) = x$ is “infinitely large” in the sense that

$$1 < x, \quad 2 < x, \quad 3 < x, \quad \dots$$

Similarly, the reciprocal function $g(x) = 1/x$ is positive but “infinitesimal” in the sense that

$$0 < \frac{1}{x} < \frac{1}{2}, \quad 0 < \frac{1}{x} < \frac{1}{3}, \quad 0 < \frac{1}{x} < \frac{1}{4}, \quad \dots$$

In particular, the Archimedean property fails to hold, because $x \not< 1 + 1 + \dots + 1$, no matter how many 1’s we add together.

Theorem 3. F , together with $+$, \cdot , 0 , 1 and \leq as defined above, forms a non-Archimedean ordered field.

Inside the field F , we have a copy of the real field. That is, the map taking each real number r to the constant function with value r embeds the real ordered field into F . So we can think of F as including a copy of the real ordered field, plus a lot of other elements.

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