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Algebraic structures with such a composition law are listed below from the most to the least general. Each step adds an assumption, so that each of these structures is a special case of the preceding structures.
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1. **Semigroups**: no assumptions. **Example**: the set of all finite words written with two letters ($ab, ba, abba,\ldots$).
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2. **Monoids**: $\exists e \in G \forall g \in G: eg = g = ge$. **Examples**: $(\text{Map}(X, X), \circ, \text{id}), (\mathbb{N}, +, 0)$. 


One binary associative composition law

Let $G$ be a non-empty set. The map $G \times G \ni (a, b) \mapsto ab \in G$ is called an associative composition law if and only if

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   Examples: $(\text{Map}(X, X), \circ, \text{id})$, $(\mathbb{N}, +, 0)$.

3. **Groups**: $\forall g \in G \ \exists g^{-1} \in G: g^{-1}g = e = gg^{-1}$.
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4. **Abelian groups**: $\forall g, h \in G: gh = hg.$
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Two distributive composition laws

Let $R$ be a non-empty set with two binary associative composition laws satisfying the distributivity condition:

$$\forall r, s, t \in R: (r + s)t = rt + st, \ r(s + t) = rs + rt$$.
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1. **Rings**: $(R, +, 0)$ is an Abelian group and $(R, \cdot, 1)$ is a monoid.

   Examples: $(\text{End}_\mathbb{Z}(G); \circ, \text{id}; \text{poinwise} +, 0 \text{ function})$, where $(G, +, 0)$ is an Abelian group, matrix ring $M_n(\mathbb{Z})$. 

2. **Commutative rings**: $\forall r, s \in R: rs = sr$.

   Examples: $(\mathbb{Z}/n\mathbb{Z}; \cdot, 1; +, 0)$, polynomial ring $(\mathbb{Z}/n\mathbb{Z})[N]$. 

3. **Integral domains**: $rs = 0 \Rightarrow (r = 0 \text{ or } s = 0)$.

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   Examples: $(\mathbb{Z}/p\mathbb{Z}; \cdot, 1; +, 0)$, where $p$ is a prime number. 

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1. **Modules**: $\forall m, n \in M, r, s \in R$:
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   **Examples**: $\mathrm{End}_\mathbb{Z}(G) \times G \ni (f, g) \mapsto f(g) \in G$

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2. **Free modules:** there exists a basis of $M$. **Example:** $\bigoplus_{\mathbb{N}} R$. 


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4. **Finite-dimensional vector spaces:** there exists a finite basis of \( M. \) **Example:** \( R^n. \)
Let $A$ be a module over a commutative ring $k$ equipped with a $k$-bilinear associative multiplication

$$A \times A \ni (a, b) \mapsto ab \in A.$$ 

Then $A$ is called an algebra over $k$. 

It is called a unital algebra over $k$ if and only if $A$ is a ring with respect to its Abelian group structure and multiplication. In other words, a unital algebra is a module with a linear ring structure, or a ring with a compatible module structure. Every ring is a unital algebra over the ring $\mathbb{Z}$ of all integers.
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Let $A$ be a unital algebra over a commutative ring $k$. The **spectrum** of $a \in A$ is the following subset of $k$:

$$\text{spec}_A(a) := \{ \lambda \in k \mid \not\exists (a - \lambda 1)^{-1} \in A \}.$$

In particular, when $M$ is a module over $k$, we can take $A = \text{End}_k(M)$. Then $v \in M$ is called an eigenvector of $a \in \text{End}_k(M)$ corresponding to $\lambda \in k$ if and only if $av = \lambda v$.

Note that $v \neq 0 \Rightarrow \lambda \in \text{spec}_A(a)$. All elements of $\text{spec}_A(a)$ coming from a non-zero eigenvector of $a$ are called eigenvalues.

If $M$ is a finite-dimensional free module over a non-zero commutative ring $k$, then all elements of the spectrum of any endomorphism $a$ are eigenvalues. They are roots of the characteristic polynomial $\det(a - \lambda 1)$.
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