Compact quantum groups

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Abstract

In this paper we review the theory of compact quantum groups.

0 Introduction

There are two possible approaches to the theory of groups. In the first one, more concrete, from the very beginning we deal with groups with elements of definite mathematical nature like matrices or transformations. The group multiplication is then some mathematical operation that is known in advance. Similarly the topology of the group is taken from the surrounding space. In this approach we need not to postulate that the group multiplication is associative and continuous. These are satisfied automatically. Instead, we have to assume that multiplying two elements of a group G we get the result belonging to G, that all elements of G have inverses belonging to G and that G is a closed subset of the surrounding space. For example, introducing a $(N \times N)$-matrix group G we postulate, that G is a closed subset of the set of all $(N \times N)$-matrices, closed under matrix multiplication and that all elements of G are invertible matrices. G is automatically locally compact and the matrix multiplication is associative and continuous. The second example is a transformation group of a space X. In this case G is a set of bijections of X. One has to assume that the composition $g \circ g'$ and the inverse bijection g^{-1} belong to G for any $g, g' \in G$. If X is equipped with a topology, then there is a topology on G induced by that on X. Again, the group multiplication (composition of mappings in this case) is automatically associative and continuous. This approach to the group theory is preferred in elementary handbooks for physicist.

The second approach is more abstract and more general. We start with a set G endowed with a topological structure and a binary operation. In this case one has to postulate, that the binary operation is associative and continuous, that there exists a

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neutral element and that any element of G has an inverse. All the theory is based on these axioms. For compact groups, after a long deduction one is able to prove the existence of finite-dimensional representations establishing in this way the equivalence of the two approaches.

The two approaches are also possible in the theory of quantum groups. In the papers [13, 18] the first approach is used. It is assumed that the C*-algebra of functions on G is generated by N^2-elements u_{kl} $(k, l = 1, 2, \ldots, N)$ organized in a $N \times N$-matrix u. In other words, G is a 'quantum space of $N \times N$-matrices'. The comultiplication is of standard form:

$$\Phi(u_{kl}) = \sum_{m=1}^{N} u_{km} \otimes u_{ml}.$$

It means that matrix multiplication is the group rule on G. The reader should notice, that one need not assume the coassociativity of Φ; it follows immediately from the above equation.

In this approach, from the very beginning we have at our disposal a large class of finite-dimensional representations (the fundamental representation u, its conjugate and their tensor products). This fact essentially simplifies the theory.

In the present paper we have chosen the second more ambitious approach. We start with a unital C*-algebra A. Elements of the algebra are interpreted as continuous functions on a quantum space G. The group structure on G is described by a C*-algebra homomorphism (comultiplication) Φ acting from A into $A \otimes A$. We do not assume any particular form of Φ, instead we demand Φ to be coassociative. The second axiom that is used corresponds to the cancellation low in the classical group theory. Then the main result is the existence of a rich set of finite-dimensional representations. To have the paper as selfconsistent as possible we reproduce most of the results obtained in [13]. Comparing the present paper with [13] one should stress one point. As it was indicated by T. Koornwinder in an unpublished manuscript, the theory of C*-algebras play in [13] rather decorative role. The whole theory could be easily formulated on the level of *-algebra $\mathcal{A}$ generated by $\{u_{kl} : k, l = 1, 2, \ldots, N\}$. Passing to the closure gives no essentially new results. On the other hand, in the present approach we are able to introduce the *-algebra $\mathcal{A}$ only after the great part of the theory is developed, so the C*-algebra language is inherent in our theory.

The theory of compact quantum groups is now well established. There is a common agreement on the basic concepts. The situation in the theory of non-compact quantum groups is rather unsatisfactory. We believe that the approach using the multiplicative unitaries [2, 20] is the most promising. More traditional approach will be presented in [6]. The essential defeat of the last paper consists in lack of the proof of the existence of the Haar measure. The authors were forced to include the existence of the Haar measure in one of the axioms.

A few words about the content of the paper. At the end of this section we recall the basic notions of the theory of C*-algebras that are used in the subsequent sections. In Section 1 one can find the definition of compact quantum group. Next we present the main results. Probably the most interesting is the theorem saying that the C*-

algebra of all ‘continuous functions on a compact quantum group contains a dense Hopf *-subalgebra. The other theorems concern the Haar measure and its modular properties. A special attention is paid to the groups with faithful Haar measure. In Section 2 we prove the existence of the Haar measure. Section 3 contains the theory of unitary representations of a compact quantum group. We consider representations acting on infinite-dimensional Hilbert spaces. The main result says that any representation is equivalent to a direct sum of irreducible representations and that any irreducible representation is finite-dimensional. We use this result in Section 4, where the right regular representation is constructed and investigated. Decomposing this representation into irreducible ones we obtain a rich set of finite-dimensional representations establishing in this way a link with the theory of compact matrix groups. In Section 5 we show, that the linear span of matrix elements of finite-dimensional unitary representations is a dense Hopf *-subalgebra. Section 6 is devoted to the modular properties of the Haar measure. We show that the modular group is determined by an analytic family of linear multiplicative functionals on the Hopf *-subalgebra. The same family enters the formula for the square of the coinverse. At the end of this section the Peter-Weyl orthonormality relations for matrix elements of irreducible representations are derived. The last section is devoted to the groups with faithful Haar measure.

Dealing with compact quantum groups we mainly work with unital C*-algebras. However the Pontryagin dual of a compact group is not compact and this fact introduces non-unital algebras into our considerations. For these algebras the concept of multiplier algebra [7, 17] is of great importance.

Let A be a C*-algebra and a and a' be linear bounded operators acting on the Banach space A. We say that a' is the adjoint of a if $a(b)^*c = b^*a(c)$ for any $b, c \in A$. The adjoint operator will be denoted by a^*. Its existence is a non-trivial requirement. By definition the multiplier algebra M(A) is the subalgebra of B(A) consisting of all operators on A that have the adjoint. Then M(A) is a unital C*-algebra. Each element $a \in A$ defines (by left multiplication) an operator on A. Identifying $a \in A$ with the left multiplication by a we embed $A \subset M(A)$. One can easily show that A is an essential ideal in $M(A)$. We shall use the following simple

Proposition 0.1 *Let A be a C*-algebra and $v : A \to A$ be a linear map such that $v(A) = A$ and $v(a)^*v(b) = a^*b$ for any $a, b \in A$. Then v is an unitary element of $M(A)$.*

Proof: It follows immediately from the assumptions, that v is an isometry mapping A onto itself and that v^{-1} is the adjoint of v. Therefore v has the adjoint and $v \in M(A)$. Clearly v is unitary.

Q.E.D.

For any Hilbert space K, $B(K)$ will denote the C^*-algebra of all bounded operators acting on K and $CB(K) \subset B(K)$ will denote the subalgebra consisting of all compact operators. It is well known, that $B(K) = M(CB(K))$.

If A and B are linear subsets of the same algebra, then AB will always denote the set of all linear combinations of elements $\{ab : a \in A,\ b \in B\}$.

For any Hilbert space K we denote by $C^*(K)$ the set of all separable C*-algebras of operators acting on K in a nondegenerate way: for any $A \in C^*(K)$, $0 \in K$ is the only vector killed by all $a \in A$. The tensor product of C*-algebras appearing in this paper is always the minimal tensor product: For any $A \in C^*(H)$ and $B \in C^*(K)$ (H and K are Hilbert spaces), $A \otimes B \in C^*(H \otimes B)$.

If $B \in C^*(K)$ then $M(B)$ may be identified with the set

$$M(B) = \left\{ a \in B(K) : \begin{array}{c} ab \in B \text{ and } ba \in B \\ \text{for any } a \in B \end{array} \right\}. \tag{0.1}$$

The elements of (0.1) act on B by left multiplication. More generally: let B be a C*-subalgebra of $M(A)$. We say that B is A-non-degenerate if the closure of AB coincides with A. If this is the case, then $M(B)$ may be identified with the set

$$M(B) = \left\{ a \in M(A) : \begin{array}{c} ab \in B \text{ and } ba \in B \\ \text{for any } a \in B \end{array} \right\}. \tag{0.2}$$

The elements of (0.2) act on B by left multiplication. In particular if $B \in C^*(K)$ and A is a C*-algebra, then $B \otimes A \subset B(K) \otimes A \subset M(CB(K) \otimes A)$, $B \otimes A$ is $(CB(K) \otimes A)$-non-degenerate and $M(A \otimes B)$ may be identified with the set

$$M(B \otimes A) = \left\{ a \in M(CB(K) \otimes A) : \begin{array}{c} ab, ba \in B \otimes A \\ \text{for any } a \in B \otimes A \end{array} \right\}. \tag{0.3}$$

The elements of (0.3) act on $B \otimes A$ by left multiplication.

Let K be a Hilbert space. $C^*(K)$ is equipped with a natural order relation. Let $A, B \in C^*(K)$. We say that A is smaller than B if AB is a dense subset of B:

$$A \leq B \Longleftrightarrow \begin{array}{c} \text{The closure of } AB \\ \text{coincides with } B. \end{array} \tag{0.4}$$

In other words, $A \leq B$ if $B \in M(A)$ and B is A-nondegenerate. One can easily show, that $(C^*(K), \leq)$ is a partially ordered set. The algebra $CB(K)$ of all compact operators acting on K is the largest element of $C^*(K)$ and $\mathbf{C}I$ is the smallest one. In general unital algebras are smaller than non-unital.

Passing to the multiplier algebra does not preserve the inclusion. If $A, B \in C^*(K)$ and $A \subset B$, then in general $M(A)$ is not a subset of $M(B)$. However

$$A \leq B \quad \Longrightarrow \quad M(A) \subset M(B)$$

If moreover C is another C* algebra, then

$$A \leq B \quad \Longrightarrow \quad M(A \otimes C) \subset M(B \otimes C),$$

where $M(A \otimes C)$ and $M(B \otimes C)$ are considered as subsets of $M(CB(K) \otimes C)$ (cf (0.3)). This nice behavior of the ordering in $C^*(K)$ is the reason, why it is more useful then the inclusion relation.

Let A and B be C*-algebras. A *-algebra homomorphism $\phi : A \to M(B)$ is called a morphism from A to B, if the image $\phi(A)$ is B-non-degenerate. The set of all morphisms from A to B will be denoted by $\mathrm{Mor}(A,B)$. Any $\phi \in \mathrm{Mor}(A,B)$ extends in a unique way to the *-algebra homomorphism from $M(A)$ into $M(B)$. Due to this fact, the morphisms may be composed. One may also consider the tensor product of morphisms: For any $\phi \in \mathrm{Mor}(A,B)$ and $\phi' \in \mathrm{Mor}(A',B')$ (A, B, A' and B' are C*-algebras) there exists unique $\phi\otimes\phi' \in \mathrm{Mor}(A\otimes A', B\otimes B')$ such that $(\phi\otimes\phi')(a\otimes b) = \phi(a)\otimes\phi'(b)$ for any $a\in A$ and $b \in B$. For further details see [17].

1 Definitions and results

We shall use symbol Φ for comultiplication reserving Δ for the modular operator of Tomita–Takesaki theory.

Definition 1.1 *Let $G = (A, \Phi)$, where A is a separable unital C^*-algebra and $\Phi : A \longrightarrow A\otimes A$ is a unital *-algebra homomorphism. We say that G is a compact quantum group if*

1. The diagram

$$\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d, "\Phi"'] & A\otimes A \arrow[d, "\Phi\otimes\mathrm{id}"] \\ A\otimes A \arrow[r, "\mathrm{id}\otimes\Phi"'] & A\otimes A\otimes A \end{tikzcd}$$

is commutative,

2. The sets

$$\{(b\otimes I)\Phi(c) \ : \ b, c\in A\}, \tag{1.1}$$

$$\{(I\otimes b)\Phi(c) \ : \ b, c\in A\} \tag{1.2}$$

are linearly dense subsets of $A\otimes A$.

Remark 1. Let $G = (A, \Phi)$ be a compact quantum group, $s_A : A\otimes A \to A\otimes A$ be the flip: $s_A(a\otimes b) = b\otimes a$ for all $a, b\in A$ and $\Phi^{\mathrm{opp}} = s_A \circ \Phi$. Then one can easily verify that $G^{\mathrm{opp}} = (A, \Phi^{\mathrm{opp}})$ is a compact quantum group. We say that G^{opp} is the group opposite to G.

Remark 2. If (A, u) is a compact matrix quantum group (pseudogroup) in the sense of [13, 18] and Φ is the corresponding comultiplication, then due to the Theorem 4.9 of [13], (A, Φ) is a compact quantum group in the sense of the above definition.

Remark 3. If in Definition 1.1 the algebra A is commutative then (Gelfand – Naimark theory [5]) $A = C(\Lambda)$, where Λ is a compact space and for any $a \in C(\Lambda)$ and λ_1, $\lambda_2 \in \Lambda$ we have $(\Phi a)(\lambda_1, \lambda_2) = a(\lambda_1\cdot\lambda_2)$, where

$$\Lambda\times\Lambda \ni (\lambda_1, \lambda_2) \mapsto \lambda_1\cdot\lambda_2 \in \Lambda \tag{1.3}$$

is a continuous mapping. In this case Condition 1 of Definition 1.1 means that

$$(\lambda_1 \cdot \lambda_2) \cdot \lambda_3 = \lambda_1 \cdot (\lambda_2 \cdot \lambda_3) \tag{1.4}$$

for any λ_1, λ_2, $\lambda_3 \in \Lambda$. One can easily verify that the density of (1.1) is equivalent to the following implication:

$$(\lambda \cdot \lambda_1 = \lambda \cdot \lambda_2) \Longrightarrow (\lambda_1 = \lambda_2) \tag{1.5}$$

for any λ, λ_1, $\lambda_2 \in \Lambda$. Similarly the density of (1.2) is equivalent to the implication:

$$(\lambda_1 \cdot \lambda = \lambda_2 \cdot \lambda) \Longrightarrow (\lambda_1 = \lambda_2) \tag{1.6}$$

for any λ, λ_1, $\lambda_2 \in \Lambda$. The results of the present paper show that any compact space Λ endowed with a continuous binary operation (1.3) satisfying conditions (1.4) – (1.6) is a topological group.

Let $G = (A, \Phi)$ be a compact quantum group. Applying the hermitian conjugation to the elements of (1.1) and (1.2) we show that

$$\{\Phi(c)(b \otimes I) \ : \ b, c \in A\}, \tag{1.7}$$

$$\{\Phi(c)(I \otimes b) \ : \ b, c \in A\} \tag{1.8}$$

are linearly dense subsets of $A \otimes A$.

Let $G = (A, \Phi)$ be a compact quantum group and $v = (v_{kl})_{k,l=1,2,\dots,N}$ be an $N \times N$ matrix with entries belonging to A. We recall [13] that v is an N–dimensional unitary representation of G if v is a unitary element of $M_N(A) = M_N(\mathbf{C}) \otimes A$ and

$$\Phi(v_{kl}) = \sum_r v_{kr} \otimes v_{rl}$$

for all $k, l = 1, 2, \dots, N$. More general notion of unitary representation of G is introduced in Section 3, where infinite–dimensional representations are also considered.

Let $\mathcal{A}$ be a unital *-algebra and $\Phi : \mathcal{A} \to \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$ be a unital *-algebra homomorphism such that $(\Phi \otimes \mathrm{id})\Phi = (\mathrm{id} \otimes \Phi)\Phi$ (coassociativity). We recall that $(\mathcal{A}, \Phi)$ is a Hopf *-algebra if there exist linear mappings $e : \mathcal{A} \to \mathbf{C}$ and $\kappa : \mathcal{A} \to \mathcal{A}$ such that

$$(e \otimes \mathrm{id})\Phi(a) = a \tag{1.9}$$

$$(\mathrm{id} \otimes e)\Phi(a) = a \tag{1.10}$$

$$m(\kappa \otimes \mathrm{id})\Phi(a) = e(a)I \tag{1.11}$$

$$m(\mathrm{id} \otimes \kappa)\Phi(a) = e(a)I \tag{1.12}$$

for any $a \in \mathcal{A}$. In the above formulae m denotes the multiplication map $m : \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A} \to \mathcal{A}$ i.e. the linear map such that $m(a \otimes b) = ab$ for any $a, b \in \mathcal{A}$.

It is known that e (called counit) and κ (called coinverse or antipode) are uniquely determined. e is a unital *-algebra homomorphism. κ is antimultiplicative, anticomultiplicative and

$$\kappa\left(\kappa(a^*)^*\right) = a$$

for any $a \in \mathcal{A}$.

The main result of this paper is contained in the following theorem.

Theorem 1.2 *Let $G = (A, \Phi)$ be a compact quantum group and $\mathcal{A}$ be the set of all linear combinations of matrix elements of all finite-dimensional unitary representations of G. Then $\mathcal{A}$ is a dense *-subalgebra of A and $\Phi(\mathcal{A}) \subset \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$. Moreover $(\mathcal{A}, \Phi|_{\mathcal{A}})$ is a Hopf *-algebra.*

We shall prove this theorem in Section 5. To this and we shall use the right regular representation of G introduced in Section 4.. The latter notion is in turn closely related to the Haar measure.

Let $G = (A, \Phi)$ be a compact quantum group, A' be the set of all continuous linear functionals defined on A, ξ, $\xi' \in A'$ and $a \in A$. We shall use the convolution products (cf [12, 13]):

$$\xi * a = (\mathrm{id} \otimes \xi)\Phi(a) \in A \tag{1.13}$$

$$a * \xi' = (\xi' \otimes \mathrm{id})\Phi(a) \in A \tag{1.14}$$

$$\xi' * \xi = (\xi' \otimes \xi)\Phi \in A \tag{1.15}$$

Due to the condition 1 of Definition 1.1, the convolution product is associative:

$$\begin{aligned} (a * \xi') * \xi'' &= a * (\xi' * \xi'') \\ (\xi * a) * \xi' &= \xi * (a * \xi') \\ (\xi'' * \xi) * a &= \xi'' * (\xi * a) \\ (\xi * \xi') * \xi'' &= \xi * (\xi' * \xi'') \end{aligned}$$

for any ξ, ξ', $\xi'' \in A'$ and $a \in A$. Moreover

$$(\xi' * \xi)(a) = \xi(a * \xi') = \xi'(\xi * a) \tag{1.16}$$

for any ξ, $\xi' \in A'$ and $a \in A$.

Taking into account the inclusion $\Phi(\mathcal{A}) \subset \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A}$ we see that the right-hand-sides of (1.13) – (1.15) are meaningful for any linear functionals ξ, ξ' defined on $\mathcal{A}$ and any $a \in \mathcal{A}$. In this case $\xi * a$, $a * \xi' \in \mathcal{A}$ and $\xi' * \xi$ is a linear functional defined on $\mathcal{A}$.

In the next Section we shall prove the following

Theorem 1.3 *Let $G = (A, \Phi)$ be a compact quantum group. Then there exists unique state (positive normalized linear functional) h on A such that*

$$a * h = h * a = h(a)I \tag{1.17}$$

for any $a \in A$. In what follows h is called the Haar measure.

Using the two above theorems one can easily reproduce all the essential results of [13, 14, 15] and [8] – Sections 2, 3 and 4. The only difference between the theory of compact quantum matrix groups developed in these papers and the theory of compact quantum groups based on Definition 1.1 lies in the fact that in the latter theory we do not distinguish any particular finite-dimensional (so called fundamental) representation. Consequently in all statements of [13] the phrase '$\mathcal{A}$ is the *-subalgebra generated by matrix elements of the fundamental representation' should be replaced by '$\mathcal{A}$ is the *-subalgebra introduced in Theorem 1.2'. In particular we have

Theorem 1.4 *Let $G = (A, \Phi)$ be a compact quantum group, h be the Haar measure on G, $\mathcal{A}$ be the dense Hopf *-algebra related to G via Theorem 1.2 and e and κ be the counit and coinverse on $\mathcal{A}$. Then there exists one and only one family $(f_z)_{z \in \mathbf{C}}$ of linear multiplicative functionals defined on $\mathcal{A}$ such that*

1. $f_z(I) = 1$ $z \in \mathbf{C}$.
2. *For any $a \in \mathcal{A}$, the mapping*

$$\mathbf{C} \ni z \mapsto f_z(a) \in \mathbf{C}$$

is an entire holomorphic function.

3. $f_{z_1} * f_{z_2} = f_{z_1+z_2}$ *for any $z_1, z_2 \in \mathbf{C}$. Moreover $f_0 = e$.*
4. *For any $z \in \mathbf{C}$ and any $a \in \mathcal{A}$,*

$$f_z(\kappa(a)) = f_{-z}(a), \tag{1.18}$$

$$f_{\overline{z}}(a^*) = \overline{f_{-z}(a)}. \tag{1.19}$$

In particular for purely imaginary z, f_z is a *-character defined on $\mathcal{A}$.

5. *For any $a \in \mathcal{A}$,*

$$\kappa^2(a) = f_{-1} * a * f_1. \tag{1.20}$$

6. *The formula*

$$\sigma_t(a) = f_{it} * a * f_{it}$$

defines a one parameter group $\sigma = (\sigma_t)_{t \in \mathbf{R}}$ of modular automorphisms of $\mathcal{A}$. h is a σ-KMS state: for any $a \in \mathcal{A}$ and $b \in A$,

$$h(ab) = h(b(f_1 * a * f_1)). \tag{1.21}$$

The proof of this Theorem will be given in Section 6. Let us also notice the following result implicitly contained in [13] and [8] (Section 3):

Theorem 1.5 *With the notation introduced in Theorem 1.4, the following conditions are equivalent*

1. $f_z = e$ *for all $z \in \mathbf{C}$.*
2. *h is central.*
3. $\kappa^2 = \mathrm{id}$.
4. *The Pontryagin dual $\hat{G}$ of G is unimodular.*
5. *The left Haar measure on $\hat{G}$ is central.*
6. *The right Haar measure on $\hat{G}$ is central.*

For each $a \in A$ we shall denote by ha (ah respectively) the linear functional on A such that $(ha)(b) = h(ab)$ ($(ah)(b) = h(ba)$ respectively) for any $b \in a$.

Like in [13], in the theory based on Definition 1.1, the Haar measure h need not to be faithful (We know only that the restriction of h to the dense *-subalgebra $\mathcal{A}$ is faithful (cf Proposition 3.2). We believe that this (in a sense pathological) possibility will be removed in the future, more satisfactory theory of compact quantum groups based on a set of axioms stronger than the one used in Definition 1.1. From this point of view the following result is very interesting.

Theorem 1.6 *Let $G = (A, \Phi)$ be a compact quantum group, $(\mathcal{A}, \Phi)$ be the Hopf *-algebra related to G via Theorem 1.2 and κ be the coinverse acting on $\mathcal{A}$. Assume that the Haar measure on G is faithful. Then*

1. There exist closed operators r, s acting on $A \otimes A$ such that $A \otimes_{\mathrm{alg}} A$ is a core for r and s and

$$\begin{aligned} r(a \otimes b) &= (a \otimes I)\Phi(b), \\ s(a \otimes b) &= (I \otimes a)\Phi(b) \end{aligned}$$

for all a, $b \in A$. Moreover $\ker r = \ker s = \{0\}$.

2. $\mathcal{A} = \{a \in A : \; \Phi(a) \in A \otimes_{\mathrm{alg}} A\}$.

3. There exists one parameter group $(\sigma_t)_{t\in\mathbf{R}}$ of *-automorphisms of the C^*-algebra A such that the Haar measure h is a σ-KMS state (cf [4]).

4. The coinverse κ (considered as a linear operator acting on A) is closeable and its closure $\overline{\kappa}$ admits the following polar decomposition:

$$\overline{\kappa} = R \circ \tau_{i/2} \tag{1.22}$$

where $\tau_{i/2}$ is the analytic generator of a one parameter group $(\tau_t)_{t\in\mathbf{R}}$ of *-automorphisms of the C^*-algebra A and R is a linear antimultiplicative, commuting with the hermitian conjugation, norm preserving involution acting on A such that $\tau_t \circ R = R \circ \tau_t$ for all $t \in \mathbf{R}$. In particular $D(\overline{\kappa}) = D(\tau_{i/2})$. Moreover $b * ha$, $bh * a \in D(\overline{\kappa})$, $\overline{\kappa}(b * ha) = a * bh$ and $\overline{\kappa}(bh * a) = ha * b$ for all $a, b \in A$.

Let us recall that the analytical generator $\tau_{i/2}$ of a (pointwise continuous) one parameter group $(\tau_t)_{t\in\mathbf{R}}$ of *-automorphisms of a C^*-algebra A is the linear operator acting on A in the following way:

For any $a, b \in A$: $a \in D(\tau_{i/2})$ and $b = \tau_{i/2}(a)$ if and only if there exists a mapping $z \mapsto a_z \in A$ continuous on the strip $\{z \in \mathbf{C} : \; \Im z \in [0, 1/2]\}$ and holomorphic on the interior of this strip such that $a_t = \tau_t(a)$ for all $t \in \mathbf{R}$ and $a_i = b$.

It is known that $\tau_{i/2}$ is a closed linear mapping, $D(\tau_{i/2})$ is a dense subalgebra and $\tau_{i/2}$ is multiplicative. Moreover $\tau_{i/2}(a)^* \in D(\tau_{i/2})$ and $\tau_{i/2}(\tau_{i/2}(a)^*)^* = a$ for any $a \in D(\tau_{i/2})$.

2 The Haar measure

In this section we prove Theorem 1.3. We start with the following

Lemma 2.1 *Let $G = (A, \Phi)$ be a compact quantum group and h, ρ be states on A. Assume that ρ is faithful and*

$$h * \rho = \rho * h = h \tag{2.1}$$

Then h is the Haar measure.

Proof: Let

$$L_{h\otimes\rho} = \{q \in A \otimes A : \ (h \otimes \rho)(q^*q) = 0\}$$

be the left ideal related to the state $h \otimes \rho$. For any $c \in A$ we set

$$\Psi_L(c) = h * c - h(c)I \tag{2.2}$$

Clearly Ψ_L is a linear mapping acting on A. By definition Ψ_L is completely bounded (Ψ_L is the difference of two completely positive mappings). We shall prove that

$$(\mathrm{id} \otimes \Psi_L)\Phi(c) \in L_{h\otimes\rho} \tag{2.3}$$

for any $c \in A$. Indeed, denoting the above element by q we have:

$$\begin{aligned} q &= (\mathrm{id} \otimes \Psi_L)\Phi(c) \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes h)(\mathrm{id} \otimes \Phi)\Phi(c) - (\mathrm{id} \otimes h)\Phi(c) \otimes I \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes h)(\Phi \otimes \mathrm{id})\Phi(c) - (\mathrm{id} \otimes h)\Phi(c) \otimes I \\ &= \Phi(h * c) - (h * c) \otimes I. \end{aligned}$$

Therefore

$$q^*q = \Phi((h*c)^*(h*c)) - \Phi(h*c)^*[(h*c) \otimes I] - [(h*c)^* \otimes I]\Phi(h*c) + [(h*c)^*(h*c)] \otimes I$$

and

$$(h \otimes \rho)(q^*q) = \mathrm{I} - \mathrm{II} - \mathrm{III} + \mathrm{IV},$$

where

$$\begin{aligned} \mathrm{I} &= (h \otimes \rho)\Phi((h*c)^*(h*c)) = (h * \rho)((h*c)^*(h*c)), \\ \mathrm{II} &= \mathrm{III}^*, \\ \mathrm{III} &= (h \otimes \rho)\left\{[(h*c)^* \otimes I]\, \Phi(h*c)\right\} = h((h*c)^*(\rho * h * c)), \\ \mathrm{IV} &= (h \otimes \rho)((h*c)^*(h*c) \otimes I) = h((h*c)^*(h*c)). \end{aligned}$$

Now, using (2.1) we get $\mathrm{I} = \mathrm{II} = \mathrm{III} = \mathrm{IV}$, $(h \otimes \rho)(q^*q) = 0$ and (2.3) follows.

Let $a \in A$. Using the density of (1.1), for any $\epsilon > 0$, one can find $b_1, b_2, \ldots, b_n$, c_1, $c_2, \ldots, c_n \in A$ such that

$$\|I \otimes a - \sum (b_k \otimes I)\Phi(c_k)\| \leq \epsilon$$

Remembering that Ψ_L is completely bounded (with the bound smaller or equal 2) we have

$$\|I \otimes \Psi_L(a) - \sum (b_k \otimes I)(\mathrm{id} \otimes \Psi_L)\Phi(c_k)\| \leq 2\epsilon$$

Using (2.3) and remembering that $L_{h\otimes\rho}$ is a closed left ideal we get $I \otimes \Psi_L(a) \in L_{h\otimes\rho}$. It means that

$$(h \otimes \rho)(I \otimes \Psi_L(a)^* \Psi_L(a)) = \rho(\Psi_L(a)^* \Psi_L(a)) = 0$$

We assumed that ρ is faithful. Therefore $\Psi_L(a) = 0$ and (cf (2.2)) $h * a = h(a)I$ for any $a \in A$.

In the similar way introducing the completely bounded mapping $\Psi_R : A \to A$ by the formula

$$\Psi_R(c) = c * h - h(c)I$$

one can show that $(\Psi_R \otimes \mathrm{id})\Phi(c) \in L_{\rho\otimes h}$. Then using the density of (1.2) one obtains $\Psi_R(a) = 0$ for any $a \in A$. The latter means that $a * h = h(a)I$. Combining the two results we see that h is a Haar measure

Q.E.D.

Proof of Theorem 1.3. Let ρ be a faithful state on A (the existence of such a state follows from the separability of A),

$$\rho^{*n} = \rho * \rho * \ldots * \rho$$

be the convolution product of n-copies of ρ,

$$h_n = \frac{1}{n} \sum_{k=1}^{n} \rho^{*k}$$

be the Cesaro sum and h be a weak accumulation point of the sequence $(h_n)_{n=1,2,\ldots}$ (the set of states of any unital C^*-algebra is compact with respect to the weak topology, so the accumulation point always exists).

One can easily verify that

$$h_n * \rho = \rho * h_n = h_n + \frac{1}{n}(\rho^{*(n+1)} - \rho).$$

Therefore $h * \rho = \rho * h = h$ and (cf Lemma 2.1) h is a Haar measure.

To prove the uniqueness assume that h and h' are Haar measures. Then (cf (1.16)) for any $a \in A$ we have

$$h'(a) = h(h'(a)I) = h(h' * a) = h'(a * h) = h'(h(a)I) = h(a).$$

Q.E.D.

Remark: In this way we showed that the sequence $(h_n)_{n=1,2,\ldots}$ has only one accumulation point. Therefore (the state space is weakly compact) h_n converges weakly to h. In other words

$$h = C - \lim_{n\to\infty} \rho^{*n}$$

where $C-\lim$ denotes the Cesaro weak limit (cf formula (4.18) of [13]).

Let $a \in A$. Relation (1.17) means that

$$(\mathrm{id} \otimes h)\Phi(a) = h(a)I, \tag{2.4}$$

$$(h \otimes \mathrm{id})\Phi(a) = h(a)I. \tag{2.5}$$

3 Unitary representations

Let $G = (A, \Phi)$ be a compact quantum group and K be a Hilbert space. We use the notation introduced in Section 0. We recall ([11], [8] Remark on page 397, [19]) that v is a (strongly continuous) unitary representations of G acting on K if v is a unitary element of $M(CB(K) \otimes A)$ such that

$$(\mathrm{id} \otimes \Phi)v = v_{12}v_{13}. \tag{3.1}$$

The leg numbering notation used in the above formula is explained in [8] page 385.

We shall prove the following

Theorem 3.1 *Let v be a unitary representation of a compact quantum group $G = (A, \Phi)$, h be the Haar measure on G, K be the carrier Hilbert space of v and $B \subset B(K)$ be the norm closure of the set of all operators of the form (cf [8, 18])*

$$\mathcal{F}_v(a) = (\mathrm{id} \otimes ha)v^*, \tag{3.2}$$

where $a \in A$. Then

1. $B \in C^*(K)$

2. $v \in M(B \otimes A)$

This relation should be understood in the sense of (0.3).

3. B is the smallest (in the sense of the ordering introduced by (0.4)) element of $C^*(K)$ such that the Statement 2 holds.

4. An operator $X \in B(K)$ intertwines v with itself if and only if X commutes with all elements of B.

Proof:

Ad 1. According to (2.4)

$$(\mathrm{id} \otimes \mathrm{id} \otimes h)(\mathrm{id} \otimes \Phi)[(I \otimes a)v^*] = (\mathrm{id} \otimes h)[(I \otimes a)v^*] \otimes I$$

for any $a \in A$. The right-hand-side equals to $\mathcal{F}_v(a) \otimes I$. To compute the left-hand-side we use the multiplicativity of Φ and formula (3.1). We get

$$(\mathrm{id} \otimes \mathrm{id} \otimes h)[(I \otimes \Phi(a))v_{13}{}^*]v^* = \mathcal{F}_v(a) \otimes I$$

and (v is unitary)

$$(\mathrm{id} \otimes \mathrm{id} \otimes h)[(I \otimes \Phi(a))v_{13}{}^*] = (\mathcal{F}_v(a) \otimes I)v \tag{3.3}$$

Let $b \in A$. Then $(b^*h \otimes \mathrm{id})\Phi(a) = a * b^*h$, $(\mathrm{id} \otimes b^*h)v = \mathcal{F}_v(b)^*$ and applying $(\mathrm{id} \otimes b^*h)$ to the both sides of (3.3) we obtain

$$\mathcal{F}_v(a * b^*h) = \mathcal{F}_v(a)\mathcal{F}_v(b)^* \tag{3.4}$$

Let

$$C = BB^* = \begin{array}{c}\text{the linear envelope}\\ \text{of } \{mn^* : \ m, n \in B\}\end{array}$$

Remembering that (1.2) is dense in $A \otimes A$ one can easily show that the set of linear combinations of elements of the form $a * b^*h$ (where $a, b \in A$) is dense in A and using (3.4) we see that C is a dense subset of B. Clearly C is invariant under hermitian conjugation, so is B: $B^* = B$. Moreover $BB = BB^* = C \subset B$. It shows that B is a *-subalgebra of $B(K)$.

To end this part of the proof we have to show that the embedding $B \hookrightarrow B(K)$ is non-degenerate, i.e. that

$$\bigcap_{a \in A} \ker \mathcal{F}_v(a) = \{0\}. \tag{3.5}$$

Let $(\psi_n)_{n=1,2,\ldots}$ be an orthonormal basis in K. Then (using Dirac notation) for any compact operator $x \in CB(K)$ we have the norm convergence

$$\left[\sum_{n=1}^{N} |\psi_n)(\psi_n|\right] x \longrightarrow x$$

for $N \to \infty$. Using this fact one can easily show that the sequence

$$v\left[\sum_{n=1}^{N} |\psi_n)(\psi_n| \otimes I\right] v^*$$

converges (for $N \to \infty$) to $I \in M(CB(K) \otimes A)$ in the sense of almost uniform topology. Therefore for any $\varphi \in K$

$$\sum_{n=1}^{\infty} (\varphi|(\mathrm{id} \otimes h)\,[v(|\psi_n)(\psi_n| \otimes I)v^*]\,\varphi) = (\varphi|\varphi). \tag{3.6}$$

Let $\rho_{n\varphi}$ be the linear functional on $CB(K)$ introduced by the formula $\rho_{n\varphi}(m) = (\varphi|m\psi_n)$ ($m \in CB(K)$) and $a_{n\varphi} = (\rho_{n\varphi} \otimes \mathrm{id})v$. Then using (3.2) we may rewrite (3.6) in the following way:

$$\sum_{n=1}^{\infty} (\psi_n|\mathcal{F}_v(a_{n\varphi})\varphi) = (\varphi|\varphi)$$

and (3.5) follows.

Ad 2. Let s_A be the flip automorphism acting on $A \otimes A$: $s_A(a \otimes b) = b \otimes a$ for all $a, b \in A$. One can easily check that the mapping $\mathcal{F}_v : A \to B$ introduced by (3.2) is completely bounded. Therefore we may consider the tensor product map

$$\mathcal{F}_v \otimes \mathrm{id} : \; A \otimes A \longrightarrow B \otimes A. \tag{3.7}$$

This mapping is continuous and its range is dense in $B \otimes A$. For any $a, b \in A$ we have

$$(\mathcal{F}_v \otimes \mathrm{id}) s_A(a \otimes b) = \mathcal{F}_v(b) \otimes a = (\mathrm{id} \otimes \mathrm{id} \otimes h) \left[(I \otimes a \otimes b) v_{13}{}^* \right].$$

Comparing this formula with (3.3) we get

$$(\mathcal{F}_v \otimes \mathrm{id}) s_A \Phi(a) = (\mathcal{F}_v(a) \otimes I) v$$

for any $a \in A$. Therefore

$$(\mathcal{F}_v \otimes \mathrm{id}) s_A \left[(b \otimes I) \Phi(a) \right] = (\mathcal{F}_v(a) \otimes b) v$$

for any $a, b \in A$. This formula shows that $(B \otimes A)v \subset B \otimes A$ for any $p \in B \otimes A$. Moreover using the density of (1.1) and the properties of (3.7) we see that $(B \otimes A)v$ is dense in $B \otimes A$. Therefore the right multiplication by v maps $B \otimes A$ onto itself. So does the inverse map: $(B \otimes A)v^* = B \otimes A$. Passing to the hermitian conjugate elements we get $v(B \otimes A) = B \otimes A$. Now Statement 2 follows from (0.3).

Ad 3. Let B_1 be a C^*-algebra of operators acting on K such that $v \in M(B_1 \otimes A)$. Then

$$(m \otimes a)v^* \in B_1 \otimes A \tag{3.8}$$

for any $m \in B_1$ and $a \in A$. Moreover the set of all linear combinations of elements of the above form is dense in $B_1 \otimes A$ $(= (B_1 \otimes A)v^*)$. Applying $(\mathrm{id} \otimes h)$ to (3.8) we see that

$$m\mathcal{F}_v(a) \in B_1$$

for any $m \in B_1$ and $a \in A$ and the set of linear combinations of elements of this form is dense in B_1. Therefore $B_1 B$ is a dense subset of B_1 and relation $B \leq B_1$ follows (cf definition (0.4).

Ad 4. If $m \in B(K)$ intertwines v with itself then $(m \otimes I)v^* = v^*(m \otimes I)$ and applying $(\mathrm{id} \otimes ha)$ to the both sides of this relation we get $m\mathcal{F}_v(a) = \mathcal{F}_v(a)m$. It shows that m belongs to the commutant of B. The converse follows immediately from Statement 2.

Q.E.D.

We shall prove that the Haar measure restricted to the matrix elements of a unitary representation is faithful.

Proposition 3.2 *Let $G = (A, \Phi)$ be a compact quantum group, h be the Haar measure, v be a unitary representation of G acting on a Hilbert space K, ρ be a linear functional on $CB(K)$ and $b = (\rho \otimes \mathrm{id})v^*$. Assume that $h(b^*b) = 0$. Then $b = 0$.*

Proof: We use the notation introduced in Theorem 3.1. By the Schwarz inequality $|h(ab)|^2 \leq h(aa^*)h(b^*b) = 0$ and $\rho(\mathcal{F}_v(a)) = h(ab) = 0$ for any $a \in A$. It shows that ρ kills all elements of the algebra B. Therefore $(\rho \otimes \mathrm{id})q = 0$ for any $q \in B \otimes A$. By the continuity argument the same relation holds for any $q \in M(B \otimes A)$ and (cf Theorem 3.1.2) $b = (\rho \otimes \mathrm{id})v^* = 0$.

Q.E.D.

Using the same technic we shall prove

Proposition 3.3 *Let $v \in M(CB(K) \otimes A)$ be a unitary representation of G acting on a Hilbert space K and ρ be a linear functional on $CB(K)$. Then*

$$\Big((\rho \otimes \mathrm{id})v^* = 0\Big) \Longleftrightarrow \Big((\rho \otimes \mathrm{id})v = 0\Big)$$

Proof: Assume that $(\rho \otimes \mathrm{id})v^* = 0$. Then $\rho(\mathcal{F}_v(a)) = (ha)[(\rho \otimes \mathrm{id})v^*] = 0$. Therefore (cf the previous proof) $(\rho \otimes \mathrm{id})q = 0$ for any $q \in M(B \otimes A)$. Setting $q = v$ we see that $(\rho \otimes \mathrm{id})v = 0$. This way the '$\Rightarrow$' part of our statement is proved. To prove the converse implication it is sufficient to replace ρ by ρ^*.

Q.E.D.

It is well known that any finite-dimensional representation of a C^*-algebra decomposes into a direct sum of irreducible representations. Using Theorem 3.1.4 we conclude that any finite-dimensional unitary representation of a compact quantum group is a direct sum of irreducible representations. In fact, like in the classical theory of compact groups we have much stronger result:

Theorem 3.4 *Let v be a unitary representation of a compact quantum group $G = (A, \Phi)$ acting on a Hilbert space of any dimension. Then v is a direct sum of finite-dimensional irreducible representations.*

Proof: By the remark preceding the text of the theorem it is sufficient to show that v is a direct sum of finite-dimensional representations. The latter statement reduces easily to the following one

$$\left.\begin{array}{l}\text{For any unitary representation } v \text{ of } G \text{ acting on a Hilbert space} \\ K \text{ there exists a non-zero finite-dimensional orthogonal projection} \\ P \in B(K) \text{ intertwining } v \text{ with itself.}\end{array}\right\} \qquad (3.9)$$

Indeed then PK is a finite-dimensional subspace of K, $K = PK \oplus (I-P)K$, PK and $(I-P)K$ are v-invariant and $v = v_1 \oplus v'$, where v_1 and v' are restrictions of v to PK and $(I-P)K$ respectively. Using the transfinite induction we obtain the desired decomposition.

Let $\psi \in K$ and

$$Q = (\mathrm{id} \otimes h)\,\{v\,[|\psi)(\psi| \otimes I]\,v^*\}$$

Using (3.1) and (2.4) we compute

$$\begin{aligned} v(Q \otimes I)v^* &= (\mathrm{id} \otimes \mathrm{id} \otimes h)\,\{v_{12}v_{13}\,[|\psi)(\psi| \otimes I \otimes I]\,v_{13}^*v_{12}^*\} \\ &= (\mathrm{id} \otimes \mathrm{id} \otimes h)(\mathrm{id} \otimes \Phi)\,\{v\,[|\psi)(\psi| \otimes I]\,v^*\} \\ &= (\mathrm{id} \otimes h)\,\{v\,[|\psi)(\psi| \otimes I]\,v^*\} \otimes I \\ &= Q \otimes I. \end{aligned}$$

It shows that Q intertwines v with itself. Clearly $Q \geq 0$. Remembering that $v \in M(CB(K) \otimes A)$ we get $v\,[|\psi)(\psi| \otimes I]\,v^* \in CB(K) \otimes A$ and $Q \in CB(K)$. According to (3.6) there exists ψ such that $Q \neq 0$.

This way we showed that there exists a non-zero positive compact operator Q intertwining v with itself. Let P be the spectral projection of Q corresponding to a strictly positive eigenvalue. Then $P \neq 0$, P is finite-dimensional, P intertwines v with itself and (3.9) follows.

Q.E.D.

We end this Section with the following

Proposition 3.5 *Let v be a unitary representation of a compact quantum group $G = (A, \Phi)$ acting on a Hilbert space K and h be the Haar measure on G. Then $P = (\mathrm{id} \otimes h)v$ is the orthogonal projection onto the subspace of all v-invariant vectors (A vector $x \in K$ is said to be v-invariant if $[(\mathrm{id} \otimes \xi)v]x = \xi(I)x$ for any $\xi \in A'$).*

Proof: According to Theorem 3.4 we may assume that v is irreducible. Applying $\mathrm{id} \otimes \mathrm{id} \otimes h$ to the both sides of (3.1) we see that $v(P \otimes I) = P \otimes I$. It shows that the range of P consists of v-invariant elements. If v is a nontrivial irreducible representation then K contains no v-invariant element and $P = 0$. Otherwise $\dim K = 1$, $v = I_{B(K)} \otimes I$ and $P = I_{B(K)}$. In both cases our statement holds.

Q.E.D.

4 Right regular representation

Let $G = (A, \Phi)$ be a compact quantum group, h be the Haar measure, π be the representation of the C^*-algebra A obtained by the GNS construction applied to the state h, H be the carrier Hilbert space of π and $\Omega \in H$ be the cyclic vector. Then

$$\{\pi(a)\Omega : \; a \in A\} \tag{4.1}$$

is dense in H and

$$h(a) = (\Omega|\pi(a)\Omega) \tag{4.2}$$

for any $a \in A$. Let A' ($CB(H)'$ respectively) be the set of all continuous linear functionals defined on A (on $CB(H)$ respectively).

The right regular representation of G is the unitary representation u introduced in the following

Theorem 4.1 *With the notation introduced above*

1. *There exists unique $u \in M(CB(H) \otimes A)$ such that*

$$[(\mathrm{id} \otimes \xi)u]\,\pi(a)\Omega = \pi(\xi * a)\Omega. \tag{4.3}$$

2. *u is a unitary representation of G.*

3. The set

$$\{(\rho\otimes \mathrm{id})u:\ \rho\in CB(H)'\} \tag{4.4}$$

is dense in A.

Proof:

Ad 1. Using the density of (4.1) one can easily show that the linear envelope of all elements of the form

$$\pi(a)|\Omega)(x|\otimes c, \tag{4.5}$$

where $a, c\in A$, $x\in H$ is a dense subset of $CB(H)\otimes A$.

We know that $\pi\in \mathrm{Mor}(A, CB(H))$. Hence $(\pi\otimes\mathrm{id})\in\mathrm{Mor}(A\otimes A, CB(H)\otimes A)$, $(\pi\otimes\mathrm{id})\Phi\in\mathrm{Mor}(A, CB(H)\otimes A)$ and $(\pi\otimes\mathrm{id})\Phi(a)\in M(CB(H)\otimes A)$ for all $a\in A$. On the other hand $|\Omega)(x|\otimes c\in CB(H)\otimes A$ for all $x\in H$ and $c\in A$. Therefore

$$[(\pi\otimes\mathrm{id})\Phi(a)]\,[|\Omega)(x|\otimes c]\in CB(H)\otimes A$$

for all $a, c\in A$ and $x\in H$. Using the density of (1.8) in $A\otimes A$ one can show that the linear envelope of the above elements is dense in $CB(H)\otimes A$.

Let us notice that using (4.2), (2.5) and again (4.2) we obtain

$$\begin{array}{l}\{[(\pi\otimes\mathrm{id})\Phi(a)]\,[|\Omega)(x|\otimes c]\}^*\,\{[(\pi\otimes\mathrm{id})\Phi(a')]\,[|\Omega)(x'|\otimes c']\}\\[2ex]
\begin{aligned}
&= [|x)(\Omega|\otimes c^*]\,(\pi\otimes\mathrm{id})\Phi(a^*a')\,[|\Omega)(x'|\otimes c']\\
&= |x)(x'|\otimes c^*(h\otimes\mathrm{id})\Phi(a^*a')c'\\
&= |x)(x'|\otimes c^*h(a^*a')c'\\
&= |x)(x'|\otimes c^*(\Omega|\pi(a^*a')\Omega)c'\\
&= \{\pi(a)|\Omega)(x|\otimes c\}^*\,\{\pi(a')|\Omega)(x'|\otimes c'\}
\end{aligned}\end{array}$$

Now we use Proposition 0.1. It shows that there exists a unitary $u\in M(CB(H)\otimes A)$ such that

$$u\,[\pi(a)|\Omega)(x|\otimes c]=[(\pi\otimes\mathrm{id})\Phi(a)]\,[|\Omega)(x|\otimes c] \tag{4.6}$$

for any $a, c\in A$ and $x\in H$.

Let $\xi\in A'$. Inserting in the above relation $c=I$ and applying $(\mathrm{id}\otimes\xi)$ to the both sides we obtain

$$[(id\otimes\xi)u]\,\pi(a)|\Omega)(x|=\pi(\xi*a)|\Omega)(x|$$

and (4.3) follows. Obviously (4.3) determines u in the unique way. The proof of Statement 1 is complete.

Ad 2. We have to show that

$$(\mathrm{id}\otimes\Phi)u=u_{12}u_{13}. \tag{4.7}$$

Let $\xi,\xi'\in A'$ and $a\in A$. Then using (1.15) we have

$$\begin{aligned}
[(\mathrm{id}\otimes\xi\otimes\xi')(\mathrm{id}\otimes\Phi)u]\,\pi(a)\Omega &= [(\mathrm{id}\otimes\xi*\xi')u]\,\pi(a)\Omega\\
&= \pi(\xi*\xi'*a)\Omega.
\end{aligned}$$

On the other hand

$$\begin{aligned}[(\mathrm{id}\otimes\xi\otimes\xi')u_{12}u_{13}]\,\pi(a)\Omega &= [(\mathrm{id}\otimes\xi)u]\,[(\mathrm{id}\otimes\xi')u]\,\pi(a)\Omega\\ &= [(\mathrm{id}\otimes\xi)u]\,\pi(\xi'*a)\Omega\\ &= \pi(\xi*\xi'*a)\Omega.\end{aligned}$$

Remembering that (4.1) is dense in H and that ξ and ξ' are arbitrary elements of A' we get (4.7).

Ad 3. Let $b,c\in A$ and ρ_{bc} be the linear functional on $CB(H)$ introduced by the formula

$$\rho_{bc}(m) = (\pi(b^*)\Omega|m\pi(c)\Omega) \tag{4.8}$$

for all $m\in CB(H)$. Let $\xi\in A'$. Then using (4.3), (4.1) and (1.16) we obtain

$$\begin{aligned}\xi\,[(\rho_{bc}\otimes\mathrm{id})u] &= \rho_{bc}\,[(\mathrm{id}\otimes\xi)u]\\ &= (\pi(b^*)\Omega|\,[(\mathrm{id}\otimes\xi)u]\,\pi(c)\Omega)\\ &= (\pi(b^*)\Omega|\pi(\xi*c)\Omega) = h(b(\xi*c)) = (hb)(\xi*c)\\ &= \xi(c*hb).\end{aligned}$$

It shows that

$$(\rho_{bc}\otimes\mathrm{id})u = c*hb. \tag{4.9}$$

Let us notice that $c*hb = (\mathrm{id}\otimes h)[(I\otimes b)\Phi(c)]$. Remembering that (1.2) is dense in $A\otimes A$ we get the required result.

Q.E.D.

The C*-algebra B related (via Theorem 3.1) to the right regular representation u of a compact quantum group G will be denoted by $C^*(G)$:

$$C^*(G) = \left\{(\mathrm{id}\otimes ha)u^* : a\in A\right\}^{\text{norm closure}}.$$

Then

$$u\in M(C^*(G)\otimes A).$$

Inserting ba instead of a in (4.6) and remembering that the linear envelope of the elements (4.5) is dense in $CB(H)\otimes A$ we get

$$u\,[\pi(a)\otimes I]\,u^* = (\pi\otimes\mathrm{id})\Phi(a) \tag{4.10}$$

for any $a\in A$.

Let K be a Hilbert space and $v\in M(CB(K)\otimes A)$ be a unitary representation of G acting on K. Using the last formula one can easily show that

$$u_{23}\,[(\mathrm{id}\otimes\pi)(x)\otimes I]\,{u_{23}}^* = (\mathrm{id}\otimes\pi\otimes\mathrm{id})(\mathrm{id}\otimes\Phi)(x).$$

for any $x \in CB(H) \otimes A$. The same relation holds for all $x \in M(CB(H) \otimes A)$. Setting $x = v$ and using (3.1) we get

$$u_{23} V_{12} u_{23}{}^* = V_{12} v_{13},$$

where V is a unitary operator acting on $K \otimes H$ introduced by the formula $V = (\mathrm{id} \otimes \pi) v$. Rearranging the factors we get

$$v_{13} u_{23} = V_{12}{}^* u_{23} V_{12} \tag{4.11}$$

It shows that the right regular representation has the following absorption property: the tensor product of any unitary representation with the right regular one is equivalent to a multiple of the right regular representation. Applying $(\mathrm{id} \otimes \mathrm{id} \otimes \pi)$ to the both sides of the above relation we get

$$V_{13} U_{23} = V_{12}{}^* U_{23} V_{12},$$

where U is a unitary operator acting on $H \otimes H$ introduced by the formula $U = (\mathrm{id} \otimes \pi) u$. In particular case $K = H$ and $v = u$ we obtain

$$U_{13} U_{23} = U_{12}{}^* U_{23} U_{12}$$

This is the important pentagonal relation. It shows that U is a 'multiplicative unitary' in the sense of Baaj and Scandalis [2]. According to the last but one relation, V is a unitary adapted to U (cf [20]). It turns out that the multiplicative unitary U is manageable in the sense of [20]. The proof is given in [6]

Let $b, c \in A$ and ρ_{bc} be the linear functional on $CB(H)$ introduced by (4.8). Applying the hermitian conjugation to the both sides of (4.9) and using the obvious relation ${\rho_{bc}}^* = \rho_{c^* b^*}$ we get:

$$(\rho_{c^* b^*} \otimes \mathrm{id}) u^* = c^* * b^* h.$$

Inserting in this formula c^* and b^* instead of b and c respectively we have:

$$(\rho_{bc} \otimes \mathrm{id}) u^* = b * ch.$$

Comparing this formula with (4.9) and taking into account Proposition 3.3 we see that

$$\Big(c * hb = 0\Big) \Longleftrightarrow \Big(b * ch = 0\Big) \tag{4.12}$$

for any $b, c \in A$. In what follows we shall use the similar formula:

$$\Big(hb * c = 0\Big) \Longleftrightarrow \Big(ch * b = 0\Big) \tag{4.13}$$

for any $b, c \in A$. This is the formula (4.12) written for the opposite group G^{opp} (cf the Remark 1 following the Definition 1.1).

Let a be an element of A such that $\Phi(a) \in A \otimes_{\mathrm{alg}} A$ i.e. $\Phi(a)$ is a sum of finite number of elements of the form $a' \otimes a''$ ($a', a'' \in A$). Then

$$\operatorname{codim}\{b \in A : \ hb * a^* = 0\} < \infty$$

and, by virtue of (4.13)

$$\operatorname{codim}\{b \in A : \ a^*h * b = 0\} < \infty$$

It shows that

$$\dim\{a^*h * b : \ b \in A\} < \infty \tag{4.14}$$

With this inequality we shall prove the following generalization of Riemann-Lebesgue lemma.

Proposition 4.2 *Let $G = (A, \Phi)$ be a compact quantum group and H be H be the carrier Hilbert space of the right regular representation of G. Then*

$$C^*(G) \subset CB(H) \tag{4.15}$$

Proof: We have to show, that $\mathcal{F}_u(a) \in CB(H)$ for any $a \in A$. According to Theorem 3.4 (applied to the right regular representation) and Theorem 4.1.3 the set $\mathcal{A}$ of all linear combinations of matrix elements of finite dimensional representations of G is dense in A. For any $a \in \mathcal{A}$ we have $\Phi(a) \in A \otimes_{\text{alg}} A$.

Combining (3.2) and (4.3) we get

$$\mathcal{F}_u(a)^* \pi(b)\Omega = \pi(a^*h * b)\Omega.$$

The relation (4.14) shows that the operator $\mathcal{F}_u(a)^*$ is finite-dimensional for any $a \in \mathcal{A}$. Therefore $\mathcal{F}_u(a)^*$ is compact for any $a \in \mathcal{A}^{\text{norm closure}} = A$. So is $\mathcal{F}_u(a)$.

Q.E.D.

By virtue of Theorem 3.4 and Theorem 3.1.4, any $C^*(G)$-irreducible subspace of H is finite-dimensional. Using Theorem 1.4.5 of [1] we get

Corollary 4.3 *$C^*(G)$ is a direct sum of finite-dimensional full matrix algebras.*

Proposition 4.2 implies that each irreducible representation of G enters the right regular representation with finite multiplicity. In Section 6 we shall prove

Proposition 4.4 *The right regular representation u contains all irreducible representations of G. Each irreducible representation enters u with the multiplicity equal to its dimension.*

Continuing this line of research one can reproduce the theory of Pontryagin duality of compact quantum groups presented in [8] (Sections 2 and 3).

5 The Hopf algebras

This section is devoted to the proof of Theorem 1.2.

Let $G = (A, \Phi)$ be a compact quantum group and $(u^\alpha)_{\alpha\in\hat{G}}$ be the complete family of irreducible unitary representations of G: The representations u^α ($\alpha \in \hat{G}$) are pairwise inequivalent and any unitary irreducible representation of G is equivalent to u^α for one $\alpha \in \hat{G}$.

Let H_α be the carrier Hilbert space of u^α and $N_\alpha = \dim H_\alpha$. We know that all $N_\alpha < \infty$. Introducing an orthonormal basis in H_α we may identify u^α with a matrix $(u^\alpha_{kl})_{k,l=1,2,\dots,N_\alpha}$, where u^α_{kl} are elements of A such that:

$$\sum u^\alpha_{kr} u^{\alpha\,*}_{lr} = \delta_{kl}, \tag{5.1}$$

$$\sum u^{\alpha\,*}_{rk} u^\alpha_{rl} = \delta_{kl}, \tag{5.2}$$

$$\Phi(u^\alpha_{kl}) = \sum u^\alpha_{kr} \otimes u^\alpha_{rl}, \tag{5.3}$$

where the summation index r runs over $1, 2, \dots, N_\alpha$ and $k, l = 1, 2, \dots, N_\alpha$.

Proposition 5.1 *The set*

$$\{u^\alpha_{kl} : \ \alpha \in \hat{G},\ k, l = 1, 2, \dots, N_\alpha\} \tag{5.4}$$

is a linear basis in $\mathcal{A}$.

Proof: Any finite-dimensional unitary representation of G decomposes into direct sum of irreducible representations. Therefore any element of $\mathcal{A}$ is a linear combination of elements (5.4).

Let F be a finite subset of $\hat{G}$ and

$$u^F = \sum_{\alpha\in F}{}^{\oplus} u^\alpha.$$

Then u^F is a unitary representation of G acting on

$$H_F = \sum_{\alpha\in F}{}^{\oplus} H_\alpha.$$

Elements $m \in B(H_F)$ are represented by matrices $(m_{\alpha\beta})_{\alpha,\beta\in F}$, where $m_{\alpha\beta} \in B(H_\beta, H_\alpha)$. Remembering that u^α ($\alpha \in F$) are irreducible and pairwise inequivalent and using the Schur lemma one can easily show that m intertwines u^F with itself if and only if

$$m_{\alpha\beta} = \delta_{\alpha\beta} I_{B(H_\alpha)} \tag{5.5}$$

for all $\alpha, \beta \in F$.

Let B_F be the C^*-algebra related to u^F via Theorem 3.1. By virtue of Theorem 3.1.4 the commutant of B_F coincides with the set of all $m \in B(H_F)$ satisfying relation

(5.5). Therefore, using the bicommutant theorem (H_F is finite-dimensional) we see that $m \in B_F$ if and only if $m_{\alpha\beta} = \delta_{\alpha\beta} m_\alpha$, where $m_\alpha \in B(H_\alpha)$, $\alpha, \beta \in F$. It shows that

$$\dim B_F = \sum_{\alpha \in F} N_\alpha^2. \tag{5.6}$$

On the other hand $B_F = \{(\mathrm{id} \otimes ha) u^{F*} : \ a \in A\}$. Therefore the number of linearly independent matrix elements of u^F is larger or equal to $\dim B_F$. In other words the set

$$\{u^\alpha_{kl} : \ \alpha \in F, \ k, l = 1, 2, \ldots, N_\alpha\} \tag{5.7}$$

contains at least $\dim B_F$ linearly independent elements. According to (5.6) the total number of elements (5.7) equals to $\dim B_F$. It shows that elements (5.7) are linearly independent. Using the freedom of choice of $F \subset \hat{G}$ we see that elements (5.4) are linearly independent.

Q.E.D.

Proposition 5.2 *For any $\alpha \in \hat{G}$ there exists $\beta \in \hat{G}$ such that ${u^\alpha_{kl}}^*$ ($k, l = 1, 2, \ldots, N_\alpha$) are linear combinations of matrix elements of u^β.*

Proof: Let $\alpha \in \hat{G}$. Then

$$\sum_{k,l=1}^{N_\alpha} h(u^\alpha_{kl} {u^\alpha_{kl}}^*) = N_\alpha h(I) = N_\alpha.$$

We already know (cf Proof of Proposition 4.2) that the set $\mathcal{A}$ of all linear combinations of elements (5.4) is dense in A. Therefore there exist $k, l \in \{1, 2, \ldots, N_\alpha\}$, $\beta \in \hat{G}$ and $m, n \in \{1, 2, \ldots, N_\beta\}$ such that

$$E_{mk} = h(u^\alpha_{kl} u^\beta_{mn}) \neq 0 \tag{5.8}$$

According to (5.3)

$$\Phi(u^\alpha_{kl} u^\beta_{mn}) = \sum_{r=1}^{N_\alpha} \sum_{s=1}^{N_\beta} u^\alpha_{kr} u^\beta_{ms} \otimes u^\alpha_{rl} u^\beta_{sn}$$

Applying $\mathrm{id} \otimes h$ to the both sides and using (2.4) we obtain

$$E_{mk} = \sum_{rs} u^\alpha_{kr} u^\beta_{ms} E_{sr}$$

and (u^α is unitary)

$$\sum_s u^\beta_{ms} E_{sr} = \sum_k {u^\alpha_{kr}}^* E_{mk} \tag{5.9}$$

for any $m = 1, 2, \ldots, N_\beta$ and $n = 1, 2, \ldots, N_\alpha$.

Let λ_r ($r = 1, 2, \ldots, N_\alpha$) be complex numbers such that $\sum_r E_{sr} \lambda_r = 0$ for any $s = 1, 2, \ldots, N_\beta$. Then $\sum_{rk} E_{mk} {u^\alpha_{kr}}^* \lambda_r = 0$ and using the linear independence of (5.4) and relation (5.8) we get $\lambda_r = 0$ for all r. It shows that $\mathrm{Rank}\, E = N_\alpha$. Similarly if μ_m ($m = 1, 2, \ldots, N_\beta$) are complex numbers such that $\sum_m \mu_m E_{mk} = 0$ for any $k = 1, 2, \ldots, N_\alpha$ then $\sum_{sm} \mu_m u^\beta_{ms} E_{sr} = 0$ and $\mu_m = 0$ for all m. Therefore $\mathrm{Rank}\, E = N_\beta$.

This way we showed that E is a quadratic invertible matrix and (5.9) implies that

$$u^{\alpha\ *}_{kr} = \sum_{sm} (E^{-1})_{km} u^{\beta}_{ms} E_{sr} \tag{5.10}$$

for any $k, r = 1, 2, \ldots, N_\alpha$.

Q.E.D.

Let $\mathcal{A}$ be the set of all linear combinations of matrix elements of finite-dimensional representations of G. Then $ab \in \mathcal{A}$ for any $a, b \in \mathcal{A}$ (the tensor product of unitary representations is a unitary representation). By virtue of Theorem 3.4, Theorem 4.1.3, Proposition 5.2 and Proposition 5.1, $\mathcal{A}$ is a dense *-subalgebra of A and (5.4) is a basis (in the sense of the vector space theory) of $\mathcal{A}$.

Let $e : \mathcal{A} \to \mathbf{C}$ and $\kappa : \mathcal{A} \to \mathcal{A}$ be linear mappings introduced by the formulae

$$e(u^\alpha_{kl}) = \delta_{kl} \tag{5.11}$$

$$\kappa(u^\alpha_{kl}) = u^{\alpha\ *}_{lk} \tag{5.12}$$

for any $\alpha \in \hat{G}$ and $k, l = 1, 2, \ldots, N_\alpha$. Using these definitions and formulae (5.1) – (5.3) one can easily verify relations (1.9) – (1.12) for $a = u^\alpha_{kl}$ ($\alpha \in \hat{G}$, $k, l = 1, 2, \ldots, N_\alpha$). By linearity, these relations hold for any $a \in \mathcal{A}$. The proof of Theorem 1.2 is complete.

6 Peter-Weyl theory

In this Section we derive the orthonormality relations for the matrix elements of irreducible representations of a compact quantum group G and investigate the modular properties of the Haar measure. It follows from Proposition 5.2 that the irreducible representations of G appear in conjugate pairs: We say that u^α and u^β are conjugate if the matrix elements of u^α are hermitian conjugation of matrix elements of u^β. The case $u^\alpha = u^\beta$ is not excluded.

Let $G = (A, \Phi)$ be a compact quantum group and $(u^\alpha)_{\alpha \in \hat{G}}$ be the complete family of irreducible unitary representations of G. The carrier Hilbert space of u^α will be denoted by K^α ($\alpha \in \hat{G}$). Then $\dim K^\alpha < \infty$ and $u^\alpha \in B(K^\alpha) \otimes A$. In what follows we shall use more explicit notation for the matrix elements of irreducible representations of G. For any $\alpha \in \hat{G}$, $w \in B(K^\alpha) \otimes A$ and $x, y \in K^\alpha$ we set:

$$(x|w|y) = (\rho_{xy} \otimes \mathrm{id})w,$$

where ρ_{xy} is the linear functional on $B(K^\alpha)$ introduced by the formula $\rho_{xy}(m) = (x|m|y)$, ($m \in B(K^\alpha)$). The reader should notice that $(x|u^\alpha|y) \in \mathcal{A}$. With this notation formulae (5.11) and (5.12) take the following form: for any $x, y \in K^\alpha$ we have

$$e((x|u^\alpha|y)) = (x|y),$$

$$\kappa((x|u^\alpha|y)) = (y|u^\alpha|x)^* \,. \tag{6.1}$$

Let $\alpha, \beta \in \hat{G}$ and $u^{\alpha\beta}$ be the tensor product of u^α and u^β: $u^{\alpha\beta} = {u^\alpha}_{13} {u^\beta}_{23}$. $u^{\alpha\beta}$ is a unitary representation of G acting on $K^\alpha \otimes K^\beta$. It follows immediately from the proof of Proposition 5.2 that u^α and u^β is a pair of conjugate representations if and only if there exists a non-zero $u^{\alpha\beta}$-invariant element $E \in K^\alpha \otimes K^\beta$. If this is the case, then E is of maximal rank: $\operatorname{rank} E = \dim K^\alpha = \dim K^\beta$.

It is well known that $K^\alpha \otimes K^\beta$ may be identified with the space of all antilinear maps $K^\beta \to K^\alpha$: for any element $E \in K^\alpha \otimes K^\beta$ there exists unique antilinear map $J : K^\beta \to K^\alpha$ such that

$$(x \otimes y|E) = (x|Jy) = (y|J^*x) \tag{6.2}$$

for all $x \in K^\alpha$ and $y \in K^\beta$ (the second equality in (6.2) defines the hermitian conjugation for antilinear operators). J is invertible if and only if $\operatorname{rank} E = \dim K^\alpha = \dim K^\beta$. Using the orthonormal basis expansion one can easily compute the square of the norm of E:

$$(E|E) = \operatorname{Tr} J^* J \tag{6.3}$$

Let $E \in K^\alpha \otimes K^\beta$, $E' \in K^\beta \otimes K^\alpha$ and $J : K^\beta \to K^\alpha$ and $J' : K^\alpha \to K^\beta$ be corresponding antilinear maps. For any $x \in K^\alpha$ we set:

$$\left(E \otimes I_{B(K^\alpha)}\right) x = E \otimes x,$$

$$\left(I_{B(K^\alpha)} \otimes E'\right) x = x \otimes E',$$

Then $E \otimes I_{B(K^\alpha)}$ and $I_{B(K^\alpha)} \otimes E'$ belong to $B(K^\alpha, K^\alpha \otimes K^\beta \otimes K^\alpha)$. Consequently $\left(I_{B(K^\alpha)} \otimes E'\right)^* \left(E \otimes I_{B(K^\alpha)}\right) \in B(K^\alpha)$. Using again the orthonormal basis expansion one can easily verify that

$$\left(I_{B(K^\alpha)} \otimes E'\right)^* \left(E \otimes I_{B(K^\alpha)}\right) = JJ'.$$

If E is $u^{\alpha\beta}$-invariant and $E' \neq 0$ is $u^{\beta\alpha}$-invariant then $E \otimes I_{B(K^\alpha)}$ and $I_{B(K^\alpha)} \otimes E'$ intertwin u^α and $u^{\alpha\beta\alpha}$. Consequently $\left(I_{B(K^\alpha)} \otimes E'\right)^* \left(E \otimes I_{B(K^\alpha)}\right)$ intertwins u^α with itself and using irreducibility of u^α we see that JJ' is a multiple of $I_{B(K^\alpha)}$. Therefore $J = \lambda\,(J')^{-1}$, where $\lambda \in \mathbf{C}$, $\lambda \neq 0$. It shows that J (and consequently E) is determined up to a numerical factor. It means that the subspace of $u^{\alpha\beta}$-invariant elements is one dimensional.

For any $\beta \in \hat{G}$ we set:

$\overline{\beta}$ = the element of $\hat{G}$ such that $u^{\overline{\beta}}$ and u^β are conjugate,

E^β = a non-zero $u^{\overline{\beta}\beta}$-invariant element of $K^{\overline{\beta}} \otimes K^\beta$,

J^β = the invertible antilinear map: $K^\beta \to K^{\overline{\beta}}$ related to E^β via (6.2),

$F^\beta = \left(J^\beta\right)^* J^\beta$.

Till now the invariant elements E^β and $E^{\overline{\beta}}$ are defined up to a non-zero multiplicative factors. We choose these factors in such a way, that

$$\left(E^\beta \middle| E^\beta\right) = \left(E^{\overline{\beta}} \middle| E^{\overline{\beta}}\right) \tag{6.4}$$

and
$$J^{\overline{\beta}} = \left(J^\beta\right)^{-1}.$$

Then we have: $F^{\overline{\beta}} J^\beta = \left(J^{\overline{\beta}}\right)^* J^{\overline{\beta}} J^\beta = J^\beta J^{\overline{\beta}} \left(J^{\overline{\beta}}\right)^* = J^\beta \left(F^\beta\right)^{-1}$. Remembering, that J^β is antilinear we see that
$$\left(F^{\overline{\beta}}\right)^{it} J^\beta = J^\beta \left(F^\beta\right)^{it} \tag{6.5}$$
for any $t \in \mathbf{R}$

Let $\beta \in \hat{G}$. The $u^{\overline{\beta}\beta}$-invariance of E^β means that $u^{\overline{\beta}}{}_{13} u^\beta{}_{23} E^\beta{}_{12} = E^\beta{}_{12}$. Remembering that $u^{\overline{\beta}}$ is unitary we get (cf (5.9))
$$u^\beta{}_{23} E^\beta{}_{12} = \left(u^{\overline{\beta}}\right)^*_{13} E^\beta{}_{12}.$$

Computing the 'scalar product' with $x \otimes y$ (where $x \in K^{\overline{\beta}}$ and $y \in K^\beta$) we obtain:
$$\sum_k \left(y\middle|u^\beta\middle|e_k\right)\left(x \otimes e_k\middle|E^\beta\right) = \sum_l \left(x\middle|\left(u^{\overline{\beta}}\right)^*\middle|\epsilon_l\right)\left(\epsilon_l \otimes y\middle|E^\beta\right).$$

where (e_k) ((ϵ_l) - respectively) is an orthonormal basis in K^β ($K^{\overline{\beta}}$ - respectively). Taking into account (6.2) we have:
$$\sum_k \left(y\middle|u^\beta\middle|e_k\right)\left(e_k\middle|\left(J^\beta\right)^* x\right) = \sum_l \left(x\middle|\left(u^{\overline{\beta}}\right)^*\middle|\epsilon_l\right)\left(\epsilon_l\middle|J^\beta y\right).$$

Therefore
$$\left(y\middle|u^\beta\middle|\left(J^\beta\right)^* x\right) = \left(x\middle|\left(u^{\overline{\beta}}\right)^*\middle|J^\beta y\right)$$

and finally
$$\left(y\middle|u^\beta\middle|\left(J^\beta\right)^* x\right) = \left(J^\beta y\middle|u^{\overline{\beta}}\middle|x\right)^*. \tag{6.6}$$

This is our basic formula. It holds for any $x \in K^{\overline{\beta}}$ and $y \in K^\beta$.

Now we are able to derive the orthogonality relations for matrix elements of irreducible unitary representations. Setting $v = u^{\overline{\alpha}\beta}$ in Proposition 3.5 we get
$$(\mathrm{id} \otimes h) u^{\overline{\alpha}}{}_{13} u^\beta{}_{23} = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \dfrac{1}{M^\beta} \left|E^\beta\right)\left(E^\beta\right| & \text{if } \alpha = \beta \end{cases}$$

where
$$M^\beta = \left(E^\beta\middle|E^\beta\right) = \operatorname{Tr} F^\beta$$

Let $x', y' \in K^{\overline{\alpha}}$ and $x, y \in K^\beta$. Computing the matrix elements between vectors $x' \otimes x$ and $y' \otimes y$ we get:
$$h\left(\left(x'\middle|u^{\overline{\alpha}}\middle|y'\right)\left(x\middle|u^\beta\middle|y\right)\right) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \dfrac{1}{M^\beta}\left(x'\middle|J^\beta x\right)\left(\left(J^\beta\right)^* y'\middle|y\right) & \text{if } \alpha = \beta. \end{cases}$$

Replacing x by $J^{\overline{\beta}}x$ and y' by $\left(J^{\overline{\alpha}}\right)^* y'$ we see that

$$h\left(\left(x'\middle|u^{\overline{\alpha}}\middle|\left(J^{\overline{\alpha}}\right)^* y'\right)\left(J^{\overline{\beta}}x\middle|u^{\beta}\middle|y\right)\right)=\begin{cases} 0 & \text{if } \alpha\neq\beta \\ \dfrac{1}{M^{\beta}}\,(x'|x)\,(y'|y) & \text{if } \alpha=\beta \end{cases}$$

for any $x'\in K^{\overline{\alpha}}$, $y'\in K^{\alpha}$, $y\in K^{\beta}$ and $x\in K^{\overline{\beta}}$. Now using (6.6) (with β replaced by $\overline{\alpha}$) we get:

$$h\left(\left(J^{\overline{\alpha}}x'\middle|u^{\alpha}\middle|y'\right)^*\left(J^{\overline{\beta}}x\middle|u^{\beta}\middle|y\right)\right)=\begin{cases} 0 & \text{if } \alpha\neq\beta \\ \dfrac{1}{M^{\beta}}\,(x'\otimes y'|x\otimes y) & \text{if } \alpha=\beta. \end{cases} \tag{6.7}$$

For any $\beta\in\hat{G}$ we shall consider the linear map $\psi^{\beta}: K^{\overline{\beta}}\otimes K^{\beta}\to\mathcal{A}$ introduced by the formula:

$$\psi^{\beta}(x\otimes y)=\left(J^{\overline{\beta}}x\middle|u^{\beta}\middle|y\right)$$

for any for any $y\in K^{\beta}$ and $x\in K^{\overline{\beta}}$. Then

$$h\left(\psi^{\alpha}(z')^*\psi^{\beta}(z)\right)=\begin{cases} 0 & \text{if } \alpha\neq\beta \\ \dfrac{1}{M^{\beta}}\,(z'|z) & \text{if } \alpha=\beta. \end{cases} \tag{6.8}$$

for any $z'\in K^{\overline{\alpha}}\otimes K^{\alpha}$ and $z\in K^{\overline{\beta}}\otimes K^{\beta}$.

Let π be the GNS-representation of the algebra A produced by the Haar measure h, H be the carrier Hilbert space of π and Ω be the corresponding cyclic vector (cf Section 4). For any $\beta\in\hat{G}$ and $z\in K^{\overline{\beta}}\otimes K^{\beta}$ we set:

$$\Psi^{\beta}(z)=\pi(\psi^{\beta}(z))\Omega.$$

Then $\Psi^{\beta}\in B(K^{\overline{\beta}}\otimes K^{\beta},H)$. Relation (6.8) shows that

$$\left(\Psi^{\alpha}\right)^*\Psi^{\beta}=\frac{\delta^{\alpha\beta}}{M^{\beta}}\,I_{B(K^{\overline{\beta}}\otimes K^{\beta})},$$

where $\delta^{\alpha\beta}$ is the Kronecker symbol. It means that Ψ^{β} is a multiple of an isometry and that the ranges $H^{\beta}=\Psi^{\beta}\left(K^{\overline{\beta}}\otimes K^{\beta}\right)$ for different β are orthogonal. By definition elements of $\mathcal{A}$ are linear combinations of matrix elements of finite-dimensional unitary representations of G: any $a\in\mathcal{A}$ is of the form: $a=\sum\psi^{\beta}(z^{\beta})$, where $z^{\beta}\in K^{\overline{\beta}}\otimes K^{\beta}$ and only finite number of z^{β} are not zero. Remembering that $\mathcal{A}$ is dense in A and that Ω is cyclic we see that

$$H=\sum_{\beta\in\hat{G}}^{\oplus}H^{\beta}. \tag{6.9}$$

Moreover

$$\sum_{\beta\in\hat{G}}^{\oplus}\sqrt{M^{\beta}}\,\Psi^{\beta}:\sum_{\beta\in\hat{G}}^{\oplus}K^{\overline{\beta}}\otimes K^{\beta}\longrightarrow H \tag{6.10}$$

is a unitary map. It will be denoted by Ψ.

Let $\beta \in \hat{G}$, $x \in K^{\overline{\beta}}$, $y \in K^{\beta}$ and $\xi \in A'$. We compute:

$$\begin{aligned}
\xi * \psi^{\beta}(x \otimes y) &= (\mathrm{id} \otimes \xi)\Phi\left(\left(J^{\overline{\beta}}x\middle|u^{\beta}\middle|y\right)\right) \\
&= (\mathrm{id} \otimes \xi)\left(J^{\overline{\beta}}x\middle|{u^{\beta}}_{12}{u^{\beta}}_{13}\middle|y\right) \\
&= \left(J^{\overline{\beta}}x\middle|{u^{\beta}}_{12}\middle|my\right) = \psi^{\beta}(x \otimes my)
\end{aligned}$$

where $m = (\mathrm{id} \otimes \xi)u^{\beta}$. Let u be the right regular representation of G (cf Section 4). By virtue of (4.3) we have:

$$\begin{aligned}
[(\mathrm{id} \otimes \xi)u]\Psi^{\beta}(x \otimes y) &= [(\mathrm{id} \otimes \xi)u]\pi(\psi^{\beta}(x \otimes y))\Omega \\
&= \pi\left(\xi * \psi^{\beta}(x \otimes y)\right)\Omega \\
&= \pi\left(\psi^{\beta}(x \otimes my)\right)\Omega = \Psi^{\beta}(x \otimes my)
\end{aligned}$$

Therefore taking into account the formula for m we obtain:

$$u\left(\Psi^{\beta} \otimes I\right) = \left(\Psi^{\beta} \otimes I\right)\left(I_{B(K^{\overline{\beta}})} \otimes u^{\beta}\right).$$

It means that Ψ^{β} intertwins u and $I_{B(K^{\overline{\beta}})} \otimes u^{\beta}$. The latter representation is equivalent to the sum of $\dim K^{\overline{\beta}} = \dim K^{\beta}$ copies of u^{β}. Remembering that (6.10) is unitary we get Proposition 4.4.

Now we have to use elements of the Tomita–Takesaki theory [4, 10]. For any $a \in \mathcal{A}$ we set:

$$S\pi(a)\Omega = \pi(a)^{*}\Omega.$$

S is an antilinear (unbounded) operator acting on H. Using (6.6) one can easily show that $\psi^{\beta}(y \otimes x)^{*} = \psi^{\overline{\beta}}\left(J^{\overline{\beta}}y \otimes \left(J^{\beta}\right)^{*}x\right)$ for any $y \in K^{\overline{\beta}}$ and $x \in K^{\beta}$. Therefore, for any $z \in K^{\overline{\beta}} \otimes K^{\beta}$ we have:

$$S_{\circ}\Psi^{\beta}(z) = \Psi^{\overline{\beta}}{}_{\circ}\left[J^{\overline{\beta}} \otimes \left(J^{\beta}\right)^{*}\right](z).$$

We see that S is not decomposable in the sense of (6.9). However it respects the decomposition (6.9) in the following sense: for any $\beta \in \hat{G}$, H^{β} is contained in the domain of S and $S(H^{\beta}) = H^{\overline{\beta}}$. Remembering that all H^{β} are finite dimensional one can easily show, that S is closeable (hence $\mathcal{A}$ equipped with the scalar product induced by h is a generalized Hilbert algebra in the sense of [10]). So, we can consider the modular operator:

$$\Delta = S^{*}S.$$

Clearly Δ is decomposable in the sense of (6.9): For any $z \in K^{\overline{\beta}} \otimes K^{\beta}$ we have:

$$\begin{aligned}
\Delta_{\circ}\Psi^{\beta}(z) &= \Psi^{\beta}{}_{\circ}\left[J^{\overline{\beta}} \otimes \left(J^{\beta}\right)^{*}\right]^{*}{}_{\circ}\left[J^{\overline{\beta}} \otimes \left(J^{\beta}\right)^{*}\right](z) \\
&= \Psi^{\beta}{}_{\circ}\left[F^{\overline{\beta}} \otimes \left(F^{\beta}\right)^{-1}\right](z)
\end{aligned}$$

and

$$\Delta^{it} \circ \Psi^\beta(z) = \Psi^\beta \circ \left[\left(F^{\overline{\beta}}\right)^{it} \otimes \left(F^\beta\right)^{-it} \right](z)$$

for any $t \in \mathbf{R}$.

According to the Tomita–Takesaki theory the modular automorphism group $(\sigma_t)_{t\in\mathbf{R}}$ is introduced by the formula:

$$\pi\left(\sigma_t(a)\right)\Omega = \Delta^{it}\pi(a)\Omega.$$

In general the modular automorphisms act within the weak closure of the considered algebra, it turns out however, that in our case $\sigma_t(\mathcal{A}) = \mathcal{A}$. Indeed, comparing the two last formulae we see that for any $z \in K^{\overline{\beta}} \otimes K^\beta$

$$\sigma_t(\psi^\beta(z)) = \psi^\beta(z'). \tag{6.11}$$

where $z' = \left[\left(F^{\overline{\beta}}\right)^{it} \otimes \left(F^\beta\right)^{-it} \right](z)$

Let $x, y \in K^\beta$ and $z = J^\beta x \otimes y$. Then (cf (6.5)) $z' = J^\beta \left(F^\beta\right)^{it} x \otimes \left(F^\beta\right)^{-it} y$. Formula (6.11) shows now, that

$$\sigma_t\left(\left(x\middle|u^\beta\middle|y\right)\right) = \left(\left(F^\beta\right)^{it} x\middle|u^\beta\middle|\left(F^\beta\right)^{-it} y\right).$$

Therefore

$$(\mathrm{id}\otimes\sigma_t)\left(u^\beta\right) = \left[\left(F^\beta\right)^{-it} \otimes I\right] u^\beta \left[\left(F^\beta\right)^{-it} \otimes I\right]. \tag{6.12}$$

Now we are able to give

Proof of Theorem 1.4: At first we introduce the family of linear functionals $(f_z)_{z\in\mathbf{C}}$. Let $z \in \mathbf{C}$. By definition f_z is the linear functional on $\mathcal{A}$ such that

$$(\mathrm{id}\otimes f_z)u^\beta = \left(F^\beta\right)^{-z} \tag{6.13}$$

for any $\beta \in \hat{G}$. The existence of such a functional follows easily from Proposition 5.1.

The right hand side of the above equation is a holomorphic function of $z \in \mathbf{C}$, so Statement 2 of Theorem 1.4 holds.

We shall prove, that

$$\sigma_t(a) = f_{it} * a * f_{it} \tag{6.14}$$

for any $a \in \mathcal{A}$ and $t \in \mathbf{R}$. Indeed, denoting by $\sigma'_t(a)$ the right hand side of (6.14) we easily verify that $(\mathrm{id}\otimes\sigma'_t)u^\beta = (\mathrm{id}\otimes f_{it}\otimes\mathrm{id}\otimes f_{it})u^\beta{}_{12}u^\beta{}_{13}u^\beta{}_{14}$ coincides with the right hand side of (6.12). This way Statement 6 of Theorem 1.4 is proven (formula (1.21) coincides with the KMS-condition for the Haar measure).

Applying the counit e to the both sides of (6.14) we get $f_{2it}(a) = e(\sigma_t(a))$. It shows that for any $t \in \mathbf{R}$, the functional f_{2it} is a *-character: for any $a, b \in \mathcal{A}$

$$f_{2it}(ab) = f_{2it}(a) f_{2it}(b) \tag{6.15}$$

and

$$f_{2it}(a^*) = \overline{f_{2it}(a)}. \tag{6.16}$$

By analytical continuation in (6.15), all functionals f_z ($z \in \mathbf{C}$) are multiplicative. In particular Statement 1 holds: $f_z(I) = 1$ (otherwise f_z would be zero in contradiction with definition (6.13)).

To prove Statement 3 we compute:

$$\begin{aligned}
(\mathrm{id} \otimes f_{z_1} * f_{z_2})u^\beta &= (\mathrm{id} \otimes f_{z_1} \otimes f_{z_2})(\mathrm{id} \otimes \Phi)u^\beta \\
&= (\mathrm{id} \otimes f_{z_1} \otimes f_{z_2}){u^\beta}_{12}{u^\beta}_{13} = \left(F^\beta\right)^{-z_1}\left(F^\beta\right)^{-z_2} \\
&= \left(F^\beta\right)^{-(z_1+z_2)} = (\mathrm{id} \otimes f_{z_1+z_2})u^\beta
\end{aligned}$$

and Statement 3 of Theorem 1.4 follows.

The formula (1.19) is the analytical continuation of (6.16). The formula (1.18) follows from (5.12). Indeed:

$$\begin{aligned}
(\mathrm{id} \otimes f_{z} \circ \kappa)u^\beta &= (\mathrm{id} \otimes f_z)\left(u^\beta\right)^* = \left[(\mathrm{id} \otimes f_{-\overline{z}})u^\beta\right]^* \\
&= \left[\left(F^\beta\right)^{\overline{z}}\right]^* = \left(F^\beta\right)^z = (\mathrm{id} \otimes f_{-z})u^\beta.
\end{aligned}$$

This way Statement 4 is proved.

Combining (6.1) with (6.6) one can easily show that

$$\begin{aligned}
\kappa\left((x|u^\alpha|y)\right) &= (y|u^\alpha|x^*) \\
&= \left(J^\alpha y \middle| u^{\overline{\alpha}} \middle| \left(J^{\overline{\alpha}}\right)^* x\right)
\end{aligned}$$

for any $x, y \in K^\alpha$. Iterating this formula we get:

$$\begin{aligned}
\kappa^2\left((x|u^\alpha|y)\right) &= \left(J^{\overline{\alpha}}\left(J^{\overline{\alpha}}\right)^* x \middle| u^\alpha \middle| \left(J^{\overline{\alpha}}\right)^* J^\alpha y\right) \\
&= \left((F^\alpha)^{-1} x \middle| u^\alpha \middle| F^\alpha y\right)
\end{aligned}$$

Therefore

$$\left(\mathrm{id} \otimes \kappa^2\right)(u^\alpha) = \left[(F^\alpha)^{-1} \otimes I\right] u^\alpha \left[F^\alpha \otimes I\right] \tag{6.17}$$

Now, denoting by $\tau(a)$ the right hand side of (1.20) we easily verify that $(\mathrm{id} \otimes \tau)u^\alpha = (\mathrm{id} \otimes f_1 \otimes \mathrm{id} \otimes f_{-1}){u^\alpha}_{12}{u^\alpha}_{13}{u^\alpha}_{14}$ coincides with the right hand side of (6.17). This way Statement 5 of Theorem 1.4 is proven.

Q.E.D.

The matrix elements of irreducible representations satisfy the famous Peter–Weyl orthonormality relations. Inserting $J^\alpha x'$ and $J^\beta x$ instead of x' and x in (6.7) we obtain:

$$h\left(\left(x' \middle| u^\alpha \middle| y'\right)^* \left(x \middle| u^\beta \middle| y\right)\right) = \begin{cases} 0 & \text{if } \alpha \neq \beta \\ \dfrac{1}{M^\beta}\left(x' \middle| \left(F^\beta\right)^{-1} \middle| x\right)(y'|y) & \text{if } \alpha = \beta. \end{cases} \tag{6.18}$$

Combining this result with (1.21) we get

$$h\left(\left(x\middle|u^{\beta}\middle|y\right)\left(x'\middle|u^{\alpha}\middle|y'\right)^{*}\right)=\begin{cases} 0 & \text{if } \alpha\neq\beta \\ \dfrac{1}{M^{\beta}}\ (x'|x)\left(y'\middle|F^{\beta}\middle|y\right) & \text{if } \alpha=\beta. \end{cases} \tag{6.19}$$

These formulae coincide with the ones in [13, Theorem 5.7.4]. They hold for any $\alpha,\beta\in\hat{G}$, $x,y\in K^{\beta}$ and $x',y'\in K^{\alpha}$.

7 Groups with faithful Haar measure

Throughout this Section we assume that $G=(A,\Phi)$ is a compact quantum group and that the Haar measure h on G is faithful.

Proof of Theorem 1.6:

Ad 1. For any $\omega\in A'$, $x\in A$ and $Q\in A\otimes A$ we set

$$\rho_{\omega x}(Q) = (\omega\otimes h)(Q\Phi(x)) \tag{7.1}$$

$$\rho'_{\omega x}(Q) = (\omega\otimes h)(Q(I\otimes x)) \tag{7.2}$$

Clearly $\rho_{\omega x}$ and $\rho'_{\omega x}$ are continuous linear functionals defined on $A\otimes A$. Let r_0 be the linear operator acting on $A\otimes A$, defined on the domain $D(r_0)=A\otimes_{\mathrm{alg}}A$ by the formula

$$r_0(a\otimes b)=(a\otimes I)\Phi(b)$$

One can easily verify that

$$\rho_{\omega x}(r_0(Q))=\rho'_{\omega x}(Q) \tag{7.3}$$

for any $Q\in D(r_0)$. We have to show that r_0 is closeable. Assume that $(Q_n)_{n=1,2,\ldots}$ is a sequence of elements of $D(r_0)$ such that $Q_n\to 0$ and $r_0(Q_n)\to R\in A\otimes A$. Using (7.2), (7.3) and (7.1) we get

$$\begin{aligned}(\omega\otimes h)(R\Phi(x)) &= \rho_{\omega x}(R)=\lim\rho_{\omega x}(r_0(Q_n))\\ &= \lim\rho'_{\omega x}(Q_n)=0\end{aligned}$$

for any $x\in A$ and $\omega\in A'$. Therefore $(\mathrm{id}\otimes h)(R\Phi(x))=0$ and $(\mathrm{id}\otimes h)(R\Phi(x)(y\otimes I))=0$ for any $x,y\in A$. Remembering that (1.7) is dense in $A\otimes A$ we get $(\mathrm{id}\otimes h)(RR^*)=0$ and $R=0$. It shows, that r_0 is closeable.

Let r be the closure of r_0. Then (cf (7.3))

$$\rho_{\omega x}(r(Q))=\rho'_{\omega x}(Q)$$

for any $Q\in D(r)$. Let $Q\in\ker r$. Then $\rho'_{\omega x}(Q)=(\omega\otimes h)(Q(I\otimes x))=0$ for any $\omega\in A'$ and $x\in A$. Therefore $(\mathrm{id}\otimes h)(Q(I\otimes x))=0$, $(\mathrm{id}\otimes h)(Q(y\otimes x))=0$ for any $x,y\in A$, $(\mathrm{id}\otimes h)(QQ^*)=0$ and $Q=0$.

In the similar way one can prove the existence of the mapping s and the triviality of its closure. To this end, instead of (7.1) and (7.2), one has to consider the functionals

$$\rho_{\omega x}(Q) = (h \otimes \omega)(Q\Phi(x)),$$

$$\rho'_{\omega x}(Q) = (h \otimes \omega)(Q(x \otimes I)).$$

We left to the reader the details of this part of the proof.

Ad 2. Let $a \in A$ and $\Phi(a) \in A \otimes_{\mathrm{alg}} A$. Inserting $b = u^{\alpha}_{kl}$ in (4.14) we see that the set

$$\left\{ \sum_{r=1}^{N_\alpha} u^{\alpha}_{kr} h(a^* u^{\alpha}_{rl}) : \ \alpha \in \hat{G}, \ k, l = 1, 2, \ldots, N_\alpha \right\}$$

is contained in a finite-dimensional linear subspace of A. Remembering that elements (5.4) are linearly independent we see that there exists a finite subset $F \subset \hat{G}$ such that

$$h(a^* u^{\alpha}_{rl}) = 0$$

for all $\alpha \in \hat{G} - F$ and $r, l = 1, 2, \ldots, N_\alpha$. It shows that $\pi(a)\Omega \perp H^\alpha$ for all $\alpha \in \hat{G} - F$. By virtue of (6.9)

$$\pi(a)\Omega \in \sum_{\alpha \in F}^{\oplus} H^\alpha.$$

Remembering that h is faithful one can show that a is a linear combination of elements u^{α}_{rl} ($\alpha \in F$, $r, l = 1, 2, \ldots, N_\alpha$). Therefore $a \in \mathcal{A}$. This way we showed that

$$\{a \in A : \ \Phi(a) \in A \otimes_{\mathrm{alg}} A\} \subset \mathcal{A}.$$

The converse inclusion is obvious.

Ad 3 and 4. We shall use the holomorphic family $(f_z)_{z \in \mathbf{C}}$ of linear multiplicative functionals on $\mathcal{A}$ introduced in the previous section. For any $a \in \mathcal{A}$ and $t \in \mathbf{R}$ we set

$$\sigma_t(a) = f_{it} * a * f_{it}, \tag{7.4}$$

$$\tau_t(a) = f_{it} * a * f_{-it}, \tag{7.5}$$

$$R(a) = \kappa(f_{\frac{1}{2}} * a * f_{-\frac{1}{2}}). \tag{7.6}$$

Using Theorem 1.4 of [13] one can easily verify that $(\sigma_t)_{t \in \mathbf{R}}$ and $(\tau_t)_{t \in \mathbf{R}}$ are one-parameter groups of *-automorphisms and R is an antiisomorphism of the *-algebra $\mathcal{A}$. Moreover $R\tau_t(a) = \tau_t R(a)$ and

$$\begin{aligned} h(\sigma_t(a)) &= h(a), \\ h(\tau_t(a)) &= h(a), \\ h(R(a)) &= h(a) \end{aligned}$$

for any $a \in \mathcal{A}$ and $t \in \mathbf{R}$. Therefore σ_t, τ_t and R are unitarily implemented

$$\begin{aligned} \pi(\sigma_t(a)) &= \Delta^{it}\pi(a)\Delta^{-it}, \\ \pi(\tau_t(a)) &= Q^{2it}\pi(a)Q^{-2it}, \\ \pi(R(a)) &= Z\pi(a^*)Z^*, \end{aligned}$$

where π is the GNS-representation of A introduced in Section 4, Δ and Q are strictly positive selfadjoint operators and Z is an antiunitary operator acting on H.

We assumed that h is faithful. So is π and the above relations show that σ_t, τ_t and R are norm preserving and admit continuous extensions to the whole A.

Statement 3 follows immediately from the formula (1.21).

Combining (7.5) and (7.6) we see that

$$\kappa(a) = R(\tau_{i/2}(a)) \tag{7.7}$$

for any $a \in \mathcal{A}$. To show that $D(\overline{\kappa}) = D(\tau_{i/2})$, it is sufficient to prove that $\mathcal{A}$ is a core for $\tau_{i/2}$. We shall use a regularization operator. Let n be a natural number. For any $a \in A$ we set

$$R_n(a) = \frac{n}{\sqrt{\pi}} \int_{\mathbf{R}} e^{-n^2t^2} \tau_{t/2}(a)dt.$$

We know that $\tau_t(a)$ depends continuously on t. Therefore $\|R_n(a) - a\| \to 0$, when $n \to \infty$. Taking into account (7.5) one can easily show, that $R_n\mathcal{A} \subset \mathcal{A}$. One should notice that $\|R_n(a)\| \leq \|a\|$. Moreover $R_n(a) \in D(\tau_{i/2})$ and $\|\tau_{i/2}(R_n(a))\| \leq e^{n^2}\|a\|$. Indeed $\tau_{i/2}R_n = \frac{n}{\sqrt{\pi}} \int \exp[-n^2(t-i)^2]\tau_{t/2}dt$, $\|\tau_{i/2}R_n\| \leq \frac{n}{\sqrt{\pi}} \int |\exp[-n^2(t-i)^2]|dt$ and the estimate follows.

Let $a \in D(\tau_{i/2})$, $b = \tau_{i/2}(a)$ and $\epsilon > 0$. Then for sufficiently large n we have

$$\|R_n(a) - a\| \leq \frac{\epsilon}{2}, \qquad \|R_n(b) - b\| \leq \frac{\epsilon}{2}.$$

Clearly $\tau_{i/2}R_n(a) = R_n(b)$. We know that $\mathcal{A}$ is dense in A. Let a_ϵ be an element of $\mathcal{A}$ such that

$$\|a_\epsilon - a\| \leq \frac{\epsilon}{2}e^{-n^2}.$$

Then $R_n(a_\epsilon) \in \mathcal{A}$ and using the above estimates we have

$$\|R_n(a_\epsilon) - a\| \leq \|R_n(a_\epsilon - a)\| + \|R_n(a) - a\| \leq \frac{\epsilon}{2}(e^{-n^2} + 1) \leq \epsilon,$$

$$\|\tau_{i/2}(R_n(a_\epsilon)) - b\| \leq \|\tau_{i/2}R_n(a_\epsilon - a)\| + \|R_n(b) - b\| \leq \epsilon.$$

It shows that the closure of $\tau_{i/2}|_{\mathcal{A}}$ coincides with $\tau_{i/2}$ and the statement follows.

Let $a = u^{\beta}_{rs}{}^*$ and $b = u^{\alpha}_{kl}$ (where $\alpha, \beta \in \hat{G}$, $r, s = 1, 2, \ldots, N_\beta$ and $k, l = 1, 2, \ldots, N_\alpha$). Using (5.3), (5.12) and the orthonormality relations (6.18), (6.19) one can check that

$$\kappa(b * ha) = a * hb,$$

$$\kappa(bh * a) = ha * b.$$

By linearity these relations hold for any $a, b \in \mathcal{A}$. Remembering that $\mathcal{A}$ is dense in A we get the remaining part of Statement 4.

The proof of Theorem 1.6 is complete.

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