An example

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Consider an algebraic set in the plane defined by a single equation

$$x^4 - y^3 + 6x^2y + 6y^2 - 2x^2 - 9y \;=\; 0\,. \tag{1}$$

This curve has two cusp-like 'return' points $P_{\pm} = (\pm 2,\, -1)$ and a self-intersection point $P_{\mathrm{self}} = (0,\, 3)$, all of them – critical points of the polynomial on the LHS in (1). (The fourth critical point $(0,\, 1)$ lies well off the curve.) From each of $P_{\pm}$ there emerge a pair of branches.

The minimal regular separation exponent ν in each such pair is $3/2$ – it is an ordinary (simplest) cusp in singularity theory. A rabbit-from-the-hat way to see it is that the curve (1) admits a polynomial parametrization

$$x(t) = t^3 - 3t\,, \qquad y(t) = t^4 - 2t^2.$$

Its Taylor expansion about $t_0 = 1$ is

$$\begin{pmatrix} t^3 - 3t \\ t^4 - 2t^2 \end{pmatrix} = P_- + (t-1)^2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} + (t-1)^3 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + (t-1)^4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2}$$

Hence the Euclidean distance of points (2) for $t = 1 - \epsilon$ and $t = 1 + \epsilon$ is

$$2\sqrt{17}\,\epsilon^3 + O(\epsilon^4)\,,$$

while the distances of these points to the reference point P_- are asymptotically equal $5\,\epsilon^2$ when $\epsilon \to 0^+$. But

$$2\sqrt{17}\epsilon^3 + O(\epsilon(4)) = O\big((5\epsilon^2)^{3/2}\big)$$

when $\epsilon \to 0^+$. So the minimal regular separation exponent ν is $3/2$.