On Generalized Wronskians*

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Abstract

The Wronski determinant (*Wronskian*), usually introduced in standard courses in Ordinary Differential Equations (ODE), is a very useful tool in algebraic geometry to detect ramification loci of linear systems. The present survey aims to describe some "materializations" of the Wronskian and its close relatives, *generalized Wronskians*, in algebraic geometry. Emphasis will be put on the relationships between Schubert Calculus and ODE.

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Introduction

Let $\mathbf{f} := (f_0, f_1, \dots, f_r)$ be an $(r+1)$-tuple of holomorphic functions defined in a neighborhood of the complex line. The *Wronskian* of $\mathbf{f}$ is the holomorphic function $W(\mathbf{f})$ obtained by taking the determinant of the *Wronskian matrix* whose entries of the jth-row, $0 \le j \le r$, are the jth derivatives of $(f_0, f_1, \dots, f_r)$. The first appearance of Wronskians dates back to 1812, introduced by J. M. Hoene–Wronski (1776–1853) in the treatise [27] – see also [42]. The ubiquity of the Wronskian in nearly all the branches of mathematics, from analysis to algebraic geometry, from number theory to combinatorics, up to the theory of infinite dimensional dynamical systems, is definitely surprising if compared with its elementary definition. The present survey aims to draw a path connecting some different Wronskian materializations to make evident their common root. The emphasis will be put on the mutual relationships among linear Ordinary Differential Equations (ODEs), the theory of ramification loci of linear systems (e.g. Weierstrass points on curves) and the intersection theory of complex Grassmann varieties, ruled by the famous *Calculus* [48] elaborated in 1886 by H. C. H. Schubert (1848–1911), to whom the italians M. Pieri (1860–1913) and G. Z. Giambelli (1879–1953) contributed too – see [23, 38].

The notion of Wronskian belongs to mathematicians' common background because of its most popular application, which provides a method (sketched in Section 2) to find a particular solution of a non homogeneous linear ODE. It relies on the following key property: *if the Wronskian of a fundamental system of solutions of a linear homogeneous ODE does not vanish at some point, then it vanishes nowhere in its domain of definition*. The proof of the italicized proposition is due to J. Liouville (1809–1882) and N. H. Abel (1802–1829): it consists in showing that the derivative of the Wronskian is proportional to the Wronskian itself. This apparently innocuous property should be considered as the first historical appearance of Schubert Calculus. To see it, one must embed the Wronski determinant into a full family of *generalized Wronskians*, already used in 1939 by F. H. Schmidt [47] to study Weierstrass points and, in recent times and with the same motivation, by C. Towse in [49]. For a sample of applications to number theory see also [3] and [34].

If $\boldsymbol{\lambda} = (\lambda_0 \ge \lambda_1 \ge \dots \ge \lambda_r)$ is a *partition*, the *generalized Wronskian* $W_{\boldsymbol{\lambda}}(\mathbf{f})$ is the determinant of the matrix whose jth row, for $0 \le j \le r$, is the row of the derivatives of order $j + \lambda_{r-j}$ of $(f_0, f_1, \dots, f_r)$. Clearly $W_0(\mathbf{f}) = W(\mathbf{f})$, where $(0) := (0, \dots, 0)$ denotes the *null partition*. The *derivative of the Wronskian* $W(\mathbf{f})$, used in the proof of Liouville's–Abel's theorem, is the first example of a generalized Wronskian, corresponding to the partition $(1) := (1, 0, \dots, 0)$. The bridge to Schubert Calculus is our generalization of Liouville's and Abel's theorem (see [20, 21]): *Giambelli's formula for generalized Wronskians hold*. More precisely, if $\mathbf{f}$ is a fundamental system of solutions of a linear ODE *with constant coefficients*, then $W_{\boldsymbol{\lambda}}(\mathbf{f})$ is proportional to the usual Wronski determinant, $W_{\boldsymbol{\lambda}}(\mathbf{f}) = \Delta_{\boldsymbol{\lambda}}(\mathbf{h}) W_0(\mathbf{f})$, where $\Delta_{\boldsymbol{\lambda}}(\mathbf{h})$ is the Schur polynomial associated to a sequence $\mathbf{h} = (h_0, h_1, \dots)$ of explicit polynomial expressions in the coefficients of the given ODE and to the partition $\boldsymbol{\lambda}$ – see Section 8. If the characteristic polynomial of the linear differential equation splits into the product of distinct linear factors, then h_j is nothing else than the jth complete symmetric polynomial in its roots.

Let us now change the landscape for a while. A g^r_d on a smooth complex projective curve C of genus $g \geq 0$ is a pair (V, L), where $L \in Pic^d(C)$ and $V \in G(r+1, H^0(L))$ – the Grassmann variety of $(r+1)$-dimensional vector subspaces of the global holomorphic sections of L. If $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ is a basis of V, the *Wronskian* $W(\mathbf{v})$ is a holomorphic section of the bundle $\mathcal{L}_{g,r,d} := L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}$ – see Section 3. It can be constructed by gluing together local Wronskians $W(\mathbf{f})$, where $\mathbf{f} = (f_0, f_1, \ldots, f_r)$ is a $(r+1)$-tuple of holomorphic functions representing the basis $\mathbf{v}$ in some open set of C that trivializes L. As changing the basis of V has the effect of multiplying $W(\mathbf{v})$ by a non zero complex number, one obtains a well defined point $W(V) := W(\mathbf{v}) \bmod \mathbb{C}^* \in \mathbb{P}H^0(\mathcal{L}_{g,r,d})$, the *Wronskian* of V. We call *Wronski map* the holomorphic map $V \mapsto W(V)$. Two extremal cases show that, in general, it is neither injective nor surjective. If C is hyperelliptic and $L \in Pic^2(C)$ is the line bundle defining its unique g^1_2, then $G(2, H^0(L))$ is just a point and the Wronski map to $\mathbb{P}H^0(\mathcal{L}_{g,1,2})$ is trivially injective and not surjective. On the other hand, if $C = \mathbb{P}^1$ and $L = O_{\mathbb{P}^1}(d)$, the Wronski map $G(r+1, H^0(O_{\mathbb{P}^1}(d))) \to \mathbb{P}H^0(\mathcal{L}_{0,r,d})$ is a finite surjective morphism of degree equal to the Plücker degree of the grassmannian $G(r+1, d+1)$: in particular it is not injective [10].

The problem of determining the pre-image of an element of $\mathbb{P}H^0(\mathcal{L}_{0,r,d})$ through the Wronski map defined on $G(r+1, H^0(O_{\mathbb{P}^1}(d)))$ leads to an intriguing mixing of Geometry, Analysis and Representation Theory. In fact certain *non degenerate planes* of $G(r+1, H^0(O_{\mathbb{P}^1}(d)))$, defined through suitable intermediate Wronskians, correspond to critical points of some rational function occurring in the representation theory of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ – see Section 6.4 and [44, 45] for details. Relations between Wronskians, Schubert calculus and rational curves in the real framework were investigated by A. Eremenko, A. Gabrielov, L. Goldberg, V. Karlhamov & F. Sottile, and others – see [11, 12, 24, 29] and references therein. More connections between linear differential equations, projective curves and Schubert varieties appeared in a local context in the investigations of M. Kazarian on singularities of the boundary of fundamental systems of solutions of linear differential equations, [28].

Here, we take another point of view. In his Ph.D. thesis [37], A. Nigro proposes to extend the notion of ramification locus of a linear system on a curve to that of ramification locus of a holomorphic section of a Grassmmann bundle. The construction is motivated by the following observation (see also [8]): Let $\Gamma_{\mathtt{t}}(\rho_{r,d})$ be the set of all the sections $\gamma : C \to G(r+1, J^dL)$ of $\rho_{r,d}$ such that the pull back of the tautological bundle $\mathcal{S}_r$ over $G(r+1, J^dL)$ is trivial. Then each $g^r_d := (V, L)$ induces a holomorphic section $\gamma_V \in \Gamma_{\mathtt{t}}(\rho_{r,d})$, via the bundle monomorphism $C \times V \to J^dL$ (Cf. Section 5.3). The point is that the space $\Gamma_{\mathtt{t}}(\rho_{r,d})$ is larger than the space of linear systems, and so the theory becomes richer. A distinguished subvariety lives in $G(r+1, J^dL)$, called *Wronskian subvariety* in [37]. It is a Cartier divisor which occurs as the zero locus of a certain *Wronskian section* $\mathbb{W}$. The

Wronskian of any section γ of $\rho_{r,d}$ is defined to be $W_0(\gamma) := \gamma^*\mathbb{W}$ and if $\gamma = \gamma_V$ for some $V \in G(r+1, H^0(L))$, it coincides with the usual Wronskian of V – see Section 5. In particular, if $\mathcal{M}$ is a line bundle defining the unique g^1_2 over a hyperelliptic curve of genus $g \geq 2$, the extended Wronski map $\Gamma_{\mathtt{t}}(\rho_{1,2}) \to \mathbb{P}H^0(\mathcal{M}^{\otimes 2} \otimes K)$ is dominant (see [8]), a behavior closer to the surjectivity of the Wronski map defined on the space of g^r_ds on $\mathbb{P}^1$. The latter, in this case, coincides with $\Gamma_{\mathtt{t}}(\rho_{r,d})$ modulo the identification of V with γ_V.

In general, the construction works as follows. Let $\varrho : F \to X$ be a vector bundle of rank $d+1$ and let $\varrho_{r,d} : G := G(r+1, F) \to X$ be the Grassmann bundle of $(r+1)$-dimensional subspaces of fibers of ϱ. Let $0 \to \mathcal{S}_r \to \varrho^*_{r,d}F \to \mathcal{Q}_r \to 0$ be the universal exact sequence over G. As well known (see e.g. [14, Ch. 14]) the Chow group $A_*(G)$ of cycles modulo rational equivalence is a free $A^*(X)$-module generated by $\mathcal{B} := \{\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \varrho^*_{r,d}F)) \cap [G] \,|\, \boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}\}$, where $\mathcal{P}^{(r+1)\times(d-r)}$ denotes the set of all partitions $\boldsymbol{\lambda}$ such that $\lambda_0 \leq d-r$, where $\cap[G]$ denotes the *cap* product with the fundamental class of G and where $\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \varrho^*_{r,d}F))$ is a Schur polynomial involving the Chern classes of both $\mathcal{Q}_r$ and F. Let $F_\bullet := (F_i)_{d\geq i\geq 0}$ be a filtration of $F_\bullet$ by quotient bundles, such that F_i has rank i. Schubert varieties $\{\Omega_{\boldsymbol{\lambda}}(F_\bullet) \,|\, \boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}\}$ can be associated to $F_\bullet$, see Section 4.4, which play the role of *generalized Wronskian subvarieties*. In particular $\Omega_{(1)}(F^\bullet)$ is what in [37] is called $F^\bullet$-*Wronskian subvariety* of G. It is a cartier divisor which is the zero locus of a section $\mathbb{W}$ of the bundle $\bigwedge^{r+1} \varrho^*_{r,d}F_{r+1} \otimes \bigwedge^{r+1} \mathcal{S}_r^\vee$ over G, said to be $F_\bullet$-*Wronskian*. If $\gamma : X \to G$ is a holomorphic section, its Wronskian is, by definition, $W(\gamma) = \gamma^*\mathbb{W} \bmod \mathbb{C}^* \in \mathbb{P}H^0(\bigwedge^{r+1} F \otimes \bigwedge^{r+1} \gamma^*\mathcal{S}_r^\vee)$ and its class in $A_*(X)$ is nothing else than $\gamma^*[\Omega_{(1)}(F_\bullet)] \cap [G]$. In general, the generalized Wronskian class $[W_{\boldsymbol{\lambda}}(\gamma)]$ of γ in $A_*(X)$ is $\gamma^*[\Omega_{\boldsymbol{\lambda}}(F_\bullet)] \cap [X]$, which is the class of $\gamma^{-1}(\Omega_{\boldsymbol{\lambda}}(F_\bullet)$ provided that the codimension of the locus coincidew with the expected codimension $|\boldsymbol{\lambda}|$. Recall that $\Omega_{\boldsymbol{\lambda}}(F_\bullet)]$ can be easily computed as an explicit linear combination of the element of the basis $\mathcal{B}$ above, for instance the recipe indicated in Section 4, especially Theorem 4.13.

Let now $\varepsilon_i = c_i(\mathcal{S}_r) \in A^*(G)$ be the Chern classes of the tautological bundle $\mathcal{S}_r \to G$ and consider a basis $\mathbf{v} := (v_0, v_1, \ldots, v_r)$ of solutions of the differential equation

$$y^{(r+1)} - \varepsilon_1 y^{(r)} + \ldots + (-1)^{r+1}\varepsilon_{r+1} y = 0, \tag{1}$$

taken in the algebra $A^*(G)[[t]]$ of formal power series in an indeterminate t with coefficients in the Chow ring of G. In Section 8.12 we show that, for each partition $\boldsymbol{\lambda} \in \mathcal{P}^{(d-r)\times(r+1)}$:

$$\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \rho^*_{r,d}F)) = \frac{W_{\boldsymbol{\lambda}}(\mathbf{v})}{W(\mathbf{v})},$$

i.e. each element of the $A^*(X)$-basis of the Chow ring of G is the quotient of Wronskians associated to a fundamental system of solutions of an ordinary linear ODE with constant coefficients (taken in $A*(G)$), as a consequence of *Giambelli's formula* for generalized Wronskians, proven in [20, 21].

The path we wanted to walk is now complete. Indeed, it is now evident that Schubert calculus on a Grassmann bundle, generalized Wronskians of linear systems on curves and

Wronskians of fundamental systems of solutions of linear ODEs are all characters of the same play.

1 Wronskians, in General

1.1 In the next two sections let $\mathbb{K}$ be either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$ together with their usual euclidean topologies. If $U \subseteq \mathbb{K}$ is an open set of $\mathbb{K}$, we shall write $\mathcal{O}(U)$ for the $\mathbb{K}$-algebra of *regular* $\mathbb{K}$-valued functions defined over U: here *regular* means either C^∞ differentiable if $\mathbb{K} = \mathbb{R}$ or complex holomorphic if $\mathbb{K} = \mathbb{C}$. Let:

$$\mathbf{v} := (v_0, v_1, \dots, v_r) \in \mathcal{O}(U)^{r+1} \tag{2}$$

be a $(r+1)$-tuple of functions in $\mathcal{O}(U)$ for some open set $U \subseteq \mathbb{K}$. If t is a local parameter on U, we denote by $D : \mathcal{O}(U) \to \mathcal{O}(U)$ the usual derivation d/dt. The *Wronskian matrix* associated to the $(r+1)$-tuple (2) is the matrix valued regular function:

$$WM(\mathbf{v}) := \begin{pmatrix} \mathbf{v} \\ D\mathbf{v} \\ \vdots \\ D^r\mathbf{v} \end{pmatrix} = \begin{pmatrix} v_0 & v_1 & \dots & v_r \\ Dv_0 & Dv_1 & \dots & Dv_r \\ \vdots & \vdots & \ddots & \vdots \\ D^r v_0 & D^r v_1 & \dots & D^r v_r \end{pmatrix}.$$

The *Wronskian determinant* is the determinant of (2):

$$W_0(\mathbf{v}) = \det WM(\mathbf{v}).$$

It will be shortly said the *Wronskian* of $\mathbf{v} := (v_0, v_1, \dots, v_r)$ and will be often written in the form:

$$W_0(\mathbf{v}) := \mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge D^r\mathbf{v}. \tag{3}$$

In this paper, however, we want to see Wronskians as a part of a full family of natural functions generalizing them. They will be called, following the few pieces of literature where they have already appeared ([3], [49]) *generalized wronsians*.

1.2 Generalized Wronskians. Let $r \geq 0$ be an integer. A partition $\boldsymbol{\lambda}$ of length at most $r+1$ is a non increasing sequences of non-negative integers:

$$\boldsymbol{\lambda} : \quad \lambda_0 \geq \lambda_1 \geq \ldots \geq \lambda_r \geq 0. \tag{4}$$

The *weight* of $\boldsymbol{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_r)$ is $|\boldsymbol{\lambda}| = \sum_{i=0}^{r} \lambda_j$. Hence $\boldsymbol{\lambda}$ is a partition of the integer $|\boldsymbol{\lambda}|$. To each partition one may associate a Young–Ferrers diagram, an array of left justified rows, with λ_0 boxes in the first row, λ_1 boxes in the second row, …, λ_r-boxes in the $(r+1)$th row. We denote by $\mathcal{P}$ the set of all partitions and by $\mathcal{P}^{(r+1)\times(d-r)}$ the set of all partitions whose Young diagram is contained in a $(r+1)\times(d-r)$ rectangle, i.e. the set of all partitions $\boldsymbol{\lambda}$ such that:

$$d-r\geq\lambda_0\geq\lambda_1\geq\ldots\geq\lambda_0\geq 0.$$

If the last $r-h$ parts of $\boldsymbol{\lambda}\in\mathcal{P}_r$ are zeros, then we write simply $\boldsymbol{\lambda}=(\lambda_0,\lambda_1,\ldots,\lambda_h)$, omitting the last zero parts. For more on partitions, see [33].

1.3 Definition. *Let* $\mathbf{v}$ *as in (2) and* $\boldsymbol{\lambda}$ *as in (4). The* generalized Wronskian matrix *associated to* $\mathbf{v}$ *and to the partition* $\boldsymbol{\lambda}$ *is, by definition:*

$$WM_{\boldsymbol{\lambda}}(\mathbf{v}):=\begin{pmatrix} D^{\lambda_r}\mathbf{v}\\ D^{1+\lambda_{r-1}}\mathbf{v}\\ \vdots\\ D^{r+\lambda_0}\mathbf{v}\end{pmatrix}:=\begin{pmatrix} D^{\lambda_r}v_0 & D^{\lambda_r}v_1 & \ldots & D^{\lambda_r}v_r\\ D^{1+\lambda_{r-1}}v_0 & D^{1+\lambda_{r-1}}v_1 & \ldots & D^{1+\lambda_{r-1}}v_r\\ \vdots & \vdots & \ddots & \vdots\\ D^{r+\lambda_0}v_0 & D^{r+\lambda_0}v_1 & \ldots & D^{r+\lambda_0}v_r\end{pmatrix}$$

The $\boldsymbol{\lambda}$-generalized Wronskian *is the determinant of the Wronskian matrix:*

$$W_{\boldsymbol{\lambda}}(\mathbf{v}):=\det WM_{\boldsymbol{\lambda}}(\mathbf{v})$$

Coherently with (3) we shall write the $\boldsymbol{\lambda}$-generalized Wronskian in the form:

$$W_{\boldsymbol{\lambda}}(\mathbf{v}):=D^{\lambda_r}\mathbf{v}\wedge D^{1+\lambda_{r-1}}\mathbf{v}\wedge\ldots\wedge D^{r+\lambda_0}\mathbf{v}. \tag{5}$$

Clearly $W_0(\mathbf{v})=W(\mathbf{v})$.

1.4 Remark. Notation (3) and (5) is convenient because the derivative of any generalized Wronskian can be computed via a Leibniz's rule with respect to the product "$\wedge$". In other words:

$$\begin{aligned} D(W_{\boldsymbol{\lambda}}(\mathbf{v})) &= D(D^{\lambda_r}\mathbf{v}\wedge D^{1+\lambda_{r-1}}\mathbf{v}\wedge\ldots\wedge D^{r+\lambda_0}\mathbf{v})=\\ &= \sum_{\substack{0\leq i_0+i_1+\ldots+i_r=1\\ i_j\geq 0}} D^{i_0+\lambda_r}\mathbf{v}\wedge D^{1+i_1+\lambda_{r-1}}\mathbf{v}\wedge\ldots\wedge D^{r+i_r+\lambda_0}\mathbf{v}.\end{aligned}$$

A simple induction shows that the h-th derivative D^h of a generalized Wronskian $W_{\boldsymbol{\lambda}}(\mathbf{v})$ is a $\mathbb{Z}$-linear combination of generalized Wronskians. Recall, as in Section 1.2, that partitions can be described via Young–Ferrers diagrams, and that a standard *Young tableau* is a filling in the Young–Ferrers diagram of $\boldsymbol{\lambda}$ with integers $1,\ldots,|\boldsymbol{\lambda}|$ arranged in an increasing order in each column and each row [15]. The speculations in our final Section 8, where we show that Schubert Calculus can be recast in terms of *Wronski Calculus* implies the following result:

1.5 Theorem. *the hth derivative of the Wronskian can be written as*

$$D^hW(\mathbf{f})\;=\;\sum_{|\boldsymbol{\lambda}|=k}c_{\boldsymbol{\lambda}}W_{\boldsymbol{\lambda}}(\mathbf{f})\,,$$

where $c_{\boldsymbol{\lambda}}$ *is the number of the standard Young tableaux of the Young–Ferrers diagram* $\boldsymbol{\lambda}$. ∎

The coefficients $c_{\boldsymbol{\lambda}}$'s and their interpretation in terms of Schubert calculus are very well known and can be calculated by the *hook formula*:

$$c_{\boldsymbol{\lambda}} = \frac{|\lambda|!}{k_1 \cdot \ldots \cdot k_{|\lambda|}}$$

where k_j's , $1 \leq j \leq |\boldsymbol{\lambda}|$ are the *hook lengths* of the boxes of $\boldsymbol{\lambda}$ – Cf. [15, p. 53].

2 Wronskians and Linear ODEs

Wronskians are often introduced when dealing with linear *Ordinary Differential Equations* (ODEs).

2.1 Let $\mathbf{a}(t) = (a_1(t), \ldots, a_{r+1}(t)) \in \mathcal{O}(U)^{r+1}$ and let

$$D^{r+1}x - a_1(t)D^r x + \ldots + (-1)^{r+1}a_{r+1}(t)x = f \tag{6}$$

be a linear ODE where $f \in \mathcal{O}(U)$. Let $\mathcal{S}_{f,\mathbf{a}}$ be the set of solutions of (6). Simple linear algebra shows that $\mathcal{S}_{f,\mathbf{a}}$ is an affine space modeled over $\mathbb{K}^{r+1}$: if x_p is a *particular solution* of (6), then:

$$\mathcal{S}_{f,\mathbf{a}} = x_p + \ker P_{\mathbf{a}}(D),$$

where $P_{\mathbf{a}}(D) \in End_{\mathbb{K}}(\mathcal{O}(U))$ is the linear differential operator:

$$P_{\mathbf{a}}(D) := D^{r+1} - a_1(t)D^r + \ldots + (-1)^{r+1}a_{r+1}(t).$$

The celebrated Cauchy theorem ensures that given a column $\mathbf{c} = (c_i)_{0 \leq i \leq r} \in \mathbb{K}^{r+1}$, there exists a unique element $f \in \ker P_{\mathbf{a}}(D)$ such that $D^j f(0) = c_j$, for all $0 \leq j \leq r$. Assume now that $\mathbf{v}$ as in (2) is a basis of $\ker P_{\mathbf{a}}(D)$. A particular solution of (6) can be found through the method of *variation of arbitrary constants*. Assume that

$$x_p := (\mathbf{v} \cdot \mathbf{c})(t) = \mathbf{v}(t) \cdot \mathbf{c}(t) = \sum_{i=0}^{r} c_i(t)v_i(t)$$

is a solution of (6), where

$$\mathbf{c}(t) = \begin{pmatrix} c_0(t) \\ c_1(t) \\ \vdots \\ c_r(t) \end{pmatrix}$$

is a column of $r+1$ element of $\mathcal{O}(U)$ and "$\cdot$" is the usual row-by-column product. We look for solutions $\mathbf{v} \cdot \mathbf{c}$ of (6) such that $D^j \mathbf{v} \cdot D\mathbf{c} = 0$ for all $0 \leq j \leq r$. Then

$$D^{r+1}\mathbf{v} \cdot D\mathbf{c} = (e_1 D^r v - e_2 D^{r-1}\mathbf{v} + \ldots + (-1)^r e_r \mathbf{v} + f) \cdot D\mathbf{c} = f,$$

and the unknown functions $\mathbf{c}$ must satisfy the differential equations:

$$WM(\mathbf{v})\begin{pmatrix} Dc_0 \\ Dc_1 \\ \vdots \\ Dc_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f \end{pmatrix}.$$

The very important point is that the *Wronskian matrix is invertible in* $\mathcal{O}(U)$, because in this case:

$$D\mathbf{c} = WM(\mathbf{v})^{-1} \cdot \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f \end{pmatrix}$$

is a system of first order ODEs which can be solved by usual methods. The fact that the Wronskian matrix is invertible is very important from our point of view: let us show it. First, it is not identically 0 because the v_i were assumed to be linearly independent. In addition, if it does not vanish at some point of U it does vanish nowhere on U. Assume that $W_0(\mathbf{v})(P) \neq 0$ for some $P \in U$ and let us choose a local parameter t on U which is 0 at P, identifying the open set U with a neighbourhood of the origin. The proof consists in computing the derivative of the Wronskian to discover the celebrated

2.2 Liouville's Theorem ([4, p. 195, §27.6]). *The Wronskian satisfies the differential equation:*

$$DW_0(\mathbf{v}) = a_1(t)W_0(\mathbf{v}). \tag{7}$$

The proof of Theorem 2.2 is as follows. By defining $D\mathbf{v}$ as the row whose entries are the derivatives of the entries of $\mathbf{v}$, one notices that

$$P_{\mathbf{a}}(D)\mathbf{v} = (P_{\mathbf{a}}(D)v_0, P_{\mathbf{a}}(D)v_1, \ldots, P_{\mathbf{a}}(D)v_r) = \mathbf{0}$$

and then $D^{r+1}\mathbf{v} = a_1(t)D^r\mathbf{v} - a_2(t)D^{r-1}\mathbf{v} + \ldots + (-1)^r a_{r+1}(t)\mathbf{v}$. Thus:

$$\begin{aligned} DW_0(\mathbf{v}) &= D(\mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge D^r\mathbf{v}) = \mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge D^{r-1}\mathbf{v} \wedge D^{r+1}\mathbf{v} \\ &= \mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge (a_1(t)D^r\mathbf{v} - a_2(t)D^{r-1}\mathbf{v} + \ldots + (-1)^r\mathbf{v}) = \\ &= a_1(t)\mathbf{v} \wedge D\mathbf{v} \wedge \ldots \wedge D^r\mathbf{v} = a_1(t)W(\mathbf{v}). \end{aligned}$$

The Wronskian assumes then the form (Abel's formula):

$$W_0(\mathbf{v}) = W_0(\mathbf{v})(0) \cdot \exp(\int_0^t a(u)du), \tag{8}$$

where $W_0(\mathbf{v})(0)$ denotes the value of the Wronskian at $t = 0$. Equation (8) shows that if $W(\mathbf{v})(0) \neq 0$ then $W(\mathbf{v})(t) \neq 0$ for all $t \in U$. We shall see in Section 8 why the proof of Liouville's theorem is a first example of the *Schubert Calculus* formalism governing the intersection theory on Grassmann Schemes.

2.3 Generalized Wronkians of Solutions of ODEs. Using *generalized Wronskians* as in 1.2, Liouville's theorem (7) can be rephrased as:

$$W_{(1)}(\mathbf{v}) = a_1(t)W(\mathbf{v})$$

and can be easily generalized as follows.

2.4 Proposition. *Let* $1^k := (1, 1, \dots, 1)$ *be the* primitive partition *of the integer* k*, i.e. with* k *parts equal to* 1*. If* $\mathbf{v}$ *is a basis of* $\ker P_{\mathbf{a}}(D)$ *then:*

$$W_{(1^k)}(\mathbf{v}) = a_k(t)W(\mathbf{v}). \tag{9}$$

Proof. Let $v \in \ker P_{\mathbf{a}}(D)$. Then v is a $\mathbb{K}$-linear combination of $(v_0, v_1 \dots, v_r)$ and hence

$$W(v, v_0, v_1, \dots, v_r) = 0.$$

By expanding the Wronskian along the first column one obtains the equation:

$$W_0(\mathbf{v})D^{r+1}v - W_{(1)}(\mathbf{v})D^r v + \dots + (-1)^{r+1}W_{(1^{r+1})}(\mathbf{v})v = 0, \tag{10}$$

and the equation $D^{r+1}v = a_1(t)D^r v - a_2(t)D^{r-1}v + \dots + (-1)^r a_{r+1}(t)v$ implies:

$$\sum_{k=0}^{r}(-1)^k(W_{(1^k)}(\mathbf{v}) - a_k(t)W(\mathbf{v}))D^k v = 0. \tag{11}$$

For general $v \in \ker P_{\mathbf{a}}(D)$, the $(r+1)$-tuple $(v, Dv, \dots, D^r v)$ is linearly independent, and then (11) implies (9) for all $1 \le k \le r+1$.

2.5 Although it may not seem that relevant, a natural question arises. Can we conclude that any generalized Wronskian $W_{\boldsymbol{\lambda}}(\mathbf{v})$ associated to a basis of $\ker P_{\mathbf{a}}(D)$ is multiple of the Wronskian $W_0(\mathbf{v})$? The answer is obviously yes. In fact whenever one encounters one exterior factor in the generalized Wronskian of the form $D^{j+\lambda_{r-j}}\mathbf{v}$ with $j + \lambda_{r-j} \ge r+1$, one uses the differential equation to express $D^{j+\lambda_{r-j}}\mathbf{v}$ as a linear combination of lower derivatives of the vector $\mathbf{v}$, with coefficients polynomial expressions in $\mathbf{a}$ and its derivatives:

$$W_{\boldsymbol{\lambda}}(\mathbf{v}) = G_{\boldsymbol{\lambda}}(\mathbf{a}, D\mathbf{a}, D^2\mathbf{a}, \dots)W(\mathbf{v}).$$

The coefficient $G_{\boldsymbol{\lambda}}(\mathbf{a}, D\mathbf{a}, D^2\mathbf{a}, \dots)$ assumes a particular interesting form in the case the coefficients $\mathbf{a}$ of the equation are constant (so $D_i\mathbf{a} = 0$, for $i > 0$). One of our tasks will be to describe the form of the coefficients $G_{\boldsymbol{\lambda}}$ in this particular case, right after exploring more occurrences of the Wronskians in other mathematical contexts.

3 Wronskian Sections of Line bundles.

3.1 In the attempt to keep the paper as self contained as possible, we recall a few basic notions about holomorphic vector bundles. A holomorphic vector bundle of rank $d+1$ on a smooth complex projective variety X is a holomorphic locally trivial family of vector spaces, according to [25, p. 69], i.e. a holomorphic map $\rho : F \to X$, where F is a complex manifold which is locally a product in the following sense. There is an open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X and a biholomorphic map $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^{d+1}$ such that $pr_1 \circ \phi_\alpha = \pi$, where pr_1 is the projection onto the first factor. Let $f_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl_{d+1}(\mathbb{C})$ defined as:

$$\phi_\alpha \circ \phi_\beta^{-1}(P, v_\beta) = (P, f_{\alpha\beta}(P)\cdot v_\beta).$$

Then

$$\begin{cases} f_{\alpha\alpha} & = & id_{U_\alpha} \\ & & \\ f_{\alpha\beta}\cdot f_{\beta\gamma} & = & f_{\alpha\gamma} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma \end{cases} \tag{12}$$

If $P \in X$, the *fiber* of F at P is an n-dimensional complex vector space which will be denoted, throughout the paper, by F_P. A *holomorphic section* $s = (s_\alpha)$ of F is a collection of holomorphic functions $s_\alpha : U_\alpha \to \mathbb{C}^{d+1}$ such that $s_\alpha(P) = f_{\alpha\beta}(P)s_\beta(P)$, for each $\alpha, \beta \in \mathcal{A}$ and each $P \in U_\alpha \cap U_\beta$. A holomorphic section is then a holomorphic map $s : X \to F$, defined by $s(P) = \phi_\alpha^{-1}(P, s_\alpha(P))$ where $\phi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^{d+1}$ is a trivialization of F over $U_\alpha \ni P$. The definition does not depend on the open trivializing set containing P. The vector space of global holomorphic sections of F will be denoted $H^0(F) := H^0(X, F)$ (omitting the base variety when clear from the context). The expression $s(P)$ will denote the value of s at the point $P \in X$, thought of as a point of the fiber F_P of F at P, and not the image s_P of s in the stalk of the *sheaf of sections of* F at P. Conversely, given a collection of holomorphic functions $f_{\alpha\beta} : U_\alpha \cap U_\beta \to Gl_{d+1}(\mathbb{C})$ satisfying (12) there is a unique vector bundle having $\{f_{\alpha\beta}\}$ as cocycle associated to the covering $\{U_\alpha\}$, which can be easily constructed via a standard paste and glue procedure. A *line bundle* over X is a vector bundle of rank 1. The set of isomorphism classes of line bundles on X is a group under tensor product and is denoted by $Pic(X)$. If $\pi : X \to Y$ is a proper flat morphism, we denote by $Pic(X/S)$ the group of isomorphism classes of relative line bundles on X. A *relative line bundle* is an equivalence class of line bundles on X where $\mathcal{L}_1$ and $\mathcal{L}_2$ are declared equivalent if $\mathcal{L}_1 \otimes \mathcal{L}_2^{-1} = \pi^*\mathcal{N}$, for some $\mathcal{N} \in Pic(S)$. In other words $Pic(X/S) = Pic(X)/\pi^*Pic(S)$.

3.2 From now on, let C be a smooth projective complex curve. It will be often identified with a *compact Riemann surface*, i.e. with a complex manifold of complex dimension 1 equipped with a holomorphic atlas $\mathfrak{A} := \{(U_\alpha, z_\alpha) \,|\, \alpha \in \mathcal{A}\}$. In this context, denote by $\mathcal{O}_C$ the sheaf of holomorphic functions on C: if U is an open set, $\mathcal{O}_C(U)$ is the $\mathbb{C}$-algebra of complex holomorphic functions defined over U.

A *line bundle* $L \to C$ is completely determined by an open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of C together with its transition cocycle $\ell_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{C}^*$, i.e. the functions $\{\ell_{\alpha\beta}\}$ satisfy

$\ell_{\alpha\alpha} = 1$ and $\ell_{\alpha\beta} \cdot \ell_{\beta\gamma} = \ell_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. Up to shrinking the open sets of $\mathfrak{A}$ we may assume that the underlying covering of the atlas of C trivializes L as well. The compactness of C guarantees that the dimension $h^0(L)$ of the space $H^0(L) := H^0(C, L)$ of the global holomorphic sections is finite dimensional [13], [26].

The *canonical line* bundle of C is the line bundle whose transition functions with respect to the underlying covering of any holomorphic atlas $\{(U_\alpha, z_\alpha) \,|\, \alpha \in \mathcal{A}\}$ are the derivatives of the coordinate changes:

$$\kappa_{\alpha\beta} := \frac{dz_\alpha}{dz_\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*.$$

The local non zero holomorphic functions $\{\kappa_{\alpha\beta}\}$ obviously form a cocycle. A global holomorphic section $\omega \in H^0(C, K)$ is a *global holomorphic differential*, i.e. a collection $\{f_\alpha \, dz_\alpha\}$, where $f_\alpha \in \mathcal{O}(U_\alpha)$ and $f_{\alpha|_{U_\alpha \cap U_\beta}} = \kappa_{\alpha\beta} f_{\beta|_{U_\alpha \cap U_\beta}}$ and we shall write $\omega_{|_{U_\alpha}} = f_\alpha \, dz_\alpha$. The integer $g = h^0(K) := \dim_{\mathbb{C}} H^0(K)$ is said to be the *genus of the curve*.

3.3 Jets of line bundles. Let $\pi : \mathfrak{X} \longrightarrow S$ be a proper flat family of smooth projective curves of genus $g \geq 1$ parameterized by some smooth scheme S. Let $\mathfrak{X} \times_S \mathfrak{X} \to S$ be the 2-fold fiber product of $\mathfrak{X}$ over S and let $p, q : \mathfrak{X} \times_S \mathfrak{X} \to \mathfrak{X}$ be the projections onto the first and the second factor respectively. Denote by $\delta : \mathfrak{X} \to \mathfrak{X} \times_S \mathfrak{X}$ be the diagonal morphism and $\mathcal{I}$ the ideal sheaf of the diagonal in $\mathfrak{X} \times_S \mathfrak{X}$. The *relative canonical bundle* of the family π is by definition $K_\pi := \delta^*(\mathcal{I}/\mathcal{I}^2)$. For each $\mathcal{L} \in Pic(\mathfrak{X}/S)$, see 3.1, and each $h \geq 0$ let

$$J^h\mathcal{L} := p_*\left(\frac{O_{\mathfrak{X}\times_S\mathfrak{X}}}{\mathcal{I}^{h+1}} \otimes q^*\mathcal{L}\right) \tag{13}$$

be the bundle of *jets* (or *principal parts*) of L of order h. As $\mathfrak{X}$ is smooth, $J^h\mathcal{L}$ is a vector bundle on $\mathfrak{X}$ of rank $h+1$.

By definition $J^0\mathcal{L} = \mathcal{L}$. Let us set, by convention, $J^{-1}\mathcal{L} = 0$ – the vector bundle of rank 0. The fiber of $J^h\mathcal{L}$ over $P \in \mathfrak{X}$ will be denoted by $J^h_P\mathcal{L}$ – a complex vector space of dimension $h+1$. The obvious exact sequence

$$0 \longrightarrow \frac{\mathcal{I}^h}{\mathcal{I}^{h+1}} \longrightarrow \frac{O_{\mathfrak{X}\times_S\mathfrak{X}}}{\mathcal{I}^{h+1}} \longrightarrow \frac{O_{\mathfrak{X}\times_S\mathfrak{X}}}{\mathcal{I}^h} \longrightarrow 0,$$

gives rise to an exact sequence (See [31, p. 224] for details):

$$0 \longrightarrow \mathcal{L} \otimes K_\pi^{h+1} \longrightarrow J^h\mathcal{L} \xrightarrow{t_{h,h-1}} J^{h-1}\mathcal{L} \longrightarrow 0. \tag{14}$$

In the case when $C \to \{pt\}$ is a trivial family over a point, a single curve, and L is any line bundle, the exact sequence (14) for J^hL holds the same: in this case the relative canonical bundle is the same as the canonical bundle K of the curve.

3.4 Let $v = (v_\alpha)$ be a non constant holomorphic section of a line bundle L, i.e. $v_\alpha \in \mathcal{O}(U_\alpha)$ and $v_\alpha = \ell_{\alpha\beta} \cdot v_\beta$ on $U_\alpha \cap U_\beta$. Let (U_α, z_α) be a coordinate chart of C trivializing L. Denote by $D_\alpha : \mathcal{O}(U_\alpha) \to \mathcal{O}(U_\alpha)$ the derivation d/dz_α and by D_α^j the jth iterated of D_α. Then

$$D_h v = \left\{ \begin{pmatrix} v_\alpha \\ D_\alpha v_\alpha \\ \vdots \\ D_\alpha^h v_\alpha \end{pmatrix} \mid \alpha \in \mathcal{A} \right\}$$

is a section of $J^h L$ – see [8]. It may thought of as a *global derivative* of order h of the section v. In fact it locally looks like a local representation of v together with its first h derivatives.

The truncation morphism occurring in (14), $t_{h,h-1} : J^h L \to J^{h-1} L$, is defined in such a way that $t_h(D_h \lambda(P)) = (D_{h-1}\lambda)(P)$. See [8] for further details.

3.5 One says that $v \in H^0(L)$ vanishes at $P \in C$ with multiplicity at least $h + 1$ if $(\mathcal{D}_h v)(P) = 0$. Concretely, if $v_\alpha \in \mathcal{O}_C(U_\alpha)$ is the local representation of v in the open set U_α, then v vanishes at $P \in U_\alpha$ with multiplicity at least $h + 1$ if v_α vanishes at P together with all of its first h derivatives. The fact that $D_h v$ is a section of $J^h L$ says that the definition of vanishing at a point P does not depend on the open set U_α containing it.

We also say that the order of v at P is $h \geq 0$ if $D_{h-1}v(P) = 0$ and $D_h v(P) \neq 0$. To each $0 \neq v \in H^0(L)$ one may attach a divisor on C:

$$(v) = \sum_{P \in C} (ord_P v) P \tag{15}$$

The sum (15) is finite because v is locally a holomorphic functions and hence its zeros are isolated and the compactness of C implies that they are finitely many. The *degree* of v is $\sum_{P \in C} ord_P \lambda \geq 0$. As this number does not depend on the holomorphic section of L, this is by definition the *degree* of L. The degree of the canonical bundle is $2g - 2$ [1, p. 8]. The isomorphism classes of line bundles of degree d is denoted by $Pic^d(C)$. If $\pi : \mathfrak{X} \longrightarrow S$ is a smooth proper family of smooth curves of genus g, then $Pic^d(\mathfrak{X}/S)$ denotes the relative line bundles of relative degree d. A bundle $\mathcal{L} \in Pic(\mathfrak{X}/S)$ has relative degree d if $\deg(\mathcal{L}_{|\mathfrak{X}_s}) = d$ for each $s \in S$.

3.6 If U is a (finite dimensional complex) vector space, $G(k, U)$ will denote the grassmannian variety parameterizing k-dimensional vector subspaces of U. A $g^r_d(L)$ on C is a point of $G(r + 1, H^0(L))$, where $L \in Pic^d(C)$. A g^r_d is a $g^r_d(L)$ for some $L \in Pic^d(C)$. If $E = \sum e_P P$ is an effettive divisor on C, and V is a $g^r_d(L)$, let

$$V(-E) := \{v \in V \mid ord_P v \geq e_P\},$$

The subspace $V(-E)$ of V is not empty because it contains at least the zero section. If $\dim V(-P) = r$ for all $P \in C$, then the $g^r_d(L)$ is said to be *base point free*. It is very ample if $\dim_{\mathbb{C}} V(-P-Q) = r-1$ for all $(P,Q) \in C \times C$. If V is base point free and $\mathbf{v} := (v_0, v_1, \ldots, v_r)$ is a basis of V, the map

$$\begin{cases} \phi_{\mathbf{v}} & : & C & \longrightarrow & \mathbb{P}^r \\ & & P & \longmapsto & (v_0(P) : v_1(P) : \ldots : v_r(P)) \end{cases} \tag{16}$$

is a morphism whose image is a projective algebraic curve of degree d. Although the complex value of a section at a point is not well defined, the ratio of two sections is. Thus the map (16) is well defined. If V is very ample, (16) is an embedding, i.e. a biholomorphism onto its image.

3.7 Let $\boldsymbol{\omega} := (\omega_0, \omega_1, \ldots, \omega_{g-1})$ be a basis of $H^0(K)$. The map

$$\phi_{\boldsymbol{\omega}} := (\omega_0 : \omega_1 : \ldots : \omega_{g-1}) : C \to \mathbb{P}^{g-1}.$$

sending $P \mapsto (\omega_0(P) : \omega_1(P) : \ldots : \omega_{g-1}(P))$ is the *canonical morphism*: its image in $\mathbb{P}^{g-1}$ is a curve of degree $2g-2$. If the *canonical morphism* is not an embedding, the curve is called *hyperelliptic*. All the curves of genus 2 are hyperelliptic and the canonical morphism

$$(\omega_0 : \omega_1) : C \longrightarrow \mathbb{P}^1$$

is a ramified double covering of the projective line. All the hyperelliptic curves are ramified double coverings of the projective line: in fact they possess a line bundle $\mathcal{M}$ of degree 2 such that $h^0(\mathcal{M}) = 2$.

3.8 Definition. *Let V be a $g^r_d(L)$. A point $P \in C$ is a V-ramification point if there exists $0 \neq v \in V$ such that $D_r v(P) = 0$, i.e. iff there exists a non-zero $v \in V$ vanishing at P with multiplicity $r+1$ at least.*

Ramification points of a g^r_d can be detected as zero loci of suitable Wronskians. Let

$$\mathbf{v} := (v_0, v_1, \ldots, v_r)$$

be a basis of V and let $v_{i,\alpha} : U_\alpha \to \mathbb{C}$ be holomorphic functions representing the restriction of the section v_i to U_α, for $0 \leq i \leq r$. If $P \in U_\alpha$ is a V-ramification point, let $v = \sum_{i=0}^r a_i v_i$ be such that $D^r v(P) = 0$. This last condition translates into the following linear system:

$$WM_\alpha(\mathbf{v}) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_r \end{pmatrix} := \begin{pmatrix} v_{0,\alpha} & v_{1,\alpha} & \ldots & v_{r,\alpha} \\ D_\alpha v_{0,\alpha} & D_\alpha v_{1,\alpha} & \ldots & D_\alpha v_{r,\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ D^r_\alpha v_{0,\alpha} & D^r_\alpha v_{1,\alpha} & \ldots & D^r_\alpha v_{r,\alpha} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

which admits a non trivial solution if and only if the determinant of the Wronskian matrix $WM_\alpha(\mathbf{v})$:

$$W_0(\mathbf{v}_\alpha) = \mathbf{v}_\alpha \wedge D_\alpha \mathbf{v}_\alpha \wedge \ldots \wedge D^r_\alpha \mathbf{v}_\alpha \in \mathcal{O}_C(U_\alpha)$$

vanishes at P. It is easy to check that on $U_\alpha \cap U_\beta$ one has (see e.g. [13, Ch. 2-18] or also [8])

$$W_0(\mathbf{v}_\alpha) = {\ell_{\alpha\beta}}^{r+1}(\kappa_{\alpha\beta})^{\frac{r(r-1)}{2}} W_0(\mathbf{v}_\beta)$$

and thus the data $\{W_0(\mathbf{v}_\alpha) \,|\, \alpha \in \mathcal{A}\}$ glue together to give a global holomorphic section

$$W_0(\mathbf{v}) \in L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}, \tag{17}$$

said to be the *Wronskian of the basis* $\mathbf{v}$ of V. As $W_0(\mathbf{v})$ is not identically zero, because $\mathbf{v}$ is a basis, the ramification locus occurs in codimension 1. It does not depend on the choice of a basis $\mathbf{v}$ of V, for if $\mathbf{u}$ were another basis of V, there would be $A \in Gl_{r+1}(\mathbb{C})$ such that $\mathbf{u} = A\mathbf{v}$, and an easy check shows that $W_0(\mathbf{u}) = \det(A)W_0(\mathbf{v})$, i.e. any two bases of V define the same point of $\mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$, denoted in the following by $W_0(V)$.

3.9 The Wronski Map. We have so constructed a map:

$$\begin{cases} G(r+1, H^0(L)) & \longrightarrow & \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}) \\ \\ V & \longmapsto & W_0(V) \end{cases} \tag{18}$$

which associates to each $g^r_d(L)$ its Wronskian $W_0(V)$. Adopting the same terminology used in the literature when $C = \mathbb{P}^1$ and $L := O_{\mathbb{P}^1}(d)$ (see e.g. [11], [12]), the map (18) will be said *Wronski map*: it associates to each $g^r_d(L)$ its Wronskian $W_0(V) := W_0(\mathbf{v}) \mod \mathbb{C}^*$, where $\mathbf{v}$ is any basis of V. The map (18) has very different behaviour depending on the curve and on the choice of the linear system. It is, in general, neither injective nor surjective as the following two extremal cases show. If $C = \mathbb{P}^1$, the unique bundle of degree d is $O_{\mathbb{P}^1}(d)$, $K = O_{\mathbb{P}^1}(-2)$ and the the Wronski map

$$G(r+1, H^0(O_{\mathbb{P}^1}(d))) \longrightarrow \mathbb{P}H^0(O_{\mathbb{P}^1}((r+1)(d-r))),$$

in this case defined between two varieties of the same dimension, is a finite surjective morphism of degree equal to the Plücker degree of the grassmannian $G(r+1, d+1)$. In particular it is not injective – see [10, 45] and [11, 12] over the real numbers. At a general point of $\mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}((r+1)(d-r)))$ (represented by a form of degree $(r+1)(d-r)$) there correspond as many distinct linear systems V as the degree of the Grassmannian. For a closer analysis of the fibers of such a morphism see [44].

On the other hand if C is hyperelliptic and $\mathcal{M} \in Pic^2(C)$ is the line bundle defining its unique g^1_2, Cf. Section 3.7, then $G(2, H^0(\mathcal{M}))$ is just a point and the Wronski map:

$$G(2, H^0(\mathcal{M})) \to \mathbb{P}H^0(\mathcal{M}^{\otimes 2} \otimes K)$$

is trivially injective and not surjective, as by Riemann-Roch formula $h^0(\mathcal{M}^{\otimes 2} \otimes K) > 0$.

Later on we shall see how to make the situation more uniform, by enlarging in a natural way the notion of linear system on a curve. It will be one of the bridges connecting this part of the survey with the first one, regarding Wronskians of differential equations.

3.10 The V-weight of a point. Let V be a g^r_d and $P \in C$. The V-*weight* at P is:

$$wt_V(P) := ord_P W_0(V) = ord_P W_0(\mathbf{v}),$$

for some basis $\mathbf{v}$ of V. The *total weight* of the V-ramification points is:

$$wt_V = \sum_{P\in C} wt_V(P),$$

where the above sum is clearly finite. The *total weight* coincides with the degree of the bundle $L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}$, i.e. the degree of its first Chern class:

$$\begin{aligned}
\mathrm{wt}_V &= \int_C c_1(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}) \cap [C] = \\
&= (r+1)\int_C (c_1(L) \cap [C]) + \frac{r(r+1)}{2}\int c_1(K) \cap [C] = \\
&= (r+1)d + (g-1)r(r+1). \qquad (19)
\end{aligned}$$

obtaining the so-called *Brill–Segre formula*. For example, a smooth plane curve of degree d can be thought of as an abstract curve (compact Riemann surface) embedded in $\mathbb{P}^2$ via some $V \in G(3, H^0(L))$ for some $L \in Pic^d(C)$:

$$(v_0 : v_1 : v_2) : C \longrightarrow \mathbb{P}^2$$

where $\mathbf{v} := (v_0, v_1, v_2)$ is a basis of V. The V-ramification points correspond, in this case, to flexes of the image of C in $\mathbb{P}^2$. The total number of flexes, keeping multiplicities into account, is given by (19) for $r = 2$

$$f = 3d(d-2),$$

which is one of the famous *Plücker formulas* for plane curves.

3.11 Wronskians on Gorenstein Curves. Let C be an irreducible plane curve of degree d with δ nodes and κ cusp. Using the extension of the Wronskian of a linear system defined on a Gorenstein curve, due to Widland and Lax [50], the celebrated Plücker formula

$$f = 3d(d-2) - 6\delta - 8\kappa$$

can be obtained from the tautological identity (See [16] for details):

♯(smooth V–ramification points) =
=♯(ramification points) - ♯(singular ramification points).

3.12 The V-weight of a point P coincides with the weight of its *order partition*. We say that $n \in \mathbb{N}$ is a V-order at $P \in C$ if there exists $v \in V$ such that $ord_P v = n$. Each point possesses only $r+1$ distinct orders. In fact n is an order if $\dim V(-nP) > \dim V(-(n+1)P)$. We have the following sequence of inequalities:

$$r+1=\dim V \geq \dim V(-P) \geq \dim V(-2P) \geq \ldots \geq \dim V(-dP) \geq \dim V(-(d+1)P)=0$$

The last dimension is zero because the unique section of V vanishing at P with multiplicity $d+1$ is zero. At each step the dimension does not drop more than one unit and then there must be precisely $r+1$ jumps, i.e. $r+1$ orders. If

$$0 \leq i_0 < i_1 < \ldots < i_r \leq d$$

is the order sequence at some $P \in C$, the *V-order partition at P* is

$$\boldsymbol{\lambda}(P)=(i_r-r, i_{r-1}-(r-1), \ldots, i_1-1, i_0)$$

One may choose a basis $(v_0, v_1, \ldots, v_r)$ of V such that $ord_P v_j = i_j$. The use of such a basis shows that the Wronskian $W_0(\mathbf{v})$ vanishes at P with multiplicity

$$wt_V(P)=\sum_{j=0}^{r}(i_j-j)=|\boldsymbol{\lambda}(P)|$$

The following result is due to [40] (unpublished) and to [49].

3.13 Proposition. *A partition $\boldsymbol{\lambda}$ is the V-order partition of some $P \in C$ if and only if*

$$W_{\boldsymbol{\mu}}(V)(P)=0$$

for all $\boldsymbol{\mu}$ such that $|\boldsymbol{\mu}|<|\boldsymbol{\lambda}|$ and $W_{\boldsymbol{\lambda}}(V)(P) \neq 0$.

In this case the Wronskian vanishes at P with multiplicity exactly $|\boldsymbol{\lambda}|$.

3.14 A more intrinsic way to look at Wronskians and ramification points, which generalizes to the case of family of curves, is as follows. For $V \in G(r+1, H^0(L))$ one considers the vector bundle map

$$\mathcal{D}_r : C \times V \longrightarrow J^r L \tag{20}$$

defined by $\mathcal{D}_r(P, v) = D_r v(P) \in J^r_P L$. Both bundles have rank $r+1$ and since V has only finitely many ramification points, there is an open set of C where the map $\mathcal{D}_r$ has the maximal rank $r+1$. Then $P \in C$ is a V-ramification point if $\mathrm{rk}_P \mathcal{D}_r \leq r$. The rank of $\mathcal{D}_r$ is smaller than the maximum if and only if the determinant map of (20)

$$\bigwedge^{r+1} \mathcal{D}_r : O_C \longrightarrow \bigwedge^{r+1} J^r L$$

vanishes at P. The section $\bigwedge^{r+1} \mathcal{D}_r \in H^0(\bigwedge^{r+1} J^r L) = H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$ is precisely the *Wronskian section*, which vanishes precisely where the map $\mathcal{D}_r$ has not maximal rank. If $v = a_0 v_0 + \ldots + a_r v_r$, with respect to the basis $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ of V, then $D^r v = a_0 D^r v_0 + a_1 D^r v_1 + \ldots + a_r D^r v_r$. On a trivializing open set U_α of C one has the expression:

$$(D^r v)_{|U_\alpha} = \begin{pmatrix} a_0 v_{0,\alpha} + a_1 v_{1,\alpha} + \ldots + a_r v_{r,\alpha} \\ a_0 D_\alpha v_{0,\alpha} + a_1 D_\alpha v_{1,\alpha} + \ldots + a_r D_\alpha v_{r,\alpha} \\ \vdots \\ a_0 D^r_\alpha v_{0,\alpha} + a_1 D^r_\alpha v_{1,\alpha} + \ldots + a_r D^r_\alpha v_{r,\alpha} \end{pmatrix} = W_0(\mathbf{v}_\alpha) \cdot \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_r \end{pmatrix}.$$

In other words, the local representation of the map $\mathcal{D}_r$ is:

$$W_0(\mathbf{v}_\alpha) : U_\alpha \times \mathbb{C}^{r+1} \longrightarrow U_\alpha \times \mathbb{C}^{r+1}$$

from which:

$$\det(\mathcal{D}_{r|_{U_\alpha}}) = \mathbf{v}_\alpha \wedge D_\alpha \mathbf{v}_\alpha \wedge \ldots \wedge D^r_\alpha \mathbf{v}_\alpha$$

i.e. $\bigwedge^{r+1} \mathcal{D}_r$ is represented by the *Wronskian* $W_0(\mathbf{v})$. Changing the basis $\mathbf{v}$ of V, the Wronskian section gets multiplied by a non zero complex number and hence:

$$\bigwedge^{r+1} \mathcal{D}_r \mod \mathbb{C}^* = W_0(V) \in \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$$

i.e. precisely the Wronskian associated to the linear system V.

3.15 How do generalized Wronskians come into play in this picture? Here the question is more delicate. We have already mentioned that if the V-order partition of a point P is $\boldsymbol{\lambda}(P)$ then all the generalized Wronskians $W_{\boldsymbol{\mu}}(V)$ must vanish for all $\boldsymbol{\mu}$ such that $|\boldsymbol{\mu}| < |\boldsymbol{\lambda}(P)|$ and $W_{\boldsymbol{\lambda}}(P) \neq 0$. It is however well known that the general g^r_d on a general curve C has only simple ramification points, i.e. all the points have weight 1. This says that if a g^r_d has a ramification point with weight bigger than 1, the generalized Wronskians do not impose independent conditions, as the locus occurs in codimension 1 while the expected codimension is bigger than 1. To say some more geometrically significant things one can basically move along two directions. The first, that we just sketch here, consists in considering families of curves.

Let $\pi : \mathfrak{X} \longrightarrow S$ be a proper flat family of smooth curves of genus g and let $(\mathcal{V}, \mathcal{L})$ be a relative g^r_d, i.e. $\mathcal{V}$ is a locally free subsheaf of $\pi_*\mathcal{L}$ and $\mathcal{L} \in Pic^d(\mathfrak{X}/S)$. One can then study the ramification locus of the relative g^r_d which fiberwise cuts the ramification locus of $\mathcal{V}_s \in G(r+1, H^0(\mathcal{L}_{|\mathfrak{X}_s}))$ through the degeneracy locus of the map

$$\mathcal{D}_r : \pi^*\mathcal{V} \longrightarrow J^r_\pi \mathcal{L},$$

where $J^r_\pi\mathcal{L}$ denotes the jets of $\mathcal{L}$ *along the fibers* (see e.g. [22]). The map above induces a section $\mathcal{O}_\mathfrak{X} \to \bigwedge^{r+1} J^r_\pi \mathcal{L} \otimes \bigwedge^{r+1} \pi^*\mathcal{V}$, which is the relative Wronskian $W_0(\mathcal{V})$ of the family. Because of the exact sequence (14):

$$\bigwedge^{r+1} J^r_\pi \mathcal{L} \otimes \bigwedge^{r+1} \pi^*\mathcal{V} = \mathcal{L}^{\otimes r+1} \otimes K_\pi^{\otimes \frac{r(r+1)}{2}} \otimes \bigwedge^{r+1} \pi^*\mathcal{V}$$

In this case the class in $A_*(\mathfrak{X})$ of the ramification locus of $\mathcal{V}$ is

$$Z(W_0(\mathcal{V})] = c_1(\mathcal{L}^{\otimes r+1} \otimes K_\pi^{\otimes \frac{r(r+1)}{2}}) - \pi^* c_1(\mathcal{V})$$

A second approach to enrich the phenomenology of ramification points consists in keeping fix the curve and varying the linear system. This is the only possible approach with curves of genus 0: all the smooth rational curves are rational, and all the g^r_ds, with base points or not, are parameterized by the grassmannian $G(r+1, H^0(O_{\mathbb{P}^1}(d)))$. Here the situation is as nice as one would desire: all what may potentially occur it occurs indeed. For instance, if $\boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_h$ are partitions such that $\sum |\boldsymbol{\lambda}_i| = (r+1)(d-r)$ (= the total weight of the ramification points of a g^r_d) and $P_1, \ldots, P_h$ are arbitrary points on $\mathbb{P}^1$ one can count the number of all of the linear system such that the V order partition at P_i is precisely $\boldsymbol{\lambda}_i$. However if C has higher genus, such a kind of analysis is not possible anymore. For instance the general curve C of genus $g \geq 2$ has only simple Weierstrass points, i.e. all have weight 1, but each curve carries one and only one canonical system. The picture holding for linear systems on the projective line can be generalized in the case of higher genus curves provided one updates the notion of $g^r_d(L)$ to that of *section of a Grassmann bundle*, a path which was first indicated in [18] and then further developed in [37] and [8]. Go to the next two sections for a sketch of the construction.

4 Wronskians of Sections of Grassmann Bundles (in general)

This section is a survey of the contruction appeared in [37], partly published in [8], with some applications appeared in [18].

4.1 Let $\rho_d : F \to X$ be a vector bundle of rank $d+1$ over a smooth complex projective variety X of dimension $m \geq 0$. For each $0 \leq r \leq d$, let $\rho_{r,d} : G(r+1, F) \to X$ be the Grassmann bundle of $(r+1)$-dimensional subspaces of the fibers of F. For $r = 0$ we shall write $\rho_{0,d} : \mathbb{P}(F) \longrightarrow X$, where $\mathbb{P}(F) := G(1, F)$ is the *projective bundle* associated to F. The bundle $G(r+1, F)$ carries an universal exact sequence (Cf. [14, Appendix B.5.7]):

$$0 \longrightarrow \mathcal{S}_r \xrightarrow{\iota_r} \rho^*_{r,d}F \longrightarrow \mathcal{Q}_r \longrightarrow 0, \tag{21}$$

where $\mathcal{S}_r$ is the *universal subbundle* of $\rho^*_{r,d}F$ and $\mathcal{Q}_r$ is the *universal quotient bundle*.

Let

$$\Gamma(\rho_{r,d}) := \{\text{holomorphic } \gamma : X \to G(r+1, F) \,|\, \rho_{r,d} \circ \gamma = id_X\}$$

be the set of holomorphic sections of $\rho_{r,d}$. The choice of $\gamma \in \Gamma(\rho_{r,d})$ amounts to specify a vector sub-bundle of F of rank $r+1$. In fact the pull-back $\gamma^*\mathcal{S}_r$ via $\gamma \in \Gamma(\rho_{r,d})$ is a rank $r+1$ subbundle of F. Conversely, given a rank $r+1$ subbundle $\mathcal{V}$ of F, one may define the section $\gamma_{\mathcal{V}} \in \Gamma(\rho_{r,d})$ by $\gamma_{\mathcal{V}}(P) = \mathcal{V}_P \in G(r+1, F_P)$. The set $\Gamma(\rho_{r,d})$ is too huge and may have a very nasty behaviour: even the case when $X = \mathbb{P}^1$ and $F = J^dO_{\mathbb{P}^1}(d)$, is far from being trivial. In fact it is related with the small quantum cohomology of Grassmannians, see [2]. A first simplification is to fix $\xi \in Pic(X)$ to study the space

$$\Gamma_\xi(\rho_{r,d}) = \{\gamma \in \Gamma(\rho_{r,d}) \,|\, \bigwedge^{r+1} \gamma^*\mathcal{S}_r = \xi\}.$$

Again, if $\xi = O_{\mathbb{P}^1}(n)$ and $F = J^d O_{\mathbb{P}^1}(d)$, then $\Gamma_n(\rho_{r,d}) := \Gamma_{O_{\mathbb{P}^1}(n)}(\rho_{r,d})$ can be identified with the space of the holomorphic maps $\mathbb{P}^1 \to G(r+1, d+1)$ of degree n, compactified in [2] via a Quot-scheme construction. We shall see the easiest case ($n = 0$) in Section 6. In the following, for our limited purposes, we shall restrict the attention to the definitely simpler set

$$\Gamma_{\mathtt{t}}(\rho_{r,d}) := \gamma \in \Gamma(\rho_{r,d}) \,|\, \gamma^*\mathcal{S}_r \text{ is a trivial rank } (r+1) \text{ subbundle of } F\}.$$

4.2 Proposition. *The set $\Gamma_{\mathtt{t}}(\rho_{r,d})$, if non empty, can be identified with an open set of the grassmannian $G(r+1, H^0(F))$.*

Proof. If $\gamma \in \Gamma_{\mathtt{t}}(\rho_{r,d})$, there is an isomorphism $\phi : X \times \mathbb{C}^{r+1} \to \gamma^*\mathcal{S}_r$. Then $\psi := \gamma^*(\iota_r) \circ \phi : X \times \mathbb{C}^{r+1} \to F$ is a bundle monomorphism. Let $\sigma_i : X \to F$ defined by $\sigma_i(P) = \psi(P, \mathbf{e}_i)$. It is clearly a holomorphic section of F. Furthermore $\sigma_0, \sigma_1, \ldots, \sigma_r$ spans an $(r+1)$-dimensional subspace U_γ of $H^0(F)$ which does not depend on the choice of the isomorphism ϕ. Thus $\gamma^*\mathcal{S}_r$ is isomorphic to $X \times U_\gamma$ and $\gamma(P) = \{u(P) \,|\, u \in U_\gamma\} \in G(r+1, F_P)$. Conversely, if $U \in G(r+1, H^0(F))$, one constructs a vector bundle morphism $\phi : X \times U \to F$ via $(P, u) \mapsto u(P)$. This morphism drops rank if $\bigwedge^{r+1} \phi = 0$ and this is a closed condition. There is thus an open set $\mathcal{U} \subseteq G(r+1, F)$ such that for $U \in \mathcal{U}$, the map ϕ_U makes $X \times U$ into a vector subbundle of F. In this case one obtains a section γ_U by setting $\gamma_U(P) = U_P \in G(r+1, F_P)$. The easy check that $\gamma_{U_\gamma} = \gamma$ and that $U_{\gamma_U} = U$ is left to the reader. ■

4.3 Assume now that F comes equipped with a filtratrion $F_\bullet$ by means of quotient bundles, i.e. for each $-1 \leq j \leq i \leq d$, a set of bundle epimorphisms $q_{ij} : F_i \longrightarrow F_j$ such that $F_d = F$, the term F_i is a locally free quotient of F of rank $i+1$, $q_{ii} = id_{F_i}$ and $q_{ij}q_{jk} = q_{ik}$ for each triple $d \geq i \geq j \geq k \geq -1$. We set $F_{-1} = 0$ by convention. The map $q_{dj} : F \to F_j$ will be simply denoted by q_j. Let

$$\partial_i : \mathcal{S}_r \longrightarrow \rho^*_{r,d} F_i$$

be the composition of the universal monomorphism $\mathcal{S}_r \to \rho^*_{r,d} F$ with the map q_i so that the universal morphism ι_r itself can be identified with ∂_d.

4.4 For each $\boldsymbol{\lambda} \in \mathcal{P}^{(r+1)(d-r)}$ the subscheme of $G(r+1, F)$ defined by:

$$\Omega_{\boldsymbol{\lambda}}(\rho^*_{r,d} F_\bullet) = \{\Lambda \in G(r+1, F) \,|\, \mathrm{rk}_\Lambda \partial_{j+\lambda_{r-j}-1} \leq j, \qquad 0 \leq j \leq r\}, \tag{22}$$

is the $\boldsymbol{\lambda}$-*Schubert variety* associated to the filtration $F_\bullet$ and to the partition $\boldsymbol{\lambda}$. Their Chow classes modulo rational equivalence $[\Omega_{\boldsymbol{\lambda}}(\rho^*_{r,d} F_\bullet)]$ generate $A_*(G(r+1, F))$ as a module over $A^*(X)$ through the structural map $\rho^*_{r,d}$.

4.5 For each $0 \leq h \leq d+1$, let $N_h(F) := \ker(F \stackrel{q_{d-h}}{\to} F_{d+1-h})$. It is a vector bundle of rank h. One can define Schubert varieties according to such a *kernel* flag $N_\bullet(F)$ by setting, for each partition $\boldsymbol{\lambda}$ of lenght at most $r+1$:

$$\Omega_{\boldsymbol{\lambda}}(\rho^*_{r,d} N_\bullet(F)) = \{\Lambda \in G(r+1, F) \,|\, \Lambda \cap N_{d+1-(j+\lambda_{r-j})}(F) \geq r+1-j\}$$

It is a simple exercise of linear algebra to show that

$$\Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^{*}F_{\bullet}) = \Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^{*}N_{\bullet}(F)).$$

Both the descriptions are useful according to the purposes. The first description is more suited to describe Weierstrass points as in Section 3 (it gives an algebraic generalization of the rank sequence in a Brill-Nöther matrix, see [1, p. 154]), while the second is useful when dealing with linear systems on the projective line (See Section 6 below).

4.6 Definition. *The* $F_{\bullet}$-Wronskian *subvariety of* $G(r+1,F)$ *is*

$$\mathfrak{W}_0(\rho_{r,d}^{*}F_{\bullet}) := \Omega_{(1)}(\rho_{r,d}^{*}F_{\bullet}).$$

By (22), the $F_{\bullet}$-Wronskian variety $\mathfrak{W}_0(\rho_{r,d}^{*}F_{\bullet})$ of $G(r+1,F)$ is the degeneracy scheme of the natural map $\partial_r : \mathcal{S}_r \longrightarrow \rho_{r,d}^{*}F_r$, i.e. the zero scheme of the map

$$\bigwedge^{r+1}\partial_r : \bigwedge^{r+1}\mathcal{S}_r \longrightarrow \bigwedge^{r+1}\rho_{r,d}^{*}F_r.$$

The map

$$W_0(\rho_{r,d}^{*}F_{\bullet}) := \bigwedge^{r+1}\partial_r \in Hom(\bigwedge^{r+1}\mathcal{S}_r, \bigwedge^{r+1}\rho_{r,d}^{*}F_r) = H^0(X, \bigwedge^{r+1}\rho_{r,d}^{*}F_r \otimes \bigwedge^{r+1}\mathcal{S}_r^{\vee}), \tag{23}$$

is the *Wronskian section* (of the line bundle $\bigwedge^{r+1}\rho_{r,d}^{*}F_r \otimes \bigwedge^{r+1}\mathcal{S}_r^{\vee}$). The $F_{\bullet}$-Wronskian variety is then a Cartier divisor, because it is the zero scheme of the Wronskian section (23). The following is then a natural:

4.7 Definition. *The* generalized $F_{\bullet}$-Wronskian subvariety *of* $G(r+1,F)$, *associated to the partition* $\boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}$, *is*

$$\mathfrak{W}_{\boldsymbol{\lambda}}(\rho_{r,d}^{*}F_{\bullet}) = \Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^{*}F_{\bullet})$$

Among all of them one can recognize some distinguished ones. It is natural to define the $F_{\bullet}$-*base locus* subvariety of $G(r+1,F)$ as

$$\mathcal{B}(\rho_{r,d}^{*}F_{\bullet}) = \mathfrak{W}_{(1^{r+1})}(\rho_{r,d}^{*}F_{\bullet});$$

and the $F_{\bullet}$-*cuspidal locus* subvariety as

$$\mathcal{C}(\rho_{r,d}^{*}F_{\bullet}) = \mathfrak{W}_{(1^{r})}(\rho_{r,d}^{*}F_{\bullet}).$$

It is easy to see that each generalized Wronskian subvariety $\mathfrak{W}_{\boldsymbol{\lambda}}(\rho_{r,d}^{*}F_{\bullet})$ has codimension $|\boldsymbol{\lambda}|$ in $G(r+1,F)$. In particular the base locus variety $\mathcal{B}(\rho_{r,d}^{*}F_{\bullet})$ has codimension $r+1$.

4.8 Let $\gamma \in \Gamma(\rho_{r,d})$. The $F_\bullet$*-ramification locus* of γ is the subscheme $\gamma^{-1}(\mathfrak{W}_0(\rho_{r,d}^* F_\bullet))$ of X, its $F_\bullet$*-base locus* is $\gamma^{-1}(\mathcal{B}(\rho_{r,d}^* F_\bullet))$ and its $F_\bullet$*-cuspidal locus* is $\gamma^{-1}(\mathcal{C}(\rho_{r,d}^* F_\bullet))$. The definition of *Wronski map* defined on sections of Grassmann bundles equipped with filtrations, as in Section 4.3, is very natural too.

4.9 Definition. *For $\gamma \in \Gamma(\rho_{r,d})$, the section*

$$W_0(\gamma) := \gamma^*(W_0(\rho_{r,d}^* F_\bullet)) \bmod \mathbb{C}^* \in \mathbb{P}H^0(X, \bigwedge^{r+1} F_r \otimes \bigwedge^{r+1} \gamma^* \mathcal{S}_r^\vee)$$

*will be said the $F_\bullet$-*Wronskian *of γ.*

The class in $A_*(X)$ of the ramification locus of γ is:

$$\begin{aligned} [Z(W_0(\gamma))] &= [\gamma^{-1}(\mathcal{W}_0(\rho_{r,d}^* F))] = \gamma^*[\mathcal{W}_0(\rho_{r,d}^* F_\bullet)] = \\ &= c_1(\bigwedge^{r+1} F_r \otimes \bigwedge^{r+1} \gamma^* \mathcal{S}_r^\vee) \cap [X] = (c_1(F_r) - \gamma^* c_1(\mathcal{S}_r)) \cap [X]. \end{aligned} \tag{24}$$

If X is a curve, the expected dimension of the ramification locus is 0 and so, when γ is not entirely contained in the Wronski variety, the *total weight* w_γ of the ramification points of γ is by definition the degree of the cycle $[\gamma^{-1}(\mathcal{W}_0(\rho_{r,d}^* F_\bullet))]$:

$$w_\gamma = \int_X (c_1(F_r) - \gamma^* c_1(\mathcal{S})) \cap [X].$$

According to the definitions above, a point $P \in X$ is a ramification point of $\gamma \in \Gamma(\rho_{r,d})$ if $W_0(\gamma)(P) = 0$, which is equivalent to say that the map $\gamma^*\partial_r : \gamma^*\mathcal{S}_r \to F_r$ drops rank at P.

4.10 Definition. *Fix $\xi \in Pic(X)$. The holomorphic map:*

$$\begin{cases} \Gamma_\xi(\rho_{r,d}) & \longrightarrow & \mathbb{P}H^0(\bigwedge^{r+1} F_r \otimes \xi^\vee) \\ \gamma & \longmapsto & W_0(\gamma) \bmod \mathbb{C}^* \end{cases}$$

is the Wronski map *defined on $\Gamma_\xi(\rho_{r,d})$.*

Indeed $W_0(\gamma)$ is a section of $\gamma^*(\bigwedge^{r+1} \rho_{r,d}^* F_r \otimes \bigwedge^{r+1} \mathcal{S}_r^\vee) = \bigwedge^{r+1} F_r \otimes \xi^\vee$. The class of the *ramification locus* of γ, as in (24), can be now expressed as:

$$[Z(W_0(\gamma))] = (c_1(F_r) - \xi) \cap [X] \in A_*(X)$$

4.11 The Shape of the Extended Wronski Map. It is particularly easy to express the Wronskian of a section $\gamma \in \Gamma_{\mathtt{t}}(\rho_{r,d})$. Let $U \in G(r+1, F)$ such that $\gamma = \gamma_U$. The pull-back of the map $\partial_r : \mathcal{S}_r \longrightarrow \rho_{r,d}^* F_r$ is

$$\gamma^*\partial_r : X \times U \longrightarrow F_r \tag{25}$$

The Wronskian is the determinant of the map (25):

$$\bigwedge^{r+1}\gamma^*\partial_r : \bigwedge^{r+1}(X\times U) \to \bigwedge^{r+1} F_r$$

Once a basis $(u_0, u_1, \ldots, u_r)$ of U is chosen, the Wronskian

$$\bigwedge^{r+1}\gamma^*\partial_r \in H^0(X, \bigwedge^{r+1} F_r)$$

is represented by the section

$$P \mapsto q_r(u_0)(P) \wedge q_r(u_1)(P) \wedge \ldots \wedge q_r(u_r)(P) \in H^0(X, \bigwedge^{r+1} F_r).$$

Changing basis the sections gets multiplied by a non zero constant, and then the Wronski map

$$\Gamma_{\mathtt{t}}(\rho_{r,d}) \longrightarrow \mathbb{P}H^0(X, \bigwedge^{r+1} F_r)$$

defined by $\gamma \mapsto W_0(\gamma) \in \mathbb{P}H^0(X, \bigwedge^{r+1} F_r)$ coincides with the map

$$\begin{cases} G(r+1, H^0(F)) & \longrightarrow & \mathbb{P}H^0(X, \bigwedge^{r+1} F_r) \\ & & \\ U & \longmapsto & q_r(u_0)\wedge q_r(u_1)\wedge \ldots \wedge q_r(u_r) \mod \mathbb{C}^* \end{cases}$$

where $\mathbf{u} = (u_0, u_1, \ldots, u_r)$ is any basis of U.

4.12 Here is a quick review of intersection theory on $G(r+1, F)$ which is necessary for enumerative geometry purposes. First recall some basic terminology and notation. If $\mathbf{a} := \sum_{n\geq 0} a_n t^n$ is a formal power series with coefficients in some ring A and if $\boldsymbol{\lambda}$ is a partition as in (4), the $\boldsymbol{\lambda}$-*Schur polynomial* associated to $\mathbf{a}$ is, by definition:

$$\Delta_{\boldsymbol{\lambda}}(\mathbf{a}) = \det(a_{i+\lambda_{r-i}-j+1}) = \begin{vmatrix} a_{\lambda_r} & a_{\lambda_{r-1}+1} & \cdots & a_{\lambda_0+r} \\ a_{\lambda_r-1} & a_{\lambda_{r-1}} & \cdots & a_{\lambda_0+r-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\lambda_r-r} & a_{\lambda_{r-1}-(r-1)} & \cdots & a_{\lambda_r} \end{vmatrix}. \tag{26}$$

By the Basis Theorem [14, p. 268] the Chow group $A_*(G(r+1, F))$ is a free $A_*(X)$-module (via the structural morphism $\rho_{r,d}^* : A^*(X) \to A^*(G(r+1, F))$ generated by

$$\{\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \rho_{r,d}^* F)) \cap [G(r+1, F)] \,|\, \boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}\}$$

where $c_t(\mathcal{E})$ denotes the Chern polynomial of a bundle $\mathcal{E}$ and

$$c_t(\mathcal{Q}_r - \rho_{r,d}^* F) := \frac{c_t(\mathcal{Q}_r)}{c_t(\rho_{r,d}^* F)}$$

If $r = 0$, let
$$\mu^i := (-1)^i c_1(\mathcal{S}_0)^i \cap [\mathbb{P}(F)]$$
for each $i \geq 0$. Then, by [14, Ch. 14], $(\mu^0, \mu^1, \ldots, \mu^d)$ is an $A^*(X)$-basis of $A_*(\mathbb{P}(F))$ and for each $j \geq 0$ the following relation, defining the Chern classes of F, holds:

$$\mu^{d+1+j} + \rho_{0,d}^* c_1(F)\mu^{d+j} + \ldots + \rho_{0,d}^* c_{d+1}(F)\mu^j = 0 \tag{27}$$

A main result of [19] says that $\bigwedge^{r+1} A_*(\mathbb{P}(F))$ can be equipped with a structure of $A^*(G(r+1, F))$ module of rank 1 generated by $\mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^r$ in such a way that, for each $\boldsymbol{\lambda} \in \mathcal{P}^{(r+1)\times(d-r)}$:

$$\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \rho_{r,d}^* F)) \cdot \mu^0 \wedge \mu^1 \wedge \ldots \wedge \mu^r = \mu^{\lambda_r} \wedge \mu^{1+\lambda_{r-1}} \wedge \ldots \wedge \mu^{r+\lambda_0}. \tag{28}$$

We shall see in the last sections that $\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \rho_{r,d}^* F))$ are related with Wronskians associated to a fundamental system of soltions of a suitable differential equation. Define now:
$$\epsilon^i := [\Omega_{(i)}(\rho_{0,d}^* F_\bullet)] \in A_*(\mathbb{P}(F)),$$
where $\Omega_{(i)}(\rho_{0,d}^* F_\bullet)$ is nothing but the zero locus in codimension i of the map $\partial_{i-1} : \mathcal{S}_0 \to F_{i-1}$. Because of the relation:

$$\epsilon^i = \sum_{j=0}^{i} \rho_{0,d}^* c_j(F_{i-1})\mu^{i-j}, \tag{29}$$

it follows that $(\epsilon^0, \epsilon^1, \ldots, \epsilon^d)$ is a $A^*(X)$-basis of $A_*(\mathbb{P}(F))$ as well. For each $\boldsymbol{\lambda} \in \mathcal{P}^{(d+1)\times(d-r)}$ let $\epsilon^{\boldsymbol{\lambda}} := \epsilon^{\lambda_r} \wedge \epsilon^{1+\lambda_{r-1}} \wedge \ldots \wedge \epsilon^{r+\lambda_0} \in \bigwedge^{r+1} A_*(\mathbb{P}(F))$. Again by [19] $\{\epsilon^{\boldsymbol{\lambda}} \,|\, \boldsymbol{\lambda} \in \mathcal{P}^{(d+1)\times(d-r)}\}$ is an $A^*(X)$-basis of $A_*(G(r+1, F))$. Denote by $[\Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^* F_\bullet)]$ the class in $A_*(G(r+1, F))$ of the $F^\bullet$ Schubert variety $\Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^* F_\bullet)$.

4.13 Theorem. *The following equality holds:*

$$[\Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^* F_\bullet)] = [\Omega_{(\lambda_r)}(\rho_0^* F_\bullet)] \wedge [\Omega_{(1+\lambda_{r-1})}(\rho_1^* F_\bullet)] \wedge \ldots \wedge \Omega_{(r+\lambda_0)}(\rho_1^* F_\bullet) = \epsilon^{\boldsymbol{\lambda}} \tag{30}$$

modulo the identification of $A_*(G(r+1, F))$ with $\bigwedge^{r+1} A_*(\mathbb{P}(F))$.

Equality (30) is an elegant and compact re-interpretation of the determinantal formula of Schubert calculus proven by Kempf and Laksov in [30] to compute classes of degeneracy loci of maps of vector bundles. For more general and deep investigations on this subject see [42]. Formula (30) was basically discovered in [17] for trivial bundles. The present formulation is as in [37].

(Sketch of) Proof of (4.13). Let $A_0 \subsetneq A_1 \subsetneq \ldots \subsetneq A_r$ be a flag of subbundles of F such that $\mathrm{rk} A_j = j + \lambda_{r-j}$, defined through the exact sequence $0 \to A_j \to F \to F_{d+1-(j+\lambda_{r-j})} \to 0$. Then
$$\Omega(A_0, A_1, \ldots, A_r) = \{\Lambda \in G(r+1, F) \,|\, \Lambda \cap A_i \geq i\},$$

coincides with $\Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^*F_\bullet)$ defined by (22), as a simple check shows. Formula 7.9 in [32], which translates the determinantal formula proven in [30] shows, up to a change of notation, that:

$$[\Omega(A_0, A_1, \ldots, A_r)] = [\Omega(A_0)] \wedge [\Omega(A_1)] \wedge \ldots \wedge [\Omega(A_r)],$$

which is thence equivalent to (30). ∎

5 Wronskians of Sections of Grassmann Bundles of Jets

5.1 The general framework of Section 4 shows that the notion of linear system can be generalized into that of pairs $(\gamma, F^\bullet)$, where $F^\bullet$ is a vector bundle on X equipped with a filtration and γ a section of the Grassmann bundle $G(r+1, F)$. This picture can be fruitfully applied in the case of (families of) smooth complex projective curves of genus $g \geq 0$. For the time being let C be any one such, and let $L \in Pic^d(C)$. In this section we shall denote by $\rho_d : J^dL \to C$ the bundle of jets of $L \to C$ up to the order d. Accordingly, for each $0 \leq r \leq d$, we shall denote $\rho_{r,d} : G(r+1, J^dL) \to C$ the Grassmann bundle of $(r+1)$-dimensional subspaces of fibers of ρ. The natural filtration of J^dL given by the quotients $J^dL \to J^iL \to 0$, for $-1 \leq 0 \leq d$, will be denoted $J^\bullet L$ (setting $J^{-1}L = 0$).

5.2 The *kernel filtration* of J^dL

$$N_\bullet(L) : 0 \subset N_1(L) \subset \ldots \subset N_d(L) \subset N_{d+1}(L) = J^dL \tag{31}$$

is defined through the exact sequence of vector bundles $0 \to N_h(L) \to J^dL \to J^{d-h}L \to 0$, where $N_h(L)$ is a vector bundle of rank h. It will be also called the *osculating flag* – see below and Section 6. The fiber of $N_h(L)$ at $P \in C$ will be denoted by $N_{h,P}(L)$.

As in the previous section, the $\boldsymbol{\lambda}$-generalized *Wronskian* subvariety of $G(r+1, J^dL)$ is $\Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^*J^\bullet L)$, which has codimension $|\boldsymbol{\lambda}|$ in $G(r+1, J^dL)$. By virtue of Proposition 4.2, the space $\Gamma_{\mathtt{t}}(\rho_{r,d})$ of sections γ of $\rho_{r,d}$ such that $\gamma^*\mathcal{S}_r$ is a trivial subbundle of J^dL, can be identified with an open subset of $G(r+1, H^0(J^dL))$. Hence $\gamma^*\mathcal{S}_r$ is of the form $C \times U$ for some $U \in G(r+1, H^0(J^dL))$. As in section 4 we gain a Wronski map:

$$\Gamma_{\mathtt{t}}(\rho_{r,d}) \longrightarrow \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}}) \tag{32}$$

defined by $\gamma \mapsto W_0(\gamma) \in \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$. As we said, this map is the restriction to the open subset $\Gamma_{\mathtt{t}}(\rho_{r,d}) \subseteq G(r+1, H^0(J^dL))$ of the determinant map:

$$G(r+1, H^0(J^dL)) \to \mathbb{P}H^0(L^{\otimes r+1} \otimes K^{\otimes \frac{r(r+1)}{2}})$$

sending $U \mapsto t_r(u_0) \wedge t_r(u_1) \wedge \ldots \wedge t_r(u_r)$, where $(u_0, u_1, \ldots, u_r)$ is a basis of U and t_r denotes the epimorphism $J^dL \to J^rL$.

5.3 We notice now that each $g^r_d(L)$, i.e. $V \in G(r+1, H^0(L))$, can be seen in fact as an element of $\Gamma_{\mathtt{t}}(\rho_{r,d})$, because $\mathcal{D}_d : C \times V \to J^dL$ realizes $C \times V$ as a (trivial) vector subbundle of J^dL. Indeed $D_dV := \{D_dv \,|\, v \in V\}$ is an $(r+1)$-dimensional subspace of $H^0(J^dL)$ because the map $J^dL \to L \to 0$ induces the surjection $H^0(J^dL) \to H^0(L) \to 0$, see e.g. [8], and then $\mathcal{D}_dv = 0$ implies $v = 0$.

We have thus an injective map $G(r+1, H^0(L)) \hookrightarrow \Gamma_{\mathtt{t}}(\rho_{r,d}) \subseteq G(r+1, H^0(J^dL))$, sending V to γ_{D_dV}, and

$$W_0(\gamma_{\mathcal{D}_dV}) := D_ru_0 \wedge D_ru_1 \wedge \ldots \wedge D_ru_r \mod \mathbb{C}^* = W_0(V)$$

which proves that our Wronski map defined on $\Gamma_{\mathtt{t}}(\rho_{r,d})$, which is in general strictly larger than $G(r+1, H^0(L))$, coincides with the Wronskian $W_0(V)$ defined in section 3. We are so in condition of defining *generalized Wronskian subloci*. Recall the natural evaluation map

$$\mathrm{ev} : C \times \Gamma_{\mathtt{t}}(\rho_{r,d}) \longrightarrow G(r+1, J^dL)$$

sending $(P, \gamma) \mapsto \gamma(P)$. If $\Omega_{\boldsymbol{\lambda}}(\rho^*_{r,d}J^\bullet L)$ is a generalized Wronskian variety of $G(r+1, J^dL)$, then $\mathrm{ev}^{-1}(\Omega_{\boldsymbol{\lambda}}(\rho^*_{r,d}J^\bullet L))$ cuts the locus of pairs (P, γ) such that $\gamma(P) \in \Omega_{\boldsymbol{\lambda}}(J^\bullet L)$. We also set $\mathrm{ev}_P(\gamma) = \gamma(P)$, for each $P \in C$. It follows that the general section of any irreducible component of $\mathrm{ev}_P^{-1}(\Omega_{\boldsymbol{\lambda}}(J^\bullet L))$ is a section having $\boldsymbol{\lambda}$ as a ramification partition.

5.4 The map $D_{d,P} : H^0(L) \to J^d_PL$ sending $v \mapsto D_dv(P)$ is a vector space monomorphism. If $V \in G(r+1, H^0(L))$, then $v \in V \cap D^{-1}_{d,p}N_{h,P}(L)$ if and only if $D^hv(P) = 0$, i.e. if and only if v vanishes at P with multiplicity at least h. This explains the terminology *osculating flag* used in Section 5.2.

5.5 Example. More details on the present example are in [18]. Let $\pi : \mathfrak{X} \longrightarrow S$ be a proper flat family of smooth projective curves of genus $g \geq 2$. The Hodge bundle of the family is $\mathbb{E}_\pi := \pi_*K_\pi$. The vector bundle map over $\mathfrak{X}$

$$\pi^*\mathbb{E}_\pi \to J^{2g-2}K_\pi$$

is injective and then it induces a section $\gamma_K : \mathfrak{X} \to G(g, J^{2g-2}K_\pi)$. In this case the cuspidal locus of γ_K, which is by definition $\gamma_K^{-1}(\Omega_{023\ldots g})$, coincides with the locus in $\mathfrak{X}$ of the Weierstrass points of the hyperelliptic fibers of π. With the same notation of 4.12 and 4.13, its class in $A^{g-1}(\mathfrak{X})$ is given by:

$$[\gamma_K^{-1}(\Omega_{023\ldots g}(J^\bullet K)] = \gamma_K^*[\Omega_{023\ldots g}(J^\bullet K)] = \gamma_K^*(\epsilon^0 \wedge \epsilon^2 \wedge \ldots \wedge \epsilon^g).$$

which can be easily computed through straightforward computations as in [18, Section 3], where they are performed for $g = 4$. Since on each hyperelliptic fiber there are precisely $2g+2$ Weierstrass points, the class of the hyperelliptic locus in $A^{g-2}(S)$ is given by:

$$[H] = \frac{1}{2g+2} \cdot \pi_*\gamma_K^*(\epsilon^0 \wedge \epsilon^2 \wedge \ldots \wedge \epsilon^g)$$

which yields precisely the formula displayed in [36, p. 314].

5.6 If $C = \mathbb{P}^1$ and $L = O_{\mathbb{P}^1}(d)$, then $\Gamma_{\mathtt{t}}(\rho_{r,d})$ coincides in this case with $G(r+1, H^0(L))$ and our picture allows to rephrase in an elegant way the situation exposed in the first part of [10]. The *Wronski map* $\Gamma_{\mathtt{t}}(\rho_{r,d}) \to \mathbb{P}H^0(O_{\mathbb{P}^1}((r+1)(d-r))$ coincides with (18), modulo the identification of $\Gamma_{\mathtt{t}}(\rho_{r,d})$ with $G(r+1, H^0(O_{\mathbb{P}^1}(d))$. In other words, when C is not rational, the theoy exposed up to now can be seen as a cheap generalization of the theory of linear systems on the projective line, for which we want to spend some additional words in a separate section.

6 Linear Systems on $\mathbb{P}^1$ and the Intermediate Wronskians

In the case of linear systems g^r_d defined on the projective line, the picture outlined in Section 5 gets simpler but, even in this case is particularly rich of nice geometry interacting with other parts of mathematics. For sake of brevity, denote by L_d the invertible sheaf $O_{\mathbb{P}^1}(d)$, i.e. the unique line bundle on $\mathbb{P}^1$ of degree d. The elements of a basis $\mathbf{x} := (x_0, x_1)$ of $H^0(L_1)$ can be regarded as homogeneous coordinates $(x_0 : x_1)$ on $\mathbb{P}^1$. Furthermore $H^0(L_d) = \mathrm{Sym}^d H^0(L_1)$, i.e. $H^0(L_d)$ can be identified with the $\mathbb{C}$-vector space generated by the monomials $\{x_0^i x_1^{d-i}\}_{0 \le i \le d}$. A g^r_d on $\mathbb{P}^1$ is a point of $G(r+1, H^0(L_d))$. If V is any one such and $\mathbf{v} := (v_0, v_1, \ldots, v_r)$ a basis of it, the rational map

$$\varphi_V := (v_0, v_1, \ldots, v_r) : \mathbb{P}^1 \cdots \to \mathbb{P}^r \tag{33}$$

is a morphism if V has no base points, i.e. if $\dim V(-P) = \dim V - 1$, for each $P \in \mathbb{P}^1$. In this latter case the image of (33) is a non-degenerated (i.e. not contained in any hyperplane) rational curve of degree d in $\mathbb{P}^r$. If $V = H^0(L_d)$ then $\varphi_V(\mathbb{P}^1)$ is nothing else than the rational normal curve of degree d, and each curve of degree d in $\mathbb{P}^r$ can be seen as the rational normal curve in $\mathbb{P}H^0(L_d)$ composed with a projection $\mathbb{P}H^0(L_d) \cdots \to \mathbb{P}^r$ with center a complementary linear subvariety of $V \in G(r+1, H^0(L_d))$ (see e.g. [10], [29]). Keeping the same notation as in Section 5, let $\rho_d : J^d L_d \to \mathbb{P}^1$ be the bundles of d-jets of L_d. The vector bundle map (20), $\mathcal{D}_d : \mathbb{P}^1 \times H^0(L_d) \to J^d L$, is an isomorphism in this case, being an injective morphism between vector bundles of the same rank. In particular the map:

$$\begin{cases} \mathcal{D}_{d,P} & : & H^0(L_d) & \longrightarrow & J^d_P L_d \\ & & P & \longmapsto & D_d v(P) \end{cases} \tag{34}$$

is a vector space isomorphism for all $P \in \mathbb{P}^1$. Define the *osculating flag* at P of $H^0(L_d)$:

$$\mathcal{F}_{\bullet,P} : 0 \subset \mathcal{F}_{1,P} \subset \ldots \subset \mathcal{F}_{d,P} \subset \mathcal{F}_{d+1,P} = J^d_P L$$

by setting (Cf. 5.2)

$$\mathcal{F}_{h,P} = \mathcal{D}^{-1}_{d,p}(N_{h,P}(L)) \subseteq H^0(L_d).$$

In other words, $v \in V \cap \mathcal{F}_{h,P}$ if and only if $\mathcal{D}_h v(P) = 0$, i.e. if and only if v vanishes at P with multiplicity at least h. Equivalently $\mathcal{F}_{h,P}$ may be identified with the vector subspace of $H^0(L_d)$ of the homogeneous polynomials vanishing at P with multiplicity at least h or, again, with the set of all hyperplanes of $\mathbb{P}H^0(L_d)$ intersecting the rational normal curve of $\mathbb{P}H^0(L_d))$ at P with multiplicity at least $d-h$.

6.1 A simple use of Riemann-Roch formula shows that $h^0(L_d) = h^0(J^d L_d)$: then the injective "derivative map" $D_d : H^0(L_d) \to H^0(J^d L_d)$ is an isomorphism which itself induces a biholomorphism:

$$G(r+1, H^0(L_d)) \to G(r+1, H^0(J^d L_d)),$$

from which one concludes that $\Gamma_{\mathtt{t}}(\rho_{r,d}) = G(r+1, H^0(J^d L_d)) \cong G(r+1, H^0(L_d))$, i.e. that $\Gamma_{\mathtt{t}}(\rho_{r,d})$ parameterizes all the g^r_ds on $\mathbb{P}^1$ (with base points or not). In particular it is compact. If $V \in G(r+1, H^0(L_d))$, denote by γ_V the corresponding element of $\Gamma_{\mathtt{t}}(\rho_{r,d})$. The evaluation morphism $\mathbb{P}^1 \times G(r+1, H^0(L_d)) \to G(r+1, J^d L_d)$ maps (P, V) to $\gamma_V(P) \in G(r+1, J^d_P L_d)$.

6.2 By 6.1, the *Wronski map* (32), $\gamma \mapsto W_0(\gamma)$, coincides with the Wronski map (18):

$$G(r+1, H^0(L_d)) \to \mathbb{P}H^0(L_{(r+1)(d-r)}), \tag{35}$$

mapping $V \mapsto W_0(V)$. It is a finite surjective morphism (see e.g. [10], [29], [44]). Its degree is precisely the Plücker degree of the Grassmannian $G(r+1, d+1)$, namely:

$$d_{r,d} := \int \sigma_{(1)}^{(r+1)(d-r)} \cap [G(r+1, d+1)] = \frac{1!2!\ldots r! \cdot (r+1)(d-r)!}{(d-r)!(d-r+1)! \cdot \ldots \cdot d!}.$$

In other words, given a homogeneous polynomial W of degree $(d-r)(r+1)$ in two indeterminates (x_0, x_1), there are at most $d_{r,d}$ distincts g^r_ds having W as a Wronskian. Or, put otherwise, the number of preimages of W through the Wronski map (35) is $d_{r,d}$, provided one keeps multiplicities into account.

The number $d_{r,d}$ was calculated by Schubert himself in 1886, Cf. [48] and [14, p. 274]. The Wronski map in the case of real polynomials was studied in the context of real rational curves as well. Its degree was obtained by L. Goldberg for $r = 1$ in [24], and for any $r \geq 1$ by A. Eremenko and A. Gabrielov in [11]. For more considerations on real Wronski map see also [29].

6.3 Let $\boldsymbol{\lambda}$ be a partition of lenght at most $r+1$ and define

$$\Omega_{\boldsymbol{\lambda}}(P) =: \Omega_{\boldsymbol{\lambda}}(\mathcal{F}_{\bullet,P}) \subseteq G(r+1, H^0(L_d)).$$

It is a Schubert variety of codimension $|\boldsymbol{\lambda}|$ in $G(r+1, H^0(L_d))$. If $\boldsymbol{\lambda}(V, P)$ is the order partition of V at P (see Section 3.12) then

$$V \in \Omega^\circ_{\boldsymbol{\lambda}(V,P)}(P) \subseteq \Omega_{\boldsymbol{\lambda}(V,P)}(P)$$

and P is a V-ramification point if and only if $|\boldsymbol{\lambda}(V,P)| > 0$. The Wronskian $W_0(V)$ of V vanishes at all, and only at, the V-ramification points, whose total weight equals the dimension of the grassmannian $G(r+1, H^0(L_d))$, as one sees by putting $g = 0$ in formula (19). Let $\{(\underline{P}, \underline{w})\} := \{(P_0, w_0), (P_1, w_1), \ldots, (P_k, w_k))$ be a $k+1$-tuple of pairs where $P_i \in \mathbb{P}^1$ and (w_i) are positive integers such that

$$\sum w_i = (r+1)(d-r). \tag{36}$$

Suppose that:

$$D_{w_i-1}W_0(V) \in H^0(J^{w_i-1}L_{(r+1)(d-r)})$$

vanishes at P_i, for all $0 \le i \le k$. Then $(P_0, P_1, \ldots, P_k)$ are all the ramification points of V, each one of weight $wt_V(P_i) = w_i = |\lambda(P_i, V)|$, and then:

$$\begin{aligned} V &\in \Omega^{\circ}_{\boldsymbol{\lambda}(V,P_0)}(P_0) \cap \Omega^{\circ}_{\boldsymbol{\lambda}(V,P_1)}(P_1) \cap \ldots \cap \Omega^{\circ}_{\boldsymbol{\lambda}(V,P_k)}(P_k) = \\ &= \Omega_{\boldsymbol{\lambda}(V,P_0)}(P_0) \cap \Omega_{\boldsymbol{\lambda}(V,P_1)}(P_1) \cap \ldots \cap \Omega_{\boldsymbol{\lambda}(V,P_k)}(P_k) \end{aligned} \tag{37}$$

Because of (36), the "expected dimension" of the intersection (37) is zero. The intersections associated with the osculating flags of the normal rational curve were first studied by D. Eisenbud and J. Harris in the eighties in the paper [10], where the authors show that the intersection is zero-dimensional indeed and the number of its distinct points is at most:

$$\int_{G(r+1,H^0(L_d))} \sigma_{\boldsymbol{\lambda}(P_0,V)} \cdot \sigma_{\boldsymbol{\lambda}(P_1,V)} \cdot \ldots \cdot \sigma_{\boldsymbol{\lambda}(P_k,V)} \cap [G(r+1, H^0(L_d))]$$

where $\sigma_{\boldsymbol{\lambda}}$ is the Schubert cycle defined by the equality $\sigma_{\boldsymbol{\lambda}} \cap [G(r+1, H^0(L_d))] = \Omega_{\boldsymbol{\lambda}}$. This fact was used in [7] to deduce explicit formulas (and a list up to $n = 40$) for the number of space rational curves of degree $n-3$ having $2n$ *hyperstalls* at $2n$ prescribed points.

6.4 Preimages of the Wronski Map. Notice that if $P \in \mathbb{P}^1$ is a base point of V, it occurs in the V-ramification locus as well, and the Wronskian vanishes at it with weight at least $(r+1)$. The set B_P of linear systems having P as base point is a closed subset of $G(r+1, H^0(L_d))$ of codimension $(r+1)$. In fact $B_P := \mathrm{ev}_P^{-1}(\mathcal{B}(\rho^*_{r,d}J^{\bullet}L_d))$, which is a closed subset of codimension $(r+1)$ (Cf. Section 4.7).

Let $\{(\underline{P}, \underline{w})\}$ be as in 6.3. The set $\mathcal{G}_{r,d}(\underline{P})$ of all $V \in G(r+1, H^0(L_d))$ having no base point at any of the P_is is an open dense subset of codimension $(r+1)$ at least. In fact

$$\mathcal{G}_{r,d}(\underline{P}) = G(r+1, H^0(L_d)) \setminus (B_{P_0} \cap B_{P_1} \cap \ldots \cap B_{P_k})$$

i.e. it is a complement of the closed subset intersection of the B_{P_i}s, which has codimension at least $r+1$. Consider now a $(k+1)$-tuple of partitions

$$\vec{\boldsymbol{\lambda}} = (\boldsymbol{\lambda}_0, \boldsymbol{\lambda}_1, \ldots, \boldsymbol{\lambda}_k)$$

such that $\boldsymbol{\lambda}_j$ has weight w_j for each $0 \leq j \leq k$. We shall write:

$$\boldsymbol{\lambda}_j := \lambda_{j,0} \geq \lambda_{j,1} \geq \ldots \geq \lambda_{j,r}$$

The elements of the intersection in $G(r+1, H^0(L_d))$:

$$I(\vec{\boldsymbol{\lambda}}, \underline{P}) = \Omega_{\boldsymbol{\lambda}_0}(P_0) \cap \Omega_{\boldsymbol{\lambda}_1}(P_1) \cap \ldots \cap \Omega_{\boldsymbol{\lambda}_k}(P_k) \cap \mathcal{G}_{r,d}(\underline{P}) \tag{38}$$

correspond to base point free linear systems ramifying at $\underline{P}$ according to $\boldsymbol{\lambda}$.

The problem of determining $I(\vec{\boldsymbol{\lambda}}, \underline{P})$ leads to interesting analytic considerations related with Wronskians. Up to a projective change of coordinates, it is not restrictive to assume that $P_0 = \infty := (0:1)$. Using the coordinate $x = x_1/x_0$, the osculating flag at ∞ shall be denoted by $\mathcal{F}_{\bullet,\infty}$. Accordingly, the partition $\boldsymbol{\lambda}_0$ will be renamed $\boldsymbol{\lambda}_\infty$. Notice that $\mathcal{F}_{j,\infty}$ coincides with the vector space Poly_j of the polynomials of degree at most j in the variable x: in fact a polynomial $P(x)$ (thought of as the affine representation of a homogeneous polynomial of degree d in two variables) vanishes at ∞ with multiplicity j if and only if it has degree $d+1-j$. If $V \in G(r+1, H^0(L_d))$ let

$$W_V(x) := \frac{W_0(V)}{x_0^{(r+1)(d-r)}} \tag{39}$$

be the Wronskian expressed in the affine open subset of $\mathbb{P}^1$ defined by $x_0 \neq 0$. It is clear that $W_V(x)$ is a polynomial of degree less or equal than $(r+1)(d-r)$, because of possible ramifications of V at ∞. It can be written in the form:

$$W_{w,z}(x) := W_V(x) = (x-z_1)^{w_1} \cdot \ldots \cdot (x-z_k)^{w_k}$$

where now $\sum_{i=1}^k w_i = \deg W_V(x) \leq (r+1)(d-r)$ and $z_i := x(P_i)$ are the values of the coordinate x at $P_i \in \mathbb{P}^1$.

If $\mathbf{f} = (f_0, f_1, \ldots, f_r)$ defined by $f_i := v_i/x_0^d$ where $\mathbf{v} = (v_0, v_1, \ldots, v_r)$ denotes, as usual, a basis of V, then according to (3), one writes $W_V(x) = \mathbf{f} \wedge D\mathbf{f} \wedge \ldots \wedge D^r\mathbf{f}$, where

$$D^j\mathbf{f} = \left(\frac{d^j f_i}{dx^j}\right)_{0 \leq j \leq r}.$$

The space V can be realized as the set of $g \in \mathrm{Poly}_d$ satisfying the following differential equations:

$$E_V(g) = \begin{vmatrix} g & f_0 & f_1 & \ldots & f_r \\ Dg & Df_0 & Df_1 & \ldots & Df_r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ D^r g & D^r f_0 & D^r f_1 & \ldots & D^r f_r \\ D^{r+1} g & D^{r+1} f_0 & D^{r+1} f_1 & \ldots & D^{r+1} f_r \end{vmatrix} = 0\,. \tag{40}$$

6.5 Intermediate Wronskians. By contruction, any $V \in I(\vec{\boldsymbol{\lambda}}, \underline{P})$ has no base point and $W_V(x) = W_{w,z}(x)$ as in (39). Denote by $V_\bullet$ the flag obtained by the intersection of V and $\mathcal{F}_{\bullet,\infty}$:

$$V_\bullet = \{V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_r = V\}, \quad \dim V_j = j. \tag{41}$$

Here V_j is a vector subspace of Poly_d of dimension $j+1$ and all the polynomials in V_j have degree $\le d_j$, where $0 \le d_0 < d_1 < \ldots < d_r \le d$ is the order sequence of V at P (Cf. Section 3.12). Denote by $W_j(x)$ the Wronskian $W_{V_j}(x)$ of V_j, $0 \le j \le r$. We call it *the j-th intermediate Wronskian of V*. In particular, the r-th intermediate Wronskian coincides with (39). Non vanishing properties of intermediate Wronskians have been recently investigated in a more analytic context in [5] and [6] to study factorizations of linear differential operators with non constant $\mathbb{C}$-valued coefficients.

Intermediate Wronskians are important because every $V \in G(r+1, \mathrm{Poly}_d)$ is completely determined by the set of its intermediate Wronskians $W_0(x), \ldots, W_r(x)$. Indeed, the ODE (40) can be rewritten as follows:

$$\frac{d}{dx}\frac{W_{r-1}^2(x)}{W_{r-2}(x)W_r(x)} \cdot \ldots \cdot \frac{d}{dx}\frac{W_2^2(x)}{W_3(x)W_1(x)}\frac{d}{dx} \cdot \frac{W_1^2(x)}{W_2(x)W_0(x)} \cdot \frac{d}{dx}\frac{g(x)}{W_1(x)} = 0\,.$$

By [39, Part VII, Section 5, Problem 62], one can take as a basis of V the following set of $r+1$ linearly independent solutions of (40):

$$\begin{array}{rcl}
g_0(x) & = & W_0(x)\,, \\
g_1(x) & = & W_0(x)\displaystyle\int^x \frac{W_2W_0}{W_1^2}\,, \\
g_2(x) & = & W_0(x)\displaystyle\int^x \left(\frac{W_2(\xi)W_0(\xi)}{W_1^2(\xi)}\int^\xi \frac{W_0W_2}{W_1^2}\right), \\
\ldots & & \ldots \qquad \ldots \qquad \ldots \qquad \ldots \\
g_r(x) & = & W_0(x)\displaystyle\int^x \left(\frac{W_1(\xi)W_{-1}(\xi)}{W_0^2(\xi)}\int^\xi \left(\frac{W_0(\tau)W_2(\tau)}{W_1^2(\tau)}\int^\tau \ldots \int^\eta \frac{W_{r-2}W_r}{W_{r-1}^2}\right)\ldots\right),
\end{array}$$

Define now polynomials $Z_0(x), Z_1(x), \ldots, Z_r(x)$ through the following formula:

$$Z_i(x) = \prod_{j=1}^k (x - z_j)^{m_j(i)}, \quad 0 \le i \le r \tag{42}$$

where

$$m_j(i) = \lambda_{j,r} + \lambda_{j,r-1} + \ldots + \lambda_{j,r-i}, \qquad 1 \le j \le k.$$

In particular $Z_r(x) = W_V(x)$.

6.6 Lemma ([45]) *The ratio $T_{r-i}(x) := W_i(x)/Z_i(x)$ is a polynomial of degree*

$$(i+1)(d-i) - \sum_{l=0}^{i} \lambda_{r-l,\infty} - \sum_{j=1}^{k} m_j(i). \tag{43}$$

In particular, $T_0(x) = 1$. Thus we have $W_{r-j}(x) = T_j(x)Z_{r-j}(x),\ 0 \le j \le r$. The roots of $T_j(x)$ are said to be *the additional roots* of the $(r-j)$-th intermediate Wronskian. If (38) contains more than one element, then the intermediate Wronskians of these elements all differ by some additional root.

6.7 Non-degenerate planes. ([45]) The intersection (38) contains some distinguished elements, called *non-degenerate plane*. Denote by $\Delta(f)$ the discriminant of a polynomial $f(x)$ and by $\mathrm{Res}(f,g)$ the resultant of polynomials $f(x),\ g(x)$.

Definition. *We call $V \in I(\vec{\boldsymbol{\lambda}}, \underline{P})$ a non-degenerate plane if the polynomials $T_0(x), \dots, T_{r-1}(x)$*

i) do not vanish at the ramification points $P_1, \dots, P_k$, i.e. $T_i(z_j) \neq 0$ for all $0 \le i \le r-1$ and all $1 \le j \le k$;

ii) do not have multiple roots: $\Delta(T_i) \neq 0$, for all $0 \le i \le r$;

iii) for each $0 \le i \le r-1$, T_i and T_{i+1} have no common roots: $\mathrm{Res}(T_i, T_{i+1}) \neq 0$.

6.8 Relative discriminants and resultants. Non-degenerate planes correspond to critical points of a certain *generating function* which can be described in terms of *relative* discriminants and resultants. For fixed $\mathbf{z} = (z_1, \dots, z_n)$, any monic polynomial $f(x)$ can be written in a unique way as the product of two monic polynomials $T(x)$ and $Z(x)$ satisfying

$$f(x) = T(x)Z(x),\quad T(z_j) \neq 0,\quad Z(x) \neq 0 \text{ for any } x \neq z_j,\quad 1 \le j \le n. \tag{44}$$

One defines *the relative discriminant of $f(x)$ with respect to* $\mathbf{z}$ as being

$$\Delta_{\mathbf{z}}(f) = \frac{\Delta(f)}{\Delta(Z)} = \Delta(T)(\mathrm{Res}(Z,T))^2,$$

and *the relative resultant of $f_i(x) = T_i(x)Z_i(x)$, $i = 1, 2$, with respect to* $\mathbf{z}$ as

$$\mathrm{Res}_{\mathbf{z}}(f_1, f_2) = \frac{\mathrm{Res}(f_1, f_2)}{\mathrm{Res}(Z_1, Z_2)} = \mathrm{Res}(T_1, T_2)\mathrm{Res}(T_1, Z_2)\mathrm{Res}(T_2, Z_1).$$

If V is a non-degenerate plane in the intersection $I(\vec{\boldsymbol{\lambda}}, \underline{P})$ given by (38), then the decomposition $W_i(x) = T_{r-i}(x)Z_i(x)$ is exactly the same as displayed in (44). The *generating function* of $I(\vec{\boldsymbol{\lambda}}, \underline{P})$ is a rational function such that its critical points determine the non-degenerate elements in such an intersection. Its expression is (see [45]):

$$\Phi_{(\vec{\boldsymbol{\lambda}}, \mathbf{z})}(T_1, \dots, T_{p-1}) = \frac{\Delta_{\mathbf{z}}(W_0) \cdot \ldots \cdot \Delta_{\mathbf{z}}(W_{r-1})}{\mathrm{Res}_{\mathbf{z}}(W_1, W_2) \cdot \ldots \cdot \mathrm{Res}_{\mathbf{z}}(W_{r-1}, W_r)} \tag{45}$$

Part of the following theorem was originally obtained by A. Gabrielov (unpublished), along his investigations of the Wronski map.

6.9 Theorem ([45]) *There is a one-to-one correspondence between the critical points with non-zero critical values of the function* $\Phi_{(\vec{\boldsymbol\lambda},\mathbf{z})}(T_0,\dots,T_{r-1})$ *and the non-degenerate planes in the intersection* $I(\vec{\boldsymbol\lambda},\underline{P})$ *given by* (38).

Namely, every critical point defines the intermediate Wronskians, and hence a non-degenerate plane, see 6.5. Conversely, for every non-degenerate plane one can calculate the intermediate Wronskians, and the corresponding polynomials $T_i(x)$ supply a critical point of the *generating function* (45).

6.10 Relation to Bethe vectors in the Gaudin model (see [43, 45, 35]and references therein)**.** Once one re-writes the generation function (45) in terms of unknown roots of the polynomials T_j's, the same turns into the *master function* associated with the *Gaudin model* of statistical mechanics. In this model one looks for common eignevectors of certain commuting operators called *Gaudin Hamiltonians*.

In the representation theory, the partitions are the weights of the irreducible finite-dimensional $\mathfrak{sl}_p$-representations. Take $\boldsymbol\lambda$ as in Section 6.4 and recall that $\boldsymbol\lambda_\infty$ is the partition related to the point $P_0 = \infty = (0:1) \in \mathbb{P}^1$, after renaming $\boldsymbol\lambda_0$. The Gaudin Hamiltonians act in the subspace of singular vectors of the weight $\boldsymbol\lambda^*_\infty$, the *dual* to $\boldsymbol\lambda_\infty$, in the tensor product of k irreducible $\mathfrak{sl}_p$-representations of weights $\boldsymbol\lambda_1,\dots,\boldsymbol\lambda_k$, respectively, marked by $x(P_j) = z_j$ of Sec. 6.4, respectively.

The *Bethe Ansatz* gives

- a family of vectors of weight $\boldsymbol\lambda^*_\infty$; this family meromorphically depends on a number of complex parameters;
- a system of equations on these parameters;

such that

- the member of the family that corresponds to a solution of the system is a common singular eigenvector of the Gaudin Hamiltonians called the *Bethe vector*.

It turns out that the Bethe system coincides with the system on critical points with non-zero critical value of the function Φ. Thus every orbit of critical points with non-zero critical value (i.e., a set of additional roots of the intermediate Wronskians, i.e., a non-degenerate plane) defines the Bethe vector and vice versa.

7 Wronskians of (hyper)elliptic involutions

The main reference for this section is [8].

7.1 Let C be a smooth projective curve of genus $g \geq 1$ and let $\mathcal{M} \in Pic^2(C)$ such that $h^0(C,\mathcal{M}) = 2$. Then C is elliptic if $g = 1$, and $\mathcal{M} = O_C(2P)$ for some $P \in C$, or hyperelliptic if $g \geq 2$ and then $\mathcal{M}$ is of the form $O_C(2P)$, where P is some of the Weierstrass points of C. If K is the canonical bundle of a hyperelliptic curve, then $K = \mathcal{M}^{\otimes g-1}$ [1, I-D9, p. 41]. (Hyper)elliptic curves are, in a sense, the closest example to rational curves, as they are double ramified coverings of the projective line. How does the wronski map look like in this case? One of the main result of [8] is that the *extended Wronski map*

$$\Gamma_{\mathtt{t}}(\rho_{1,2}) \to \mathbb{P}H^0(C, \mathcal{M}^{\otimes 2} \otimes K) = \mathbb{P}H^0(C, \mathcal{M}^{\otimes g+1})$$

is dominant (notation as in Section 5.2). The proof relies on producing an explicit basis of the space $H^0(\mathcal{M}^{\otimes g+1})$. One also shows that if $\mathbf{v} = (v_0, v_1)$ generates $H^0(\mathcal{M})$, then a basis of $H^0(\mathcal{M}^{\otimes g+1})$ is formed by

$$\{v_0^{g+1-i} v_1^i, W_0(\mathbf{v})\}_{0 \le i \le g+1}$$

where $W_0(\mathbf{v})$ is precisely the Wronskian of the chosen basis $\mathbf{v}$ of $H^0(\mathcal{M})$. More generally one has:

7.2 Theorem. (Cf. [8, 6.6]) *The following direct sum decomposition holds:*

$$H^0(\mathcal{M}^{\otimes a}) = \mathrm{Sym}^{a-g-1} H^0(\mathcal{M}) \cdot W(\boldsymbol{\lambda}) \oplus \mathrm{Sym}^a H^0(\mathcal{M}), \tag{46}$$

where $\mathrm{Sym}^j H^0(\mathcal{M}) \cdot W(\boldsymbol{\lambda})$ *is the image of* $\mathrm{Sym}^j H^0(\mathcal{M})$ *in* $H^0(\mathcal{M}^{\otimes g+1+j})$ *through the multiplication–by–*$W(\boldsymbol{\lambda})$ *map* $H^0(\mathcal{M}^{\otimes j}) \to H^0(\mathcal{M}^{\otimes g+1+j})$.

7.3 Example. Let C be an elliptic curve and let $L = O_C(2P_0)$ for some point $P_0 \in C$. Let P_1, P_2, P_3 be the remaining ramification points of $H^0(L)$. By definition of ramification points there exists v_0 and v_1 vanishing at P_0 and P_1 with multiplicity 2. Then $\mathbf{v} := (v_0, v_1)$ is a basis of $H^0(L)$ and the map

$$(W_0(\mathbf{v}) : v_0^2 : v_0 v_1 : v_1^2) : C \longrightarrow \mathbb{P}^3$$

realizes the elliptic curve as quartic curve in $\mathbb{P}^3$, a complete intersection of two quadrics.

If $(X_0 : X_1 : X_2 : X_3)$ are homogeneous coordinates of $\mathbb{P}^3$, one of the two quadrics is the cone in $\mathbb{P}^3$ of equation $X_1 X_3 - X_2^2 = 0$. To find the second quadric one argues as follows. Let v_2 and v_3 such that $Dv_2(P_2) = 0$ and $Dv_3(P_3) = 0$. Since (v_0, v_1) form a basis of $H^0(L)$, there are $a, b \in \mathbb{C}^*$ such that $v_2 = v_1 - av_0$ and $v_3 = v_1 - bv_0$.

The product $v_0 v_1 v_2 v_3$ is a section of $L^{\otimes 4}$ which vanishes at each ramification point with multiplicity 2. Since the Wronskian $W_0(\mathbf{v})$ vanishes at each P_i with multiplicity 1, up to a multiplicative constant one has the the relation

$$W(\mathbf{v})^2 = v_0 v_1 (v_1 - av_0)(v_1 - bv_0) = v_0 v_1 (v_1^2 - (a+b) v_1 v_0 + ab v_0^2)$$

which shows that the image of C is contained in the quadric

$$X_0^2 - X_2 X_3 - (a+b) X_2^2 + ab X_1 X_2 = 0$$

Again, one may observe that defining $x = v_1/v_0$ and $y = W(\mathbf{v})/v_0^2$ one gets the equation:

$$y^2 = x(x-a)(x-b)$$

which is the classical affine *Weierstrass equation* for an elliptic curve (which then would have the right to be called *Wronski equation*, too). One then sees that its natural compactification lives in the weighted projective space $\mathbb{P}(2,1,1)$ or as a quadric section of a quadric cone of $\mathbb{P}^3$. Notice that on a trivializing set U of C, with local parameter x, one may write $x = f_1/f_0$, where f_0, f_1 are local holomorphic functions representing v_0, v_1 on U, respectively. Then one sees that:

$$(x_{|U})' := \frac{d}{dz}\left(\left.\frac{v_1}{v_0}\right|_U\right) = \frac{d}{dz}\left(\frac{f_1}{f_0}\right) = \frac{W(f_0,f_1)}{f_0^2} = \left.\frac{W(\mathbf{v})}{v_0^2}\right| = y_{|U}$$

i.e. $y = x'$ which is of course compatible with the fact that the "parametric" equations of an elliptic curve in the affine plane are given by $x = \wp_\Lambda(z)$ and $y = \wp'_\Lambda(z)$, where $\wp_\Lambda$ is the Weierstrass $\wp$-function associated to some lattice of $\mathbb{C}$.

7.4 In [8] one shows as in Example 7.3 that a hyperelliptic curve of genus $g \geq 2$ satisfies a Weierstrass type equation expressing the relation between the Wronskian and the product of the sections vanishing twice at its Weierstrass points. It is also intrinsically shown, again using the notion of Wronskian, that the hyperelliptic curve of genus $g \geq 2$ can be realized, for each $a \geq 0$, as a curve of degree $2(g+1+a)$ lying on a rational normal scroll $S(a, g+1+a)$ of $\mathbb{P}^{g+2+a}$. If $a = 0$ it is a quadric section of a cone of degree $g+1$ in $\mathbb{P}^{g+2}$ – see also [9].

8 Wronskians, Linear ODEs and Wronski–Schubert Calculus.

The goal of this last section is to reconcile the first part of this survey, regarding Wronskians of fundamental systems of soutions of linear ODE with the geometry described in the last four sections.

8.1 We shall work in the category of (not necessarily finitely generated) $\mathbb{Q}$-algebras. If t and T are two indeterminates over A, we denote by $A[T]$ and $A[[t]]$ the corresponding A-algebras of polynomials and of formal power series. If $\phi = \sum_{n\geq 0} a_n t^n \in A[[t]]$, the constant term a_0 will be denoted by $\phi(0)$. If $P(T) \in A[T]$ is a polynomial of degree $r+1$, for each $0 \leq i \leq r+1$, we denote by $(-1)^i e_i(P)$ the coefficient of T^{r+1-i} . For instance if P is monic $e_0(P) = 1$ and one writes:

$$P(T) = T^{r+1} - e_1(P)T^r + \ldots + (-1)^{r+1} e_{r+1}(P).$$

Each $\psi \in Hom_{\mathbb{Q}}(A, B)$ induces two obvious A-algebra homomorphisms, $A[T] \to B[T]$ and $A[[t]] \to B[[t]]$. The former is defined by $e_i(\psi(P)) = \psi(e_i(P))$ and the latter by $\sum_{n\geq 0} a_n t^n \mapsto \sum_{n\geq 0} \psi(a_n) t^n$, indicated by the same letter ψ by abuse of notation.

8.2 Let $E_r := \mathbb{Q}[e_1, e_2, \ldots, e_{r+1}]$ be the polynomial $\mathbb{Q}$-algebra in the set of indeterminates $(e_1, \ldots, e_{r+1})$. We call

$$U_{r+1}(T) = T^{r+1} - e_1 T^r + \ldots + (-1)^{r+1} e_{r+1}$$

the *universal monic polynomial* of degree $r+1$, such that $e_i(U_{r+1}(T)) = e_i$, for all $0 \le i \le r+1$. If $\mathbf{a} = (a_0, a_1, \ldots)$ is any sequence in an E_r-module A, set $U_0(\mathbf{a}) = a_0$ and, for each $1 \le i \le r$

$$U_i(\mathbf{a}) = a_i - e_1 a_{i-1} + \ldots + (-1)^i e_i a_0.$$

Let $\mathbf{h} := (h_0, h_1, h_2, \ldots, h_r, h_{r+1}, \ldots)$ be the sequence in E_r defined by the equality of formal power series:

$$\sum_{n\ge 0} h_n t^n = \frac{1}{1 - e_1 t + \ldots + (-1)^{r+1} t^{r+1}} = 1 + \sum_{n\ge 1} (e_1 t - e_2 t + \ldots + (-1)^r e_{r+1} t^{r+1})^n$$

One easily sees that $h_0 = 1$, $h_1 = e_1$, $h_2 = e_1^2 - e_2, \ldots$. In general $h_n = \det(e_{j-i+1})_{1\le i,j\le n}$ (see [14]). Notice that by definition one has $U_i(\mathbf{h}_r) = 0$ for all $1 \le i \le r$.

8.3 Let $\mathbf{x} := (x_0, x_1, \ldots, x_r)$ and $\mathbf{f} := (f_n)_{n\ge 0}$ be two sets of indeterminates over $\mathbb{Q}$. We form the $\mathbb{Q}$-polynomial algebra

$$E_r[\mathbf{x}, \mathbf{f}] := E_r[x_0, x_1, \ldots, x_r, f_0, f_1, \ldots]$$

and the corresponding algebra $E_r[\mathbf{x}, \mathbf{f}][[t]]$ of formal power series. Let $D := d/dt$ be the usual formal derivative of formal power series. If D^j is its jth iterated, one has

$$D^j(\sum_{n\ge 0} a_n \frac{t^n}{n!}) = \sum_{n\ge 0} a_{n+j} \frac{t^n}{n!}.$$

Evaluating the polynomial U_{r+1} at D we get the *universal differential operator:*

$$U_{r+1}(D) = D^{r+1} - e_1 D^r + \ldots + (-1)^{r+1} e_{r+1}.$$

Let $f := \sum_{n\ge 0} f_n \frac{t^n}{n!} \in \mathbb{Q}[\mathbf{f}][[t]] \subseteq E_r[\mathbf{x}, \mathbf{f}][[t]]$. Consider the universal Cauchy problem for ordinary linear ODE with constant coefficients:

$$\begin{cases} U_{r+1}(D)y &= f \\ D^i y(0) &= x_i \end{cases}, \tag{47}$$

To solve (47) means to find an element $g \in E_r[\mathbf{x}, \mathbf{f}]$ satisfying the differential equation such that $D^i g(0) = x_i$ (the first r-terms of the formal power series g are x_i, $0 \le i \le r$).

8.4 Theorem. *Let $\sum_{n\ge 0} p_n \cdot t^n \in E_r[\mathbf{x}, \mathbf{f}][[t]]$ defined by:*

$$\sum_{n\ge 0} p_n t^n = \frac{U_0(\mathbf{x}) + U_1(\mathbf{x})t + \ldots + U_r(\mathbf{x})t^r + \sum_{n\ge r+1} f_{n-r-1} t^n}{1 - e_1 t + \ldots + (-1)^{r+1} e_{r+1} t^{r+1}} \tag{48}$$

Then

$$g := \sum_{n\ge 0} p_n \frac{t^n}{n!}$$

is the unique solution of the Cauchy problem (47).

Proof. It consists in a routine straightforward check. See [21] for details. ∎

8.5 Theorem *Let A be any $\mathbb{Q}$-algebra, $P \in A[T]$ and $\phi = \sum_{n\geq 0} \phi_n t^n/n! \in A[[t]]$. For each $(b_0, b_1, \ldots, b_r) \in A^{r+1}$, the unique $\mathbb{Q}$-algebra homomorphism mapping $x_i \mapsto a_i$, $e_i \mapsto e_i(P)$ and $f_i \mapsto \phi_i$, maps g to the unique solution of the Cauchy problem*

$$\left\{ \begin{array}{lcl} P(D)y & = & \phi \\ D^i y(0) & = & a_i \end{array} \right. \tag{49}$$

Proof. Obvious, as Cauchy problem (49) is just a specialization of the universal one. ∎

8.6 Corollary. *For each $0 \leq i \leq r$, let $\psi_i : E_r[\mathbf{x}, \mathbf{f}] \to E_r$ the unique E_r algebra homomorphism over the identity sending $\mathbf{x} \mapsto (\underbrace{0, \ldots, 0}_{i}, 1, h_1, \ldots, h_{r-i})$ and $\mathbf{f} \mapsto (0, 0, \ldots)$. Let $u_i := \psi_i(g) \in E_r[[t]]$, where g is the universal solution of the Cauchy problem (47). Then*

$$\mathbf{u}_r = (u_0, u_1, \ldots, u_r)$$

is an E_r-basis of $\ker U_{r+1}(D)$.

Proof. That each u_i is a solution of $U_{r+1}(D)y = 0$ follows from Theorem 8.4 and by specialization. Furthermore, if $u := a_0u_0 + a_1u_1 + \ldots + a_ru_r = 0$, then u is the unique solution of $U_{r+1}(D)y = 0$, with all the initial conditions equal to 0. Then it is zero, by the unicity stated in 8.5, i.e. $(u_0, \ldots, u_r)$ are linearly independent. ∎

8.7 Corollary. *Let A be any $\mathbb{Q}$-algebra and $P \in A[T]$. Let $\psi : E_r \to A$ be the unique morphism mapping $e_i \mapsto e_i(P)$. Then $(\psi(u_0), \psi(u_1), \ldots, \psi(u_r))$ is an A-basis of* $\ker P(D)$. ∎

Corollary 8.7 may be suggestively stated by saying that $\ker P(D) \cong \ker U_{r+1}(D) \otimes_{E_r} A$.

8.8 Recall the definition of Schur polynomial (26) associated to a partition and to a formal power series. If $n \in \mathbb{Z}$ and $\boldsymbol{\mu} \in \mathcal{P}$ has weight n, by $\binom{n}{\boldsymbol{\mu}}$ we mean the coefficient of $x_0^{\mu^0} x_1^{\mu^1} \ldots x_r^{\mu^r}$ in the expansion of $(x_0 + x_1 + \ldots + x_r)^n$. If $n < 0$ one has $\binom{n}{\boldsymbol{\mu}} = 0$ and if $n \geq 0$, with the convention that $0! = 1$, one has

$$\binom{n}{\boldsymbol{\mu}} = \frac{n!}{\mu_0!\mu_1!\ldots\mu_r!}.$$

8.9 Theorem. *For each $\boldsymbol{\lambda} \in \mathcal{P}$, the following equality holds:*

$$W_{\boldsymbol{\lambda}}(\mathbf{u}_r) = \sum_{n\geq 0} \sum_{|\boldsymbol{\mu}|=n} \binom{n}{\boldsymbol{\mu}} \Delta_{\boldsymbol{\lambda}+\boldsymbol{\mu}}(\mathbf{h}) \frac{t^n}{n!}$$

In particular, the expression of the constant term is:

$$W_{\boldsymbol{\lambda}}(\mathbf{u}_r)(0) = \Delta_{\boldsymbol{\lambda}}(\mathbf{h}).$$

Proof. It is a straightforward combinatorial exercise made easy by the use of the basis $\mathbf{u}_r$ found in 8.6. See [20] and/or [21] for details.

8.10 Proposition. *Giambelli's formula for Wronskians holds, i.e.*

$$W_{\boldsymbol{\lambda}}(\mathbf{u}_r) = \Delta_{\boldsymbol{\lambda}}(\mathbf{h}) \cdot W_0(\mathbf{u}_r).$$

Proof. By Remark 2.5, $W_{\boldsymbol{\lambda}}(\mathbf{u}_r)$ is proportional to $W_0(\mathbf{u}_r)$, i.e. $W_{\boldsymbol{\lambda}}(\mathbf{u}_r) = \gamma_{\boldsymbol{\lambda}} W_0(\mathbf{u}_r)$ for some $\gamma_{\boldsymbol{\lambda}} \in E_r$ and two formal power series are proportional if and only if all the coefficients of all the powers of t are proportionals. Then:

$$\gamma_{\boldsymbol{\lambda}} = \frac{W_{\boldsymbol{\lambda},0}(\mathbf{u}_r)(0)}{W_0(\mathbf{u}_r)(0)} = \Delta_{\boldsymbol{\lambda}}(\mathbf{c}),$$

because of the last remark of Theorem 8.9.

8.11 Corollary. *Pieri's formula for generalized Wronskians holds:*

$$h_i W_{\boldsymbol{\lambda}}(\mathbf{u}_r) = \sum_{\boldsymbol{\mu}} W_{\boldsymbol{\mu}}(\mathbf{u}_r)$$

where the sum is over all the partitions $\boldsymbol{\mu} = (\mu_0, \mu_1, \ldots, \mu_r)$ such that $|\boldsymbol{\mu}| = i + |\boldsymbol{\lambda}|$ and

$$\mu_0 \geq \lambda_0 \geq \mu_1 \geq \lambda_1 \geq \ldots \geq \mu_r \geq \lambda_r. \tag{50}$$

It is well known that Giambelli's and Pieri's implies each other. See e.g. [14, Lemma A.9.4].

8.12 Let now $\rho_{r,d} : G \to X$ be a Grassmann bundle, where $G := G(r+1, F)$ and F is a vector bundle of rank $d+1$. As recalled in Section 4.12, $A_*(G)$ is generated as $A^*(X)$-module by

$$\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q}_r - \rho_{r,d}^* F)) \cap [G].$$

The exact sequence (21) implies that $c_t(\mathcal{S}_r)c_t(\mathcal{Q}_r) = c_t(\rho_{r,d}^* F)$, which is equivalent to

$$1 = c_t(\mathcal{S}_r)\frac{c_t(\mathcal{Q}_r)}{c_t(\rho_{r,d}^* F)} = c_t(\mathcal{Q}_r - \rho_{r,d}^* F))c_t(\mathcal{S}_r)$$

Set $\varepsilon_i = (-1)^i c_i(\mathcal{S}_r)$ and consider the differential equation

$$D^{r+1}y - \varepsilon_1 \cdot D^r y + \ldots + (-1)^{r+1}\varepsilon_{r+1} \cdot y = 0 \tag{51}$$

by looking for solutions in $A^*(G) \otimes \mathbb{Q}[[t]]$. We know by Corollary (8.7) that the unique morphism $\psi : E_r \to A^*(G) \otimes \mathbb{Q}$ sending $e_i \mapsto \varepsilon_i$, maps the universal fundamental system $(u_0, u_1, \ldots, u_r)$ to $\mathbf{v}_r = (v_0, v_1, \ldots, v_r)$, where $v_i = \psi(u_i)$ and, as a consequence, it maps $h_i \mapsto c_i(\mathcal{Q} - \rho_{r,d}^* F)$ and $W_{\boldsymbol{\lambda}}(\mathbf{u}_r)$ to $W_{\boldsymbol{\lambda}}(\mathbf{v}_r)$. Then we have proven that

$$\Delta_{\boldsymbol{\lambda}}(c_t(\mathcal{Q} - \rho_{r,d}^* F) = \frac{W_{\boldsymbol{\lambda}}(\mathbf{v}_r)}{W_0(\mathbf{v}_r)}.$$

In other words, the Chow group $A_*(X)$ can be identified with the $A^*(X)$-module generated by the generalized Wronskians associated to the basis $\mathbf{v}_r$ of solutions of the differential equation (51). In particular we have shown that the class $[\Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^* F_\bullet)]$ of the generalized Wronskian variety $\Omega_{\boldsymbol{\lambda}}(\rho_{r,d}^* F_\bullet)$ is an $A^*(X)$-linear combination of ratios of generalized Wronskians associated to the basis $\mathbf{v}_r$ of (51), by virtue of (28), (29) and (30).

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