On Schur function expansions of Thom polynomials

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Why should we expect a city to cure us of our spiritual pains? Perhaps because we cannot help, loving our city like a family. But we still have to decide which part of the city we love and invent the reasons why.

Orhan Pamuk, Istanbul: memories and the city.

Abstract

We discuss computations of the Thom polynomials of singularity classes of maps in the basis of Schur functions. We survey the known results about the bound on the length and a rectangle containment for partitions appearing in such Schur function expansions. We describe several recursions for the coefficients. For some singularities, we give old and new computations of their Thom polynomials.

1 Introduction

A prototype of the formulas considered in the present paper, is the following classical result. Let $f : M \to N$ be a holomorphic, surjective map of compact Riemann surfaces. For $x \in M$, we set

$$e_x := \text{number of branches of } f \text{ at } x.$$

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Then the ramification divisor of f is equal to

$$\sum (e_x - 1)x \,.$$

The *Riemann-Hurwitz formula* asserts that

$$\sum_{x \in M} (e_x - 1) = 2g(M) - 2 - \deg(f)\big(2g(N) - 2\big) \,. \tag{1}$$

(See, e.g., [14].) The right-hand side of Eq.(1) can be rewritten as

$$f^* c_1(N) - c_1(M) \,,$$

and gives us the Thom polynomial of the singularity A_1 of maps between curves.

In general, according to the monograph [1], the global behavior of singularities of maps $f : M \to N$ of complex analytic manifolds, is governed by their *Thom polynomials*. Knowing the Thom polynomial of a singularity class Σ, one can compute the cohomology class represented by the Σ-points of a map f. We shall recall the definition of a Thom polynomial in Section 3.

The term "Thom polynomial" has nowadays rather wide meaning. In the present paper, however, it will mean a classical Thom polynomial of the singularity classes of maps (cf. [40]). We shall work here with *complex* manifolds[1].

An explicit[2] computation of a Thom polynomial is usually a difficult task. At first, the computations of Thom polynomials were performed in the basis of monomials in the Chern classes. But around 2004, two papers: [5] and [31] appeared independently, with computations of some Thom polynomials in the basis of *Schur functions*. (The two papers concerned different singularity classes.) One should stress that even with a powerful theory of symmetric functions from [20] and [17], a passage from the monomial basis to the Schur basis is rather difficult: it is possible "in theory" but it is rather difficult in practice (of course, we speak here about "large" expressions).

It is, by no means, reasonable to ask why to work with Schur function expansions? One of the aims of the present paper is (to try) to answer this question. Of course, an important role of Schur functions in geometry was known earlier, e.g., by the Schubert Calculus (see also [16], [28], [10] – to mention just a few references). The latter reference gives a wide geometric motivation of the importance and ubiquity of Schur functions in algebraic geometry.

A basic property of Schur function expansions of Thom polynomials is the nonnegativity of the coefficients proved by Andrzej Weber and the second named author in [35] (see also [36]). These positive coefficients often have a pleasant algebraic structure, e.g., satisfy some recursions. This allow one to organize the computations of them in a pretty systematic way. Among these coefficients, we find numbers appearing in different contexts in enumerative geometry, e.g., complete quadrics (see [32]). More, as it follows from a recent paper [22], the positivity of the coefficients of Schur function expansions of classical Thom polynomials leads to upper bounds for the coefficients of Legendrian Thom polynomials expanded in an appriopriate basis.

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[1] A manifold here is always *smooth*.

[2] Even the word "explicit" has different meanings for different authors working on Thom polynomials.

Another feature comes from the fact that Thom polynomials are closely related with degeneracy loci of the cotangent map

$$f^* : T^*N_M \to T^*M$$

(by T^*N_M we denote the cotangent bundle of N pull backed by f to M). Polynomials supported on such degeneracy loci were described using Schur functions in [28]; this helps to study the Schur function expansions of Thom polynomials of other singularity classes.

In the present article, we survey basically only those papers, where the Schur function expansions of Thom polynomials play a significant role in the process of their computations or/and help in understanding their structure.

In [5], the authors computed the Thom polynomial of the second order Thom-Boardman singularity classes

$$\Sigma^{i,j} : M^m \to N^{m-i+1}$$

via its Schur function expansion, and conjectured the positivity of Schur function expansion for all Thom-Boardman singularity classes.

In [31], the second author stated some formulas for Thom polynomials of singularities $I_{2,2}, A_3 : M^m \to N^{m+k}$ (any k) and some partial result for $A_i : M^m \to N^{m+k}$ (any i, k). These expressions had the form of Schur function expansions. The details were given in [32], [19] and [33]. In Sections 7 and 8, we discuss some essential computations from these papers.

In [23], [24], the first author computed Schur function expansions for $A_4 : M^m \to N^{m+k}$ ($k = 2, 3$) and $III_{2,3} : M^m \to N^{m+k}$ (any k).

This paper is organized as follows.

In Section 2, we recall the definition and properties of Schur functions, including: cancellation–, vanishing–, basis– , and factorization property.

In Section 3, we recall the notion of a singularity class, and, following Thom [40], attach to a singularity class its Thom polynomial.

In Section 4, we discuss the $\mathcal{P}$-polynomials of singularity classes. From the structure of the $\mathcal{P}$-ideal of Σ^i, we deduce some result on a rectangle containment for partitions appearing in the Schur function expansion of a Thom polynomial of $\Sigma \subset \Sigma^i$ (Theorem 13).

In Section 5, We discuss a way of computing of Thom polynomials of the closures of single R-L orbits in a space of jets of maps: $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$, called there "singularities" after [37]. This is a "method of restriction equations" that we learned from [37].

In Section 6, we collect formulas for the Chern and Euler classes of singularities, and show by an example, how one can compute them.

In Section 7, we state some general properties of the Schur function expansions of Thom polynomials of singularities. Theorem 13 is reinterpreted for singularities. We discuss the Thom polynomial of $III_{2,3}$ for any k. For any i, k, we give the 1-part of the Thom polynomial of A_i. We discuss also recent results of Féher and Rimányi [7] giving a bound on the lengths of partitions appearing in Schur function expansions, and certain basic recursion (on k).

In Section 8, we recall Pascal staircases, and survey Schur function expansions of Thom polynomials of $I_{2,2}$ and A_3 from [32] and [19]. Their coefficients obey some (other) recursions on k. We provide details of two computations with extensive use of the algebra of Schur functions and multi-Schur functions.

In Section 9, we discuss the Schur function expansions of the Thom polynomials of $III_{3,3}$.

In Section 10, we discuss some properties of the Thom polynomials of $I_{2,3}$.

In the appendices (Section 11 and 12), we give the Schur function expansions of the Thom polynomials of $III_{3,3}$ and $I_{2,3}$ for several k.

This is basically a survey paper. Some new material is gathered in the last four sections. We lectured on this material at IMPANGA seminars in Warsaw and Cracow.

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2 Schur functions

The main reference for this section, for the conventions and notation, is [17]. This book studies (among others) multi-Schur functions which are a useful generalization of Schur functions. We shall need them in this paper. But we start our discussion with Schur functions.

For $m \in \mathbf{N}$, by an alphabet $\mathbb{A}$ of cardinality m we shall mean a finite set of indeterminates $\mathbb{A} = \{a_1, \ldots, a_m\}$. Sometimes, to point out the cardinality of an alphabet $\{a_1, \ldots, a_m\}$ we shall denote it by $\mathbb{A}_m$.

We shall often identify an alphabet $\{a_1, \ldots, a_m\}$ with the sum $a_1 + \cdots + a_m$.

Definition 1 *Given two alphabets $\mathbb{A}$, $\mathbb{B}$, the complete functions $S_i(\mathbb{A}{-}\mathbb{B})$ are defined by the generating series (with z an extra variable):*

$$\sum S_i(\mathbb{A}{-}\mathbb{B}) z^i := \prod_{b \in \mathbb{B}} (1 - bz) / \prod_{a \in \mathbb{A}} (1 - az) \,. \tag{2}$$

We see that $S_i(\mathbb{A} - \mathbb{B})$ interpolates between $S_i(\mathbb{A})$ - the complete symmetric function of degree i in $\mathbb{A}$ and $S_i(-\mathbb{B})$ - the elementary symmetric function of degree i in $\mathbb{B}$ times $(-1)^i$. For example, $S_3(\mathbb{A} - \mathbb{B})$ is equal to

$$S_3(\mathbb{A} - \mathbb{B}) = S_3(\mathbb{A}) - S_2(\mathbb{A})\Lambda_1(\mathbb{B}) + S_1(\mathbb{A})\Lambda_2(\mathbb{B}) - \Lambda_3(\mathbb{B}) \,,$$

where $\Lambda_i(\mathbb{B})$ denotes the i-th elementary symmetric function in $\mathbb{B}$.

A weakly increasing sequence $(i_1, i_2, \ldots, i_s)$ of nonnegative integers is called a *partition*. The number it divides into parts, $|I| = i_1 + i_2 + \cdots + i_s$, is called the *weight* of I. The nonzero i_p are called the *parts* of I. The number of nonzero parts is called the *length* of I.

Given two partitions $I = (i_1, i_2, \ldots, i_s)$ and $J = (j_1, j_2, \ldots, j_t)$, we shall say that I *is contained in* J, and write $I \subset J$, if for any $p = 0, 1, 2, \ldots$, we have $i_{s-p} \le j_{t-p}$.

Following [17], we give

Definition 2 *Given a partition $I = (i_1, i_2, \ldots, i_s) \in \mathbf{N}^s$, and alphabets $\mathbb{A}$ and $\mathbb{B}$, the Schur function $S_I(\mathbb{A} - \mathbb{B})$ is*

$$S_I(\mathbb{A} - \mathbb{B}) := \left| S_{i_q+q-p}(\mathbb{A}-\mathbb{B}) \right|_{1 \le p,q \le s} . \tag{3}$$

In other words, we put on the diagonal from to bottom: $S_{i_1}, S_{i_2}, \ldots, S_{i_s}$, and then, in each column, the indices of the successive S_j's should increase by one from bottom to top. For example, if $I = (1, 3, 3, 4, 5)$, then

$$S_I(\mathbb{A}-\mathbb{B}) = \begin{vmatrix} S_1(\mathbb{A}-\mathbb{B}) & S_4(\mathbb{A}-\mathbb{B}) & S_5(\mathbb{A}-\mathbb{B}) & S_7(\mathbb{A}-\mathbb{B}) & S_9(\mathbb{A}-\mathbb{B}) \\ 1 & S_3(\mathbb{A}-\mathbb{B}) & S_4(\mathbb{A}-\mathbb{B}) & S_6(\mathbb{A}-\mathbb{B}) & S_8(\mathbb{A}-\mathbb{B}) \\ 0 & S_2(\mathbb{A}-\mathbb{B}) & S_3(\mathbb{A}-\mathbb{B}) & S_5(\mathbb{A}-\mathbb{B}) & S_7(\mathbb{A}-\mathbb{B}) \\ 0 & S_1(\mathbb{A}-\mathbb{B}) & S_2(\mathbb{A}-\mathbb{B}) & S_4(\mathbb{A}-\mathbb{B}) & S_6(\mathbb{A}-\mathbb{B}) \\ 0 & 1 & S_1(\mathbb{A}-\mathbb{B}) & S_3(\mathbb{A}-\mathbb{B}) & S_5(\mathbb{A}-\mathbb{B}) \end{vmatrix} .$$

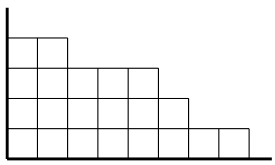
These functions are often called *supersymmetric Schur functions* or *Schur functions in difference of alphabets*. See [39], [3], [29], [34], [20] and [17] for their study.

We have the following *cancellation property*: for alphabets $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$,

$$S_I((\mathbb{A} + \mathbb{C}) - (\mathbb{B} + \mathbb{C})) = S_I(\mathbb{A} - \mathbb{B}) . \tag{4}$$

We shall use the simplified notation $i_1 i_2 \cdots i_s$ or $i_1, i_2, \ldots, i_s$ for a partition $(i_1, i_2, \ldots, i_s)$ (the latter one if $i_s \ge 10$). Also, we shall write (i^s) for the partition $(i, \ldots, i)$ (s times).

A partition I has a graphical representation due to Ferrers, called its *diagram*: it is a diagram of left packed square boxes with $i_1, i_2, \ldots, i_s$ the number of boxes in the successive rows. For example, the diagram of the partition $(2, 5, 6, 8)$ is:



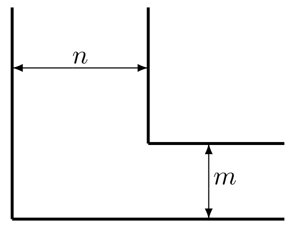
Given two partitions I and J, if we put their diagrams in such a position that they share the lowest row and the leftmost column, then " $I \subset J$ " iff the set of boxes of the diagram of I is contained in the set of boxes of the diagram of J.

We record the following property:

$$S_I(\mathbb{A} - \mathbb{B}) = (-1)^{|I|} S_J(\mathbb{B} - \mathbb{A}) = S_J(\mathbb{B}^* - \mathbb{A}^*) , \tag{5}$$

where J is the conjugate partition of I (i.e. the consecutive rows of the diagram of J are the transposed columns of the diagram of I), and $\mathbb{A}^*$ denotes the alphabet $\{-a_1, -a_2, \ldots\}$.

Fix two positive integers m and n. Let I be a partition. Suppose that the diagram of I and the following (m, n)-*hook* :

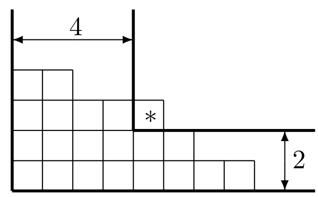


share the lowest row and the leftmost column. If the diagram of I is contained in this hook, then we say that the partition I *is contained in* the (m, n)-hook.

We record the following *vanishing property*. Given alphabets $\mathbb{A}$ and $\mathbb{B}$ of cardinalities m and n, if the diagram of a partition I is not contained in the (m, n)-hook, then

$$S_I(\mathbb{A} - \mathbb{B}) = 0\,. \tag{6}$$

For instance, $I = (2, 5, 6, 8)$ is not contained in the $(2, 4)$-hook



Therefore $S_{2568}(\mathbb{A}_2 - \mathbb{B}_4) = 0$. This vanishing property is an immediate consequence of the factorization property (see Eq.(9)).

Moreover, we have the following result.

Theorem 3 *If $\mathbb{A}$ and $\mathbb{B}$ are alphabets of cardinalities m and n, then the Schur polynomials $S_I(\mathbb{A} - \mathbb{B})$, where I runs over partitions contained in the (m, n)-hook, are* **Z***-linearly independent. (I.e., they form a basis of the abelian group of supersymmetric Schur functions in $\mathbb{A}$ and $\mathbb{B}$.)*

For a proof, see, e.g., [34, Proposition 2.3].

Note 4 We shall often identify partitions with their diagrams, as is customary.

It is handy to adopt the following

Convention 5 Instead of introducing in the argument of a symmetric function, formal variables which will be specialized, we write $\boxed{r}$ for a variable which will be specialized to r (r can be $2x_1$, $x_1 + x_2, \ldots$). For example,

$$S_2(x_1 + x_2) = x_1^2 + x_1x_2 + x_2^2 \quad \text{but} \quad S_2\left(\boxed{x_1 + x_2}\right) = (x_1 + x_2)^2 = x_1^2 + 2x_1x_2 + x_2^2\,.$$

This convention stems from [18] where the reader can find instructive examples of its use.

Definition 6 *Given two alphabets* $\mathbb{A}, \mathbb{B}$, *we set*

$$R(\mathbb{A},\mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a-b)\,, \tag{7}$$

the resultant of $\mathbb{A}, \mathbb{B}$.

Thus $R(\mathbb{A},\mathbb{B})$ is the resultant of the polynomials $R(x,\mathbb{A}) = R(\{x\},\mathbb{A})$ and $R(x,\mathbb{B})$.

We now record some properties of Schur functions that are used in our computations with Thom polynomials.

The first one is the following *linearity formula*. We have (see [17])

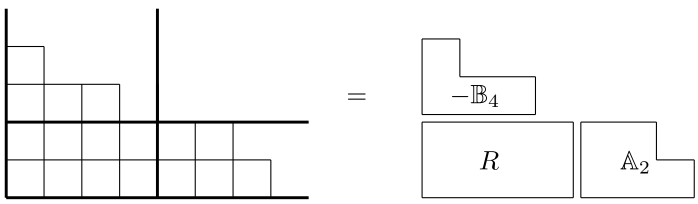
$$S_j(-\mathbb{E}-\mathbb{B}_n) = S_j(-\mathbb{E}-\mathbb{B}_{n-1}) - b_n S_{j-1}(-\mathbb{E}-\mathbb{B}_{n-1})\,. \tag{8}$$

This equality is used quite often to estimate the sizes of partitions indexing Schur function expansions of Thom polynomials (see, e.g., [32], [19], [23], [24], [25]). It serves also to establish an extremely useful *Transformation Lemma* (see Lemma 7).

The second one is the following *factorization property* [3]. For partitions $I = (i_1, \ldots, i_m)$ and $J = (j_1, \ldots, j_s)$, we have

$$S_{(j_1,\ldots,j_s,i_1+n,\ldots,i_m+n)}(\mathbb{A}_m - \mathbb{B}_n) = S_I(\mathbb{A}_m)\ R(\mathbb{A}_m,\mathbb{B}_n)\ S_J(-\mathbb{B}_n)\,. \tag{9}$$

For example, with $m = 4$, $n = 2$, $I = (2,3)$, $J = (1,3)$, we have



$$S_{1367}(\mathbb{A}_2 - \mathbb{B}_4) = S_{23}(\mathbb{A}_2) R(\mathbb{A}_2,\mathbb{B}_4) S_{13}(-\mathbb{B}_4)\,.$$

This factorization property is useful to simplify the h-parts (cf. the end of Section 4) of Thom polynomials (see Section 8, and [32], [19], [23], [24]). Cf. also [26].

We shall also need multi-Schur functions. Given s, two sets of alphabets $\{\mathbb{A}^1, \mathbb{A}^2, \ldots, \mathbb{A}^s\}$, $\{\mathbb{B}^1, \mathbb{B}^2, \ldots, \mathbb{B}^s\}$, and partition $I = (i_1, \ldots, i_s)$, we define following [17] the *multi-Schur function*

$$S_I(\mathbb{A}^1 - \mathbb{B}^1, \ldots, \mathbb{A}^s - \mathbb{B}^s) = \left| S_{i_q+q-p}(\mathbb{A}^q - \mathbb{B}^q) \right|_{1\le p,q\le s}\,. \tag{10}$$

In case where the alphabets are repeated, we indicate by a semicolon the corresponding bloc separation. For example,

$$S_{i,i;i}(\mathbb{A}-\mathbb{C};\mathbb{B}-\mathbb{D}) = S_{i,i,i}(\mathbb{A}-\mathbb{C},\mathbb{A}-\mathbb{C},\mathbb{B}-\mathbb{D})\,.$$

We record the following *Transformation Lemma* (see [17, Lemma 1.4.1])

Lemma 7 *Let $\mathbb{D}^0, \mathbb{D}^1, \ldots, \mathbb{D}^{s-1}$ be a family of alphabets such that* $\mathrm{card}(\mathbb{D}^i) \le i$ *for $0 \le i \le s-1$. Then the multi-Schur function $S_I(\mathbb{A}^1 - \mathbb{B}^1, \ldots, \mathbb{A}^s - \mathbb{B}^s)$ is equal to the determinant*

$$\left|S_{i_q+q-p}(\mathbb{A}^q - \mathbb{B}^q - \mathbb{D}^{s-p})\right|_{1\le p,q\le s}.$$

In other words, one does not change the value of a multi-Schur function by replacing in row p the difference $\mathbb{A} - \mathbb{B}$ by $\mathbb{A} - \mathbb{B} - \mathbb{D}^{s-p}$. We leave it to the reader to prove this result.

3 Thom polynomials of singularity classes of maps

Fix $m, n, p \in \mathbf{N}$. Consider the space $\mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$ of p-jets of analytic functions from $\mathbf{C}^m$ to $\mathbf{C}^n$ which map 0 to 0. Consider the natural right-left action of the group $\mathrm{Aut}^p_m \times \mathrm{Aut}^p_n$ on $\mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$, where Aut^p_n denotes the group of p-jets of automorphisms of $(\mathbf{C}^n, 0)$. By a *singularity class* we shall mean a closed algebraic right-left invariant subset of $\mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$. Given complex analytic manifolds M^m and N^n, a singularity class $\Sigma \subset \mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$ defines the subset $\Sigma(M, N) \subset \mathcal{J}^p(M, N)$, where $\mathcal{J}^p(M, N)$ is the space of p-jets from M to N.

Theorem 8 *Let $\Sigma \subset \mathcal{J}^p(\mathbf{C}^m_0, \mathbf{C}^n_0)$ be a singularity class. There exists a universal polynomial $\mathcal{T}^\Sigma$ over $\mathbf{Z}$ in $m+n$ variables $c_1, \ldots, c_m, c'_1, \ldots, c'_n$ which depends only on Σ, m and n such that for any complex analytic manifolds M^m, N^n and for almost any map*[3] *$f : M \to N$, the class of*

$$\Sigma(f) := f_p^{-1}(\Sigma(M, N))$$

is equal to

$$\mathcal{T}^\Sigma(c_1(M), \ldots, c_m(M), f^*c_1(N), \ldots, f^*c_n(N)),$$

where $f_p : M \to \mathcal{J}^p(M, N)$ is the p-jet extension of f.

This is a theorem due to Thom, see [40].

If a singularity class Σ is *stable* (e.g. closed under the contact equivalence, see, e.g., [7]), then $\mathcal{T}^\Sigma$ depends on $c_i(TM - TN_M)$.

Let $f : M \to N$ be a map of complex analytic manifolds. In the present paper, we shall work with the cotangent map

$$f^* : T^*N_M \to T^*M\,, \tag{11}$$

rather than with the tangent one. Given a partition I, we define

$$S_I(T^*M - T^*N_M)$$

to be the effect of the following specialization of $S_I(\mathbb{A}-\mathbb{B})$: the indeterminates of $\mathbb{A}$ are set equal to the Chern roots of T^*M, and the indeterminates of $\mathbb{B}$ to the Chern roots of T^*N_M.

[3] The Riemann-Hurwitz formula quoted in Introduction holds for any surjective f. In the theory of Thom polynomials we restrict ourselves only to *almost all* maps, i.e., the maps from some open subset in the space of all maps.

Given a singularity class Σ, the Poincaré dual of $\Sigma(f)$, for almost any map $f : M \to N$, will be written in the form

$$\sum_I \alpha_I S_I(T^*M - T^*N_M) \tag{12}$$

with integer coefficients α_I.

Accordingly, we shall write

$$\mathcal{T}^\Sigma = \sum_I \alpha_I S_I\,, \tag{13}$$

where S_I is identified with $S_I(\mathbb{A}{-}\mathbb{B})$ for the universal Chern roots $\mathbb{A}$ and $\mathbb{B}$.

For example, consider the singularity class $\Sigma = \Sigma^i$. So, $m - i \le n$, and looking at the $(m-i)$th degeneracy locus of the cotangent map (11), we have

$$\mathcal{T}^{\Sigma^i} = S_{(n-m+i)^i}\,,$$

the Giambelli-Thom-Porteous formula (see [27]).

A basic result on Schur function expansions of Thom polynomials of singularity classes is

Theorem 9 *([35]) Let Σ be a nontrivial stable singularity class. Then for any partition I, the coefficient α_I in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}^\Sigma = \sum \alpha_I S_I\,,$$

is nonnegative and $\sum_I \alpha_I > 0$.

This result was conjectured in [5]. Thus, it is not obvious. But its proof is almost obvious. The original proof in [35] used the classification space of singularities and the Fulton-Lazarsfeld theorem [9]. We now give an outline of another proof (of nonnegativity only), communicated to the second author by Klyachko (Ankara, 2006) and, independently, by Kazarian [12].

Sketch of proof of the theorem First, using some Veronese map, we "materialize" all singularity classes in sufficiently large Grassmannians.

We fix a singularity class Σ and take the Schur function expansion of $\mathcal{T}^\Sigma$. We take sufficiently large Grassmannian containing Σ and such that specializing $\mathcal{T}^\Sigma$ in the Chern classes of the tautological (quotient) bundle Q, we do not lose any Schur summand.

We identify by the Giambelli formula (see [8], p.146 and [10], p.18, p.27), a Schur polynomial of Q with the corresponding Schubert cycle.

To test a coefficient in the Schur function expansion of $\mathcal{T}^\Sigma$, we intersect $[\Sigma]$ with the corresponding *dual* Schubert cycle (see [8], p.150). Using the Bertini-Kleiman theorem [13], we put the cycles in a general position, so that we can reduce to set-theoretic intersection, which is nonnegative. $\square$

Note 10 If $\alpha_I \neq 0$, then we shall say that I belongs to the indexing set of the Schur function expansion of $\mathcal{T}^\Sigma$, or that the partition I appears in the Schur function expansion of $\mathcal{T}^\Sigma$, or just I appears in $\mathcal{T}^\Sigma$.

It appears that this positivity result can be used to find upper bounds for the coefficients of expansions of Legendrian Thom polynomials in a suitable basis, see [22]. For the Lagrangian Thom polynomials, this is the basis of the so called $\widetilde{Q}$-functions, see [21].

We record now a variant valid for *not necessary* stable singularity classes.

Theorem 11 *([36]) Let Σ be a nontrivial singularity class. Then for any partitions I, J, the coefficient $\alpha_{I,J}$ in the Schur function expansion of the Thom polynomial*

$$\mathcal{T}^\Sigma = \sum \alpha_{I,J} S_I(T^*M) S_J(TN_M)$$

is nonnegative, and $\sum_{I,J} \alpha_{I,J} > 0$.

(It is important that we use the cotangent bundle to the source M and the tangent bundle to the target N.) The latter result implies the former, see [36]. This last paper contains also some variations on positivity of generalized Thom polynomials, and emphasizes the role of cone classes for globally generated and ample vector bundles, following Fulton and Lazarsfeld.

4 $\mathcal{P}$-ideals of singularity classes

More generally, it is natural to consider the $\mathcal{P}$-ideal of a singularity class Σ, denoted by $\mathcal{P}^\Sigma$. This is the subset in the polynomial ring $\mathbf{Z}[c_1, \ldots, c_m, c'_1 \ldots, c'_n]$, consisting of all polynomials P which satisfy the following universality property. For any complex analytic manifolds M^m, N^n and almost any map $f : M \to N$,

$$P(c_1(M), \ldots, c_m(M), f^*c_1(N), \ldots, f^*c_n(N))$$

is supported on $\Sigma(f)$. (This means – see [28], [10] – that the class of a cycle on M in $H(M, \mathbf{Z})$ is in the image of $H(\Sigma(f), \mathbf{Z}) \to H(M, \mathbf{Z})$.)

Note 12 These ideals were first studied (1988) in [28] for the classes $\Sigma = \Sigma^i$. They were rediscovered (2004) in [6] in the context of group actions.

For $\Sigma = \Sigma^i$, $\mathcal{P}^\Sigma$ is simply the ideal of polynomials which – after specialization to the Chern classes of M and N – support cycles in the locus D, where

$$\dim\bigl(\mathrm{Ker}(f_* : TM \to TN_M)\bigr) \geq i\,.$$

for almost any map $f : M \to N$. (This means that the class of a cycle on M in $H(M, \mathbf{Z})$ is in the image of $H(D, \mathbf{Z}) \to H(M, \mathbf{Z})$.)

Note that in terms of the cotangent map, D is the locus where

$$\mathrm{rank}\bigl(f^* : T^*N_M \to T^*M\bigr) \leq m - i\,,$$

for almost any map $f : M \to N$.

Of course, the component of minimal degree of $\mathcal{P}^\Sigma$ is generated over $\mathbf{Z}$ by $\mathcal{T}^\Sigma$. Usefulness of $\mathcal{P}$-ideals come from the following observation. Suppose that $\Sigma \subset \Sigma'$, where Σ' is another singularity class. Then $\mathcal{T}^\Sigma$ belongs to $\mathcal{P}^{\Sigma'}$. Thus if one knows the algebraic structure of $\mathcal{P}^{\Sigma'}$, one can use it to compute $\mathcal{T}^\Sigma$. In this way, the degeneracy loci of the cotangent map (11) appear to be useful objects to study Thom polynomials.

Set $\mathcal{P}^i := \mathcal{P}^{\Sigma^i}$. By [28] and [29], one knows the algebraic structure of $\mathcal{P}^i$, i.e., a certain *finite* set of its algebraic generators (cf. [28, Proposition 6.1]), and its $\mathbf{Z}$-basis (cf. [28, Proposition 6.2]). The arguments combine geometry of Grassmann bundles with algebra of Schur functions.

Before proceeding further, let us state the following result which is rather useful to compute the Schur function expansions of Thom polynomials. Its setting is the same as that of Theorem 8.

Theorem 13 *([28], [32]) Suppose that a stable singularity class Σ is contained in Σ^i. Then all summands in the Schur function expansion of $\mathcal{T}^{\Sigma}$ are indexed by partitions containing $(n-m+i)^i$.*

Thus the partitions not containing this rectangle cannot appear in the Schur function expansion of $\mathcal{T}^{\Sigma}$.

This result seems to be quite obvious. However, its proof is not obvious. Let $\mathbb{A}$ and $\mathbb{B}$ be two alphabets such that

$$\sum c_i = \prod_{a\in\mathbb{A}}(1+a) \quad \text{and} \quad \sum c'_j = \prod_{b\in\mathbb{B}}(1+b).$$

We have

Proposition 14 *No nonzero $\mathbf{Z}[c_1,\ldots,c_m]$-linear combination of the Schur functions $S_I(\mathbb{A}-\mathbb{B})$'s, where all I's do not contain $(n-m+i)^i$, belongs to $\mathcal{P}^i$.*

The idea of the proof is to interpret $\mathcal{P}^i$ as a "generalized resultant", and use some specialization trick. For details, we refer the reader to the proof of "Claim" on p. 164 in [29].

Thus, in particular, no nonzero $\mathbf{Z}$-linear combination of the $S_I(\mathbb{A}-\mathbb{B})$'s, where all I's do not contain $(n-m+i)^i$, belongs to $\mathcal{P}^i$.

Also, we have

Proposition 15 *Any $S_I(\mathbb{A}-\mathbb{B})$, where I contains $(n-m+i)^i$ belongs to $\mathcal{P}^i$.*

The idea of the proof is to use a desingularization of D in the product of two Grassmann bundles, and apply appropriate pushforward formulas. For details, see [28, Proposition 3.2].

We are now ready to justify the theorem. Since Σ is contained in Σ^i, the Thom polynomial $\mathcal{T}^{\Sigma}$ belongs to $\mathcal{P}^i$. By the stability assumption, the Thom polynomial $\mathcal{T}^{\Sigma}$ is a (unique) $\mathbf{Z}$-linear combination of the $S_I(\mathbb{A}-\mathbb{B})$'s. Propositions 15 and 14 imply that only Schur functions indexed by partitions containing the rectangle $(n-m+i)^i$ appear in this sum. □

In the computations of Thom polynomials, it is convenient to "split" them into pieces supported on the consecutive degeneracy loci of the cotangent map (11). Let $\mathcal{T}$ be the Thom polynomial of a singularity class. Following [33], by the *h-part* of $\mathcal{T}$ we mean the sum of all Schur functions appearing in $\mathcal{T}$ (multiplied by their coefficients) such that the corresponding partitions satisfy the following condition: I contains the rectangle partition $(n-m+h)^h$, but it does not contain the larger diagram $(n-m+h+1)^{h+1}$. The polynomial $\mathcal{T}$ is a sum of its h-parts, $h=1,2,\ldots$.

5 Single R-L orbits

In the present paper, we shall mostly study Thom polynomials of singularities.

Let $k \geq 0$ be a fixed integer and $\bullet \in \mathbf{N}$. Two stable germs $\kappa_1, \kappa_2 : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ are said to be right-left equivalent if there exist germs of biholomorphisms φ of $(\mathbf{C}^\bullet, 0)$ and ψ of $(\mathbf{C}^{\bullet+k}, 0)$ such that $\psi \circ \kappa_1 \circ \varphi^{-1} = \kappa_2$. A suspension of a germ is its trivial unfolding: $(x, v) \mapsto (\kappa(x), v)$. Consider the equivalence relation (on stable germs $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$) generated by right-left equivalence and suspension. A *singularity* η is an equivalence class of this relation.[4]

According to Mather's classification ([4] or [1]), singularities are in one-to-one correspondence with finite dimensional (local) $\mathbf{C}$-algebras. We shall use the following notation of Mather:

– A_i will stand for the stable germs with local algebra $\mathbf{C}[[x]]/(x^{i+1})$, $i \geq 0$;

– $I_{a,b}$ (of Thom-Boardman type $\Sigma^{2,0}$) for stable germs with local algebra $\mathbf{C}[[x, y]]/(xy, x^a + y^b)$, $b \geq a \geq 2$;

– $III_{a,b}$ (of Thom-Boardman type $\Sigma^{2,0}$) for stable germs with local algebra $\mathbf{C}[[x, y]]/(xy, x^a, y^b)$, $b \geq a \geq 2$ (here $k \geq 1$).

With a singularity η, there is associated Thom polynomial $\mathcal{T}^\eta$ in the formal variables $c_1, c_2, \ldots$ which after the substitution of c_i to

$$c_i(f^*TN - TM) = [c(f^*TN)/c(TM)]_i \,, \tag{14}$$

for a general map $f : M \to N$ between complex analytic manifolds, evaluates the Poincaré dual of $[\eta(f)]$, where $\eta(f)$ is the cycle carried by the closure of the set

$$\{x \in M : \text{the singularity of } f \text{ at } x \text{ is } \eta\} \,. \tag{15}$$

By $\operatorname{codim}(\eta)$, we mean the codimension of $\eta(f)$ in X.

Codimensions of above singularities are as follows (cf. [4, Chapter 8]):

– A_i associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0))$, where $i \geq 0$ and $k \geq 0$ has codimension $(k + 1)i$.

– $I_{a,b}$ associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0))$, where $b \geq a \geq 2$ and $k \geq 0$ has codimension $(k + 1)(a + b - 1) + 1$.

– $III_{a,b}$ associated with maps $(\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0))$, where $b \geq a \geq 2$ and $k \geq 1$ has codimension $(k + 1)(a + b - 2) + 2$.

We shall now follow the approach in [37]. Let $\kappa : (\mathbf{C}^n, 0) \to (\mathbf{C}^{n+k}, 0)$ be a prototype of a singularity η. It is possible to choose a maximal compact subgroup G_η of the *right-left symmetry group*

$$\operatorname{Aut} \kappa = \{(\varphi, \psi) \in \operatorname{Aut}_n \times \operatorname{Aut}_{n+k} : \psi \circ \kappa \circ \varphi^{-1} = \kappa\} \,, \tag{16}$$

such that images of its projections to the factors Aut_n and Aut_{n+k} are linear [5]. That is, projecting on the source $\mathbf{C}^n$ and the target $\mathbf{C}^{n+k}$, we obtain representations $\lambda_1(\eta)$ and $\lambda_2(\eta)$. Let E'_η and E_η denote the vector bundles associated with the universal principal G_η-bundles $EG_\eta \to BG_\eta$ that correspond to $\lambda_1(\eta)$

[4]This terminology stems from [37]; a singularity corresponds to a single R-L orbit.

[5]By Aut_n we mean here the space of automorphisms of $(\mathbf{C}^n, 0)$.

and $\lambda_2(\eta)$, respectively. The *total Chern class*, $c(\eta) \in H^*(BG_\eta, \mathbf{Z})$, and the *Euler class*, $e(\eta) \in H^{2\,\mathrm{codim}(\eta)}(BG_\eta, \mathbf{Z})$, of η are defined by

$$c(\eta) := \frac{c(E_\eta)}{c(E'_\eta)} \qquad \text{and} \qquad e(\eta) := e(E'_\eta)\,. \tag{17}$$

We end this section by recalling the *method of restriction equations* due to Rimányi et al.

Theorem 16 *([37]) Let η be a singularity. Suppose that the number of singularities of codimension less than or equal to* $\mathrm{codim}(\eta)$ *is finite. Moreover, assume that the Euler classes of all singularities of codimension smaller than* $\mathrm{codim}(\eta)$ *are not zero-divisors. Then we have*

1. *if* $\xi \neq \eta$ *and* $\mathrm{codim}(\xi) \leq \mathrm{codim}(\eta)$*, then* $\mathcal{T}^\eta(c(\xi)) = 0$*;*
2. $\mathcal{T}^\eta(c(\eta)) = e(\eta)$*.*

This system of equations (taken for all such ξ's) determines the Thom polynomial $\mathcal{T}^\eta$ in a unique way.

Solving of these equations is rather difficult. This method is well suited for compter experiments, though the bounds of such computations are quite sharp.

6 Computing the Chern and Euler classes

The Chern and Euler classes recalled in the present section were given in: [37], [32], [23], [24] and [25].

Let $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet+k}, 0)$ be a singularity in the sense of Section 5.

For $\eta = A_i$, a suitable maximal compact subgroup can be chosen as $G_{A_i} = U(1) \times U(k)$. The Chern class is

$$c(A_i) = \frac{1 + (i+1)x}{1+x} \prod_{j=1}^{k} (1 + y_j)\,, \tag{18}$$

where x and $y_1, \ldots, y_k$ are the Chern roots of the universal bundles on $BU(1)$ and $BU(k)$. The Euler class is

$$e(A_i) = i!\ x^i \prod_{j=1}^{k} (y_j - ix) \cdots (y_j - 2x)(y_j - x)\,. \tag{19}$$

In case of $\eta = I_{2,2}$, we consider the extension of $U(1) \times U(1)$ by $\mathbf{Z}/2\mathbf{Z}$. Denoting this group by H, a maximal compact subgroup is $G_\eta = H \times U(k)$ for all $k \geq 0$. But to make computations easier, we use the subgroup $U(1) \times U(1) \times U(k)$ as G_η (cf. [37], p.502)). We have

$$c(I_{2,2}) = \frac{(1+2x_1)(1+2x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k} (1 + y_j)\,. \tag{20}$$

Here x_1, x_2 and $y_1, \ldots, y_k$ are the Chern roots of the universal bundles on two copies of $BU(1)$ and on $BU(k)$. The Euler class is

$$e(I_{2,2}) = x_1 x_2 (2x_1 - x_2)(2x_2 - x_1) \prod_{j=1}^{k} (y_j - x_1)(y_j - x_2)(y_j - x_1 - x_2)\,. \tag{21}$$

Next, we consider $\eta = III_{2,2}$. This time we use the maximal compact group $G_\eta = U(2) \times U(k-1)$ for $k \geq 1$. We have

$$c(III_{2,2}) = \frac{(1+2x_1)(1+2x_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k-1} (1 + y_j)\,, \tag{22}$$

where x_1, x_2 and $y_1, \ldots, y_{k-1}$ denote the Chern roots of the universal bundles on $BU(2)$ and $BU(k-1)$. The Euler class is

$$e(III_{2,2}) = (x_1 x_2)^2 (x_1 - 2x_2)(x_2 - 2x_1) \prod_{j=1}^{k-1} (x_1 - y_j) \prod_{j=1}^{k-1} (x_2 - y_j)\,. \tag{23}$$

For the singularity $III_{2,3}$, we can use the action of the $U(1) \times U(1) \times U(k-1)$. We have

$$c(III_{2,3}) = \frac{(1+2x_1)(1+3x_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k-1} (1 + y_j)\,. \tag{24}$$

This time x_1, x_2 and $y_1, \ldots, y_k$ are the Chern roots of the universal bundles on two copies of $BU(1)$ and on $BU(k-1)$. The Euler class is

$$\begin{aligned} e(III_{2,3}) =& 4x_1^2 x_2^3 (x_1 - x_2)(x_1 - 3x_2)(x_2 - 2x_1) \\ & \times \prod_{j=1}^{k-1} (x_1 - y_j)(x_2 - y_j)(2x_2 - y_j)\,. \end{aligned} \tag{25}$$

For the singularity $III_{3,3}$, the maximal compact group is $U(2) \times U(k-1)$. The Chern class is

$$c(III_{3,3}) = \frac{(1 + 3x_1)(1 + 3x_2)(1 + x_1 + x_2)}{(1 + x_1)(1 + x_2)} \prod_{j=1}^{k-1} (1 + y_j)\,, \tag{26}$$

where x_1, x_2 and $y_1, \ldots, y_{k-1}$ are the Chern roots of the universal bundles $BU(2)$ and $BU(k-1)$. The Euler class is

$$\begin{aligned} e(III_{3,3}) =& 4x_1^3 x_2^3 (3x_1 - x_2)(3x_1 - 2x_2)(3x_2 - x_1)(3x_2 - 2x_1) \\ & \times \prod_{j=1}^{k-1} (x_1 - y_j)(2x_1 - y_j)(x_2 - y_j)(2x_2 - y_j)\,. \end{aligned} \tag{27}$$

We display now the Chern or/and Euler classes of some other singularities (we omit to interpret the variables x_i and y_j). We have

$$c(I_{a,b}) = \frac{(1 + \frac{a+b}{\gcd(a,b)} x_1)(1 + \frac{ab}{\gcd(a,b)} x_2)}{(1 + \frac{a}{\gcd(a,b)} x_1)(1 + \frac{b}{\gcd(a,b)} x_2)} \prod_{j=1}^{k-1} (1 + y_j)\,; \tag{28}$$

$$e(I_{a,b}) = \frac{a!b!a^{b-1}b^{a-1}x^{a+b}}{\gcd(a,b)^{a+b}} \prod_{j=1}^{k} \left(\prod_{i=1}^{a} (i\frac{\gcd(a,b)}{b}x - y_j) \prod_{i=1}^{b-1} (i\frac{\gcd(a,b)}{a}x - y_j) \right) ; \tag{29}$$

$$c(III_{a,b}) = \frac{(1+ax_1)(1+bx_2)(1+x_1+x_2)}{(1+x_1)(1+x_2)} \prod_{j=1}^{k-1} (1+y_j) ; \tag{30}$$

$$\begin{aligned} e(III_{a,b}) = (a-1)!(b-1)! \prod_{i=1}^{b-1} (ax_1 - ix_2) \prod_{i=1}^{a-1} (bx_2 - ix_1) \\ \times \prod_{j=1}^{k-1} \left(\prod_{i=1}^{a-1} (y_j - ix_1) \prod_{i=1}^{b-1} (y_j - ix_2) \right) . \end{aligned} \tag{31}$$

A general strategy for computing the Chern and Euler classes of singularities was described in [37].

We show now, following [24], how to compute the Euler class of $III_{2,3}$. Assume that $k = 1$ and consider the germ $g(x,y) = (x^2, y^3, xy)$. A prototype of $III_{2,3}$ can be written as the unfolding

$$g + \sum_{i=1}^{8} u_i h_i ,$$

where h_i form a basis of the space

$$\frac{\mathfrak{m}^3_{x,y}}{\mathfrak{m}_{x,y} \cdot \{\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\} + \mathbf{C}^3 \cdot I(g)} ,$$

and where $I(g)$ is the subspace generated by the component functions of g. We shall work with the basis consisting of the following germs:

$$\begin{aligned} h_1(x,y) &= (x,0,0), & h_5(x,y) &= (0,y,0), \\ h_2(x,y) &= (y,0,0), & h_6(x,y) &= (0,y^2,0), \\ h_3(x,y) &= (y^2,0,0), & h_7(x,y) &= (0,0,x), \\ h_4(x,y) &= (0,x,0), & h_8(x,y) &= (0,0,y). \end{aligned}$$

Let ρ_{h_i} denote the representation of the action of the group $U(1) \times U(1)$ on the space generated by h_i. Then, denoting the one-dimensional representations of the first and the second copies of $U(1)$ by λ and μ, we have

$$\begin{aligned} \rho_{h_1} &= \lambda, & \rho_{h_5} &= \mu^2, \\ \rho_{h_2} &= \lambda^2 \otimes \mu^{-1}, & \rho_{h_6} &= \mu, \\ \rho_{h_3} &= \lambda^2 \otimes \mu^{-2}, & \rho_{h_7} &= \mu, \\ \rho_{h_4} &= \lambda^{-1} \otimes \mu^3, & \rho_{h_8} &= \lambda. \end{aligned}$$

Therefore for $k = 1$, using the representation $\bigoplus \rho_{h_i}$, we can write the Euler class as

$$e(III_{2,3}) = 4x_1^2x_2^3(x_1 - x_2)(x_1 - 3x_2)(x_2 - 2x_1)\,, \tag{32}$$

where x_1 and x_2 denote the Chern roots of the universal bundles on the two copies of $BU(1)$.

For $k = 2$, in addition to h_i above, we need to consider the representations of the action of the group $U(k-1) = U(1)$ on the spaces generated by $(x, y) \mapsto (0, 0, 0, x)$, $(x, y) \mapsto (0, 0, 0, y)$ and $(x, y) \mapsto (0, 0, 0, y^2)$. These can be written as $\nu \otimes \lambda^{-1}$, $\nu \otimes \mu^{-1}$ and $\nu \otimes \mu^{-2}$, where ν denotes the one-dimensional representation of this copy of $U(1)$. Hence, in this case, the Euler class can be written as

$$e(III_{2,3}) = 4x_1^2x_2^3(x_1 - x_2)(x_1 - 3x_2)(x_2 - 2x_1)(x_1 - y_1)(x_2 - y_1)(2x_2 - y_1)\,, \tag{33}$$

where x_i are as above and y_1 denotes the Chern root of the universal bundle on $BU(1)$.

For $k \geq 1$, we need to consider $U(k-1)$ instead of $U(1)$, giving rise to $y_1, \ldots, y_{k-1}$ (and respectively to the product $\prod_{j=1}^{k-1}(x_1 - y_j)(x_2 - y_j)(2x_2 - y_j)$) instead of y_1 (and respectively of $(x_1 - y_1)(x_2 - y_1)(2x_2 - y_1)$).

We shall need the following alphabets:

Definition 17 *We set*

$$\begin{aligned}
\mathbb{D} &= \boxed{x_1} + \boxed{x_2} + \boxed{x_1 + x_2}\,,\\
\mathbb{E} &= \boxed{2x_1} + \boxed{2x_2}\,,\\
\mathbb{F} &= \boxed{2x_1} + \boxed{3x_2} + \boxed{x_1 + x_2}\,,\\
\mathbb{G} &= \boxed{3x_1} + \boxed{3x_2} + \boxed{x_1 + x_2}\,,\\
\mathbb{H} &= \boxed{2x_1} + \boxed{4x_2} + \boxed{x_1 + x_2}\,.
\end{aligned}$$

Notation 18 In the rest of the paper we shall use the shifted parameter

$$r := k + 1\,. \tag{34}$$

When we need to emphasize the dependence on r we shall write $\eta(r)$ for the singularity $\eta : (\mathbf{C}^\bullet, 0) \to (\mathbf{C}^{\bullet + r - 1}, 0)$, and denote the Thom polynomial of $\eta(r)$ by $\mathcal{T}_r^\eta$, or $\mathcal{T}_r$ for short. (In this notation, the result of Thom, $\mathcal{T}_r^{A_1} = S_r$, has a transparent form.)

We now specify, with the help of these alphabets, some equations characterizing Thom polynomials $\mathcal{T}_r$ imposed by different singularities.

Note 19 The variables below will be specialized to the Chern roots of the *cotangent* bundles.

First, we give the vanishing equations coming from the Chern classes of singularities. Let $\mathbb{B}_j$ denote an alphabet of cardinality j. We have the following equations:

$$A_i(r): \qquad \mathcal{T}_r(x - \mathbb{B}_{r-1} - \boxed{(i+1)x}) = 0 \qquad \text{for } \; i = 0, 1, 2, \ldots\,; \tag{35}$$

$$I_{2,2}(r): \qquad \mathcal{T}_r\left(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}\right) = 0\,; \tag{36}$$

$$I_{2,3}(r): \qquad \mathcal{T}_r\left(\boxed{2x} + \boxed{3x} - \boxed{5x} - \boxed{6x} - \mathbb{B}_{r-1}\right) = 0\,; \tag{37}$$

$$III_{2,2}(r): \qquad \mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0\,; \tag{38}$$

$$III_{2,3}(r): \qquad \mathcal{T}_r\left(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2}\right) = 0\,; \tag{39}$$

$$III_{2,4}(r): \qquad \mathcal{T}_r\left(\mathbb{X}_2 - \mathbb{H} - \mathbb{B}_{r-2}\right) = 0\,. \tag{40}$$

Using the Chern classes displayed above, one can write down other vanishing equations.

We give now some normalizing equations coming from the Euler classes of singularities. We have

$$A_i(r): \qquad \mathcal{T}_r(x - \mathbb{B}_{r-1} - \boxed{(i+1)x}) = R(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}\,, \mathbb{B}_{r-1} + \boxed{(i+1)x})\,; \tag{41}$$

$$I_{2,2}(r): \qquad \mathcal{T}_r(\mathbb{X}_2 - \mathbb{E} - \mathbb{B}_{r-1}) = x_1 x_2 (x_1 - 2x_2)(x_2 - 2x_1)\; R(\mathbb{X}_2 + \boxed{x_1 + x_2}, \mathbb{B}_{r-1})\,; \tag{42}$$

$$\begin{aligned} I_{2,3}(r): \qquad & \mathcal{T}_r\left(\boxed{2x} + \boxed{3x} - \boxed{5x} - \boxed{6x} - \mathbb{B}_{r-1}\right) \\ & = 2xR(\boxed{2x} + \boxed{3x}\,, \boxed{5x} + \boxed{6x} + \mathbb{B}_{r-1}) \\ & \times \prod_{j=1}^{r-1} (4x - b_j)(6x - b_j)\,; \end{aligned} \tag{43}$$

$$III_{2,2}(r): \qquad \mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2})\,; \tag{44}$$

$$III_{2,3}(r): \mathcal{T}_r\left(\mathbb{X}_2 - \mathbb{F} - \mathbb{B}_{r-2}\right) = 2x_2(x_1 - x_2) R(\mathbb{X}_2, \mathbb{F} + \mathbb{B}_{r-2}) \prod_{j=1}^{r-2} (2x_2 - b_j)\,; \tag{45}$$

$$III_{3,3}(r) : \mathcal{T}_r\left(\mathbb{X}_2 - \mathbb{G} - \mathbb{B}_{r-2}\right) = x_1x_2(3x_1 - 2x_2)(3x_2 - 2x_1) \times R(\mathbb{X}_2, \mathbb{G} + \mathbb{B}_{r-2}) \prod_{j=1}^{r-2} (2x_1 - b_j)(2x_2 - b_j)\,. \tag{46}$$

Using the Euler classes displayed above, one can write down other normalizing equations.

7 Thom polynomials of singularities

In this section, we shall study, for singularities η, Schur function expansions Thom polynomials $\mathcal{T}^\eta$ written in the form (13) (cf. also (12):

$$\mathcal{T}^\eta = \sum_I \alpha_I S_I\,.$$

It is interesting to find bounds on partitions appearing in Schur function expansions of Thom polynomials of singularities. One such follows immediately from Theorem 13.

Proposition 20 *Suppose that a singularity η is of Thom-Boardman type $\Sigma^{i,\cdots}$. Then all summands in the Schur function expansion of $\mathcal{T}_r^\eta$ are indexed by partitions containing the rectangle partition $(r+i-1)^i$.*

For example, consider the singularity $III_{2,3}(r)$. As its Thom-Boardman type is $\Sigma^{2,0}$, all partitions in the Schur function expansion of $\mathcal{T}_r^{III_{2,3}}(r)$ contain the partition $(r+1, r+1)$. This Thom polynomial is characterized by the equations: (35), $i = 0, 1, 2, 3$, (38) and (45). Its Schur function expansion is given by the following expression:

Theorem 21 *([24], [7]) We have*

$$\mathcal{T}_r^{III_{2,3}} = \sum_{i=1}^{r+1} 2^i S_{r+1-i,r+1,r+i}\,. \tag{47}$$

7.1 On Morin singularities $A_i(r)$

One of the most important problems in global singularity theory is to write down the explicit Schur function expansion of the Thom polynomials for Morin singularities $A_i(r)$. We now describe, following [33], the 1-part of $\mathcal{T}_r^{A_i}$ for any i and r.

Let $\mathbb{A}$ be an alphabet of cardinality m. Consider the function $F(\mathbb{A}, -)$, defined for any difference of alphabets $\mathbb{G} - \mathbb{H}$ by

$$F(\mathbb{A}, \mathbb{G} - \mathbb{H}) := \sum_I S_I(\mathbb{A}) S_{n-i_m,\ldots,n-i_1,n+|I|}(\mathbb{G} - \mathbb{H})\,, \tag{48}$$

where the sum is over partitions $I = (i_1, i_2, \ldots, i_m)$ such that $i_m \leq n$.

A basic link of this function to resultants is given by the following result.

Lemma 22 *For a variable x and an alphabet $\mathbb{B}$ of cardinality n, we have*

$$F(\mathbb{A}, x - \mathbb{B}) = R(x + \mathbb{A}x, \mathbb{B})\,. \tag{49}$$

(*loc.cit.* Lemma 8).

Next, we define the following function $F_r^{(i)}(-)$:

$$F_r^{(i)}(\mathbb{G} - \mathbb{H}) = \sum_J S_J(\boxed{2} + \boxed{3} + \cdots + \boxed{i})S_{r-j_{i-1},\ldots,r-j_1,r+|J|}(\mathbb{G} - \mathbb{H})\,, \tag{50}$$

where the sum is over partitions $J \subset (r^{i-1})$, and for $i = 1$ we understand $F_r^{(1)}(-) = S_r(-)$.

The following result gives the key algebraic property of $F_r^{(i)}$.

Proposition 23 *We have*

$$F_r^{(i)}(x - \mathbb{B}_r) = R(x + \boxed{2x} + \boxed{3x} + \cdots + \boxed{ix}, \mathbb{B}_r)\,. \tag{51}$$

Proof. The assertion follows from Lemma 22 with $m = i - 1$, $n = r$, and $\mathbb{A} = \boxed{2} + \boxed{3} + \cdots + \boxed{i}$.

With the help of Proposition 23, the following result on Thom polynomials was established:

Theorem 24 *For any i, r, the 1-part of $\mathcal{T}_r^{A_i}$ is equal to $F_r^{(i)}$.*

(*loc.cit.* pp.173–174).

We shall now use a couple of functions $F_r^{(i)}$ to rephrase some results from [40], [38], folklore, [11] and [37], respectively:

$$F_r^{(1)} = S_r = \mathcal{T}_r^{A_1};$$

$$F_r^{(2)} = \sum_{j \le r} 2^j S_{r-j,r+j} = \mathcal{T}_r^{A_2};$$

$$F_1^{(3)} = S_{111} + 5S_{12} + 6S_3 = \mathcal{T}_1^{A_3};$$

$$F_1^{(4)} + 10S_{22} = S_{1111} + 9S_{112} + 26S_{13} + 24S_4 + 10S_{22} = \mathcal{T}_1^{A_4};$$

$$F_2^{(3)} + 5S_{33} = S_{222} + 5S_{123} + 6S_{114} + 19S_{24} + 30S_{15} + 36S_6 + 5S_{33} = \mathcal{T}_2^{A_3}$$

([33], pp.174–176). The reader can find in [33] more examples of the functions $F_r^{(i)}$. In the next section, we shall discuss the Schur function expansions of $\mathcal{T}_r^{A_3}$ for all r.

Definition 25 *For a positive integer p We denote by Φ_p the linear endomorphism on the* **Z***-module spanned by Schur functions indexed by partitions of length $\le p$ that sends a Schur function $S_{j_1,\ldots,j_p}$ to $S_{j_1+1,\ldots,j_p+1}$.*

Example 26 For any $i, r \ge 1$, we have

$$F_r^{(i)} = \overline{F_r^{(i)}} + \Phi_i(F_{r-1}^{(i)})\,, \tag{52}$$

where the first summand gathers the Schur functions indexed by partitions of length $< i$.

In [2], the author discusses another approach to Thom polynomials of Morin singularities.

7.2 A basic recursion

In the forthcoming section, we shall discuss some recursions for Thom polynomials. The following result was recently obtained in [7, Proposition 7.15, Theorem 7.14]. Let Q_η denote the local algebra of the singularity η.

Theorem 27 *Let η be a stable singularity. Then the length of any partition, appearing in the Schur function expansion of $\mathcal{T}_r^\eta$, is $\leq \dim(Q_\eta)-1$. Moreover, by erasing one column of length $\dim(Q_\eta)-1$ from all the diagrams of partitions appearing in $\mathcal{T}_r^\eta$, we get all the diagrams of partitions appearing in $\mathcal{T}_{r-1}^\eta$ (we disregard the partitions whose diagrams have no such a column).*

In other words, for $p = \dim(Q_\eta) - 1$, the following equation holds:

$$\mathcal{T}_r^\eta = \overline{\mathcal{T}}_r^\eta + \Phi_p(\mathcal{T}_{r-1}^\eta)\,, \tag{53}$$

where the first summand gathers the Schur functions indexed by partitions of length $< p$.

This result was earlier established for the singularities $I_{2,2}(r)$, $A_3(r)$, $A_4(r)$, $III_{2,3}(r)$ and $III_{3,3}(r)$ from the restriction equations which they obey, with help of Eq.(8) (see [32], [19], [23], [24] and [25]).

This recurrence relation is quite easy to observe, especially by computing examples with the help of computer. It is, however, not sufficient to compute Thom polynomials. As the matter of fact, Schur function expansions of Thom polynomials often contain many terms, where the first column is shorter than the maximal possible. So these "initial terms", denoted by $\overline{\mathcal{T}}_r^\eta$ in (53), cannot be obtained by the operation of adding a maximal possible column.

Another interesting question is to find upper bounds of the coefficients in Schur function expansions of Thom polynomials. This will be a subject of some future study.

8 Pascal staircases and two recursions

We invoke first some results from [32] and [19]. We start with useful algebraic identity associated with Pascal staircases (cf. [19]). Then we discuss the Schur function computations of the Thom polynomials of $I_{2,2}(r)$ and $A_3(r)$.

8.1 Pascal staircases

The material of this subsection stems from [19].

Consider an infinite matrix $P = [p_{s,t}]$ with rows and columns numbered by $s, t = 1, 2, \ldots$.

We suppose that $p_{1,t} = p_{2,t} = 0$ for $t \geq 2$, $p_{3,t} = p_{4,t} = 0$ for $t \geq 3$, $p_{5,t} = p_{6,t} = 0$ for $t \geq 4$ etc.

The first column is an arbitrary sequence $v = (v_1, v_2, \ldots)$. In the case when this sequence is the sequence of coefficients of the Taylor expansion of a function $f(z)$, we write P_f for the corresponding matrix P.

To define the remaining $p_{s,t}$'s, we use the recursive formula

$$p_{s+1,t} = p_{s,t-1} + p_{s,t}\,. \tag{54}$$

We visualize this definition by

$$\begin{array}{ccccc} a & b & & a & b \\ & \square & \Rightarrow & & a+b \end{array}$$

We thus get the following *Pascal staircase* $P = [p_{i,j}]_{i,j=1,2,\ldots}$:

$$\begin{array}{cccccc}
v_1 & 0 & 0 & 0 & 0 & \ldots \\
v_2 & 0 & 0 & 0 & 0 & \ldots \\
v_3 & v_2 & 0 & 0 & 0 & \ldots \\
v_4 & v_3{+}v_2 & 0 & 0 & 0 & \ldots \\
v_5 & v_4{+}v_3{+}v_2 & v_3{+}v_2 & 0 & 0 & \ldots \\
v_6 & v_5{+}v_4{+}v_3{+}v_2 & v_4{+}2v_3{+}2v_2 & 0 & 0 & \ldots \\
v_7 & v_6{+}v_5{+}v_4{+}v_3{+}v_2 & v_5{+}2v_4{+}3v_3{+}3v_2 & v_4{+}2v_3{+}2v_2 & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Given an integer $d \geq 0$, and an alphabet $\mathbb{A}$, we define the function $W(d) = W(d, \mathbb{A})$ by

$$W(d, \mathbb{A}) = \sum_{i,j} p_{d+1-i,j+1}\, S_i(-\mathbb{A})\, S_{j,d-i-j}(\mathbb{X}_2). \tag{55}$$

The function $W(d, \mathbb{A})$ is linear in the elements of the first column of P. Hence it is sufficient to restrict to the case $v = (1, y, y^2, \ldots)$, i.e., to take $P = P_{1/(1-zy)}$ to determine it.

Lemma 28 *If $P = P_{1/(1-zy)}$ and $\mathbb{A} = \boxed{x_1 + x_2}$, then $W(0) = 1$ and for $d \geq 1$*

$$W(d, \boxed{x_1 + x_2}) = (y-1)y^{d-1} S_d(\mathbb{X}_2). \tag{56}$$

For the proof, see [19].

Let $\mathbb{B}$ be another alphabet. Taking now $\mathbb{A} = \boxed{x_1 + x_2} + \mathbb{B}$ instead of $\boxed{x_1 + x_2}$, and using

$$\begin{aligned}
W(d, \mathbb{A}) = \sum_{i,j,k} p_{d+1-i-k,j+1}\, S_i\big(-\boxed{x_1 + x_2}\big)\, S_{j,d-i-j-k}(\mathbb{X}_2) S_k(-\mathbb{B}) \\
= \sum_k W\left(d-k, \boxed{x_1 + x_2}\right) S_k(-\mathbb{B}) \\
= (1 - y^{-1}) \sum_k y^{d-k} S_{d-k}(\mathbb{X}_2) S_k(-\mathbb{B}) = y^t (1 - y^{-1}) S_d(\mathbb{X}_2 - y^{-1}\mathbb{B}),
\end{aligned}$$

we get the following corollary.

Corollary 29 *If $P = P_{1/(1-zy)}$ and $\mathbb{B}$ is an arbitrary alphabet, then (apart from initial values) we have*

$$W(d, \boxed{x_1 + x_2} + \mathbb{B}) = (y-1)y^{d-1} S_d(\mathbb{X}_2 - y^{-1}\mathbb{B}). \tag{57}$$

8.2 Recursions for $I_{2,2}(r)$

The material of this subsection stems from [32].

The codimension of $I_{2,2}(r)$, $r \geq 1$, is $3r+1$. Set $\mathcal{T}_r := \mathcal{T}_r^{I_{2,2}}$ and $\overline{\mathcal{T}}_r = \overline{\mathcal{T}}_r^{I_{2,2}}$. We have $\mathcal{T}_1 = \overline{\mathcal{T}}_1 = S_{22}$.

A partition appearing in the Schur function expansion of $\mathcal{T}_r$ contains the partition $(r+1, r+1)$ and has at most three parts. In particular, if S_{i_1,i_2} appears in the Schur function expansion of $\mathcal{T}_r$, then $i_1 = r+1+p$ and $i_2 = 2r-p$, where $0 \leq 2p \leq r-1$.

Invoke the map Φ_3 from Definition 25. We have, for $r \geq 2$, the following recursive equation:

$$\mathcal{T}_r = \overline{\mathcal{T}}_r + \Phi_3(\overline{\mathcal{T}}_{r-1}) + \Phi_3^2(\overline{\mathcal{T}}_{r-2}) + \cdots + \Phi_3^{r-1}(\overline{\mathcal{T}}_1)\,. \tag{58}$$

So we are left with computation of $\overline{\mathcal{T}}_r$.

Consider the matrix whose (i,j)th entry is the partition $(i+j, 1+2i-j)$ with the convention that $(i+j, 1+2i-j)$ is the empty partition for $2j > i+1$:

$$\begin{bmatrix} 22 & \emptyset & \emptyset & \emptyset & \emptyset & \dots \\ 34 & \emptyset & \emptyset & \emptyset & \emptyset & \dots \\ 46 & 55 & \emptyset & \emptyset & \emptyset & \dots \\ 58 & 67 & \emptyset & \emptyset & \emptyset & \dots \\ 6,10 & 79 & 88 & \emptyset & \emptyset & \dots \\ 7,12 & 8,11 & 9,10 & \emptyset & \emptyset & \dots \\ 8,14 & 9,13 & 10,12 & 11,11 & \emptyset & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Note that the rth row of the above matrix consists of partitions appearing in $\overline{\mathcal{T}}_r$. It turns out that the coefficients of their Schur functions are given by the corresponding entries of the Pascal staircase $P = [P_{i,j}]_{i=1,\dots;j=1,\dots}$, associated with the sequence $\{2^i - 1\}_{i=1,2,\dots}$:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 3 & 0 & 0 & 0 & 0 & 0 & \dots \\ 7 & 3 & 0 & 0 & 0 & 0 & \dots \\ 15 & 10 & 0 & 0 & 0 & 0 & \dots \\ 31 & 25 & 10 & 0 & 0 & 0 & \dots \\ 63 & 56 & 35 & 0 & 0 & 0 & \dots \\ 127 & 119 & 91 & 35 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{59}$$

Namely, we have

$$\overline{\mathcal{T}}_r = \sum_{2j \leq r+1} P_{r,j} S_{r+j, 2r+1-j}\,.$$

Example 30 *We have the following values of* $\overline{\mathcal{T}}_1, \ldots, \overline{\mathcal{T}}_7$*:*

$$\begin{aligned}
\overline{\mathcal{T}}_1 &= S_{22} \\
\overline{\mathcal{T}}_2 &= 3S_{34} \\
\overline{\mathcal{T}}_3 &= 7S_{46} + 3S_{55} \\
\overline{\mathcal{T}}_4 &= 15S_{58} + 10S_{67} \\
\overline{\mathcal{T}}_5 &= 31S_{6,10} + 25S_{79} + 10S_{88} \\
\overline{\mathcal{T}}_6 &= 63S_{7,12} + 56S_{8,11} + 35S_{9,10} \\
\overline{\mathcal{T}}_7 &= 127S_{8,14} + 119S_{9,13} + 91S_{10,12} + 35S_{11,11}\,.
\end{aligned}$$

In this case, the algebra of Schur functions combined with one of the equations characterizing the Thom polynomial, yields quickly an expression for $\overline{\mathcal{T}}_r$. Of course, $\overline{\mathcal{T}}_r$ is uniquely determined by its value on $\mathbb{X}_2$. The following result gives this value.

Proposition 31 *For any* $r \geq 1$*, we have*

$$\overline{\mathcal{T}}_r(\mathbb{X}_2) = (x_1x_2)^{r+1}\ S_{r-1}(\mathbb{D})\,. \tag{60}$$

We show the induction step. Suppose that the assertion is true for $\overline{\mathcal{T}}_i$, where $i < r$. Let $I = (j, r+1+p, r+1+q)$ be a partition appearing in the Schur function expansion of $\mathcal{T}_r$. By the factorization property (9), we get

$$S_I(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = R \cdot S_j(-\mathbb{D} - \mathbb{B}_{r-2}) \cdot S_{p,q}(\mathbb{X}_2)\,,$$

where $R = R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2})$. Therefore, using Eq. (58), we obtain

$$\mathcal{T}_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2}) = R \cdot \Big(\sum_{j=0}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2})\ \frac{\overline{\mathcal{T}}_{r-j}(\mathbb{X}_2)}{(x_1x_2)^{r-j+1}}\Big)\,. \tag{61}$$

By the induction assumption, for positive $j \leq r-1$, we have

$$\overline{\mathcal{T}}_{r-j}(\mathbb{X}_2) = (x_1x_2)^{r-j+1}\ S_{r-1-j}(\mathbb{D})\,.$$

We use now the fact that among the equations characterizing $\mathcal{T}_r$ is (38) (because the codimension of $III_{2,2}(r)$ is smaller than $\mathrm{codim}(I_{2,2}(r))$). Substituting this to (61), we obtain

$$\sum_{j=1}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) S_{r-1-j}(\mathbb{D}) + \frac{\overline{\mathcal{T}}_r(\mathbb{X}_2)}{(x_1x_2)^{r+1}} = 0\,. \tag{62}$$

But we also have, by a formula for addition of alphabets,

$$\sum_{j=1}^{r-1} S_j(-\mathbb{D} - \mathbb{B}_{r-2}) S_{r-1-j}(\mathbb{D}) + S_{r-1}(\mathbb{D}) = S_{r-1}(-\mathbb{B}_{r-2}) = 0\,. \tag{63}$$

Combining (62) and (63), gives

$$\overline{\mathcal{T}}_r(\mathbb{X}_2) = (x_1x_2)^{r+1}\ S_{r-1}(\mathbb{D})\,,$$

that is, the induction assertion. $\square$

The Schur function expansion of $S_i(\mathbb{D})$ was described in [28], [15] and Appendix A3 in [30], in the context of the *Segre classes* of the second symmetric power of a rank 2 vector bundle. Indeed, $\mathbb{D}$ is the alphabet of the Chern roots of the second symmetric power of a rank 2 bundle with the Chern roots x_1, x_2. The recursions encoded by the Pascal diagram (59) express the recursions for the coefficients of the Segre classes of the second symmetric power of a rank 2 vector bundle (*loc.cit.*).

8.3 Recursions for $A_3(r)$

The material of this subsection stems from [19].

We set

$$F_r := \sum_{j_1 \le j_2 \le r} S_{j_1,j_2}(\boxed{2} + \boxed{3}) S_{r-j_2, r-j_1, r+j_1+j_2} \,. \tag{64}$$

This function is the 1-part of $\mathcal{T}_r^{A_3}$ (see Section 7).

In [37], the author gave Thom polynomials for $A_3(1)$ and $A_3(2)$. Their Schur function expansions are

$$\mathcal{T}_1^{A_3} = S_{111} + 5S_{12} + 6S_3 = F_1 \,. \tag{65}$$

and

$$\mathcal{T}_2^{A_3} = S_{222} + 5S_{123} + 6S_{114} + 19S_{24} + 30S_{15} + 36S_6 + 5S_{33} = F_2 + 5S_{33} \,. \tag{66}$$

Note that the 2-part of $\mathcal{T}_2^{A_3}$ is $5S_{33}$.

We now pass to the case of general r. Since $A_3(r)$ has codimension $3r$, a partition appearing in the 2-part of $\mathcal{T}_r^{A_3}$ has weight $3r$ and its diagram contains the partition $(r+1, r+1)$. Moreover, it can have at most three rows.

Consider the matrix whose (i,j)th entry is the partition $(1+i+j, 2+2i-j)$ with the convention that $(1+i+j, 2+2i-j)$ is the empty partition for $2j > i+1$:

$$\begin{bmatrix} 33 & \emptyset & \emptyset & \emptyset & \emptyset & \dots \\ 45 & \emptyset & \emptyset & \emptyset & \emptyset & \dots \\ 57 & 66 & \emptyset & \emptyset & \emptyset & \dots \\ 69 & 78 & \emptyset & \emptyset & \emptyset & \dots \\ 7,11 & 8,10 & 99 & \emptyset & \emptyset & \dots \\ 8,13 & 9,12 & 10,11 & \emptyset & \emptyset & \dots \\ 9,15 & 10,14 & 11,13 & 12,12 & \emptyset & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

We now want to define a symmetric function $\overline{H}_r$ whose Schur summands are indexed by partitions from the $(r-1)$th row of the above matrix. Their coefficients will be given by the corresponding entries of the following Pascal staircase. Consider the following Taylor expansion:

$$\begin{aligned} f(z) = & \frac{5-6z}{(1-z)(1-2z)(1-3z)} \\ & = 5 + 24z + 89z^2 + 300z^3 + 965z^4 + 3024z^5 + 9329z^6 + \dots \,. \end{aligned}$$

The Pascal staircase associated with f is the following infinite matrix:

$$P = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & \dots \\ 24 & 0 & 0 & 0 & 0 & \dots \\ 89 & 24 & 0 & 0 & 0 & \dots \\ 300 & 113 & 0 & 0 & 0 & \dots \\ 965 & 413 & 113 & 0 & 0 & \dots \\ 3024 & 1378 & 526 & 0 & 0 & \dots \\ 9329 & 4402 & 1904 & 526 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

For $r \geq 2$, we set

$$\overline{H}_r := \sum_{2j \leq r} P_{r-1,j} S_{r+j,2r-j} \,. \qquad (67)$$

Example 32 *We have the following values of $\overline{H}_r$, $r = 2, \dots, 7$:*

$$\begin{aligned} \overline{H}_2 &= 5S_{33} \\ \overline{H}_3 &= 24S_{45} \\ \overline{H}_4 &= 24S_{66}{+}89S_{57} \\ \overline{H}_5 &= 113S_{78}{+}300S_{69} \\ \overline{H}_6 &= 113S_{99}{+}413S_{8,10}{+}965S_{7,11} \\ \overline{H}_7 &= 526S_{10,11}{+}1378S_{9,12}{+}3024S_{8,13} \,. \end{aligned}$$

We define by induction on r

$$H_r = \overline{H}_r + \Phi_3(\overline{H}_{r-1}) + \Phi_3^2(\overline{H}_{r-2}) + \cdots + \Phi_3^{r-2}(\overline{H}_2) \,.$$

With this definition of H_r, we state the following result.

Theorem 33 *([19]) We have*

$$\mathcal{T}_r^{A_3} = F_r + H_r \,.$$

In other words, the function H_r is the 2-part of $\mathcal{T}_r^{A_3}$, and its h-parts are zero for $h \geq 3$. Note also that we recover the recurrence (52):

$$F_r = \overline{F}_r + \Phi_3(F_{r-1}) \,.$$

We show now, following [19], the essential computations in the proof of Theorem 33. As explained in [19], it is crucial to show the vanishing (38) of $\mathcal{T}_r^{A_3}$ at the Chern class $c(III_{2,2}(r))$. I.e., it suffices to show the equality

$$(F_r + H_r)(x_1 + x_2 - \mathbb{D} - \mathbb{B}_{r-2}) = 0 \,. \qquad (68)$$

Due to the factorization property (9), each Schur function occuring in the expansion of H_r is such that

$$S_{c,r+1+a,r+1+b}(\mathbb{X}_2{-}\mathbb{D}{-}\mathbb{B}_{r-2}) = R(\mathbb{X}_2, \mathbb{D}{+}\mathbb{B}_{r-2}) \cdot S_c(-\mathbb{D}{-}\mathbb{B}_{r-2}) \cdot S_{a,b}(\mathbb{X}_2) \,.$$

We set

$$V_r(\mathbb{X}_2;\mathbb{B}_{r-2}) := \frac{H_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2})}{R(\mathbb{X}_2, \mathbb{D} + \mathbb{B}_{r-2})}\,, \tag{69}$$

so that

$$V_r(\mathbb{X}_2;\mathbb{B}_{r-2}) = \sum_{i=0}^{r-2} \sum_{\{j\geq 0:\ i+2j\leq r-2\}} e_{r-i,j}\ S_i(-\mathbb{D}-\mathbb{B}_{r-2})\ S_{j,r-i-j-2}(\mathbb{X}_2)\,. \tag{70}$$

We have the following recursive relation which follows from the observation that the coefficient of b_{r-2} in $V_r(\mathbb{X}_2;\mathbb{B}_{r-2})$ is equal to $-V_{r-1}(\mathbb{X}_2;\mathbb{B}_{r-3})$.

Lemma 34 *For $r \geq 2$, we have*

$$V_r(\mathbb{X}_2;\mathbb{B}_{r-2}) = \sum_{i=0}^{r-2} V_{r-i}(\mathbb{X}_2;0)\ S_i(-\mathbb{B}_{r-2})\,. \tag{71}$$

Thus it is sufficient to compute $V_r(\mathbb{X}_2;0)$.

Proposition 35 *For $r \geq 2$, we have*

$$V_r(\mathbb{X}_2;0) = 3^{r-2}\Big(3S_{r-2}(\mathbb{X}_2) - 2S_{1,r-3}(\mathbb{X}_2)\Big)\,. \tag{72}$$

(In particular, $V_2(\mathbb{X}_2;0) = 5$ and $V_3(\mathbb{X}_2;0) = 9S_1(\mathbb{X}_2)$.)

We now apply Corollary 29 from Subsection (8.1) with $\mathbb{B} = \boxed{2x_1} + \boxed{2x_2}$. Expanding

$$S_d\left(\mathbb{X}_2 - y^{-1}(\boxed{2x_1} + \boxed{2x_2})\right)$$
$$= S_d(\mathbb{X}_2) - \frac{2x_1+2x_2}{y} S_{d-1}(\mathbb{X}_2) + 4\frac{x_1x_2}{y^2} S_{d-2}(\mathbb{X}_2)\,,$$

we get, for $d \geq 3$,

$$W(d,\mathbb{D}) = y^{d-2}(y-1)(y-2)S_d(\mathbb{X}_2)) - 2y^{d-3}(y-1)(y-2)S_{1,d-1}(\mathbb{X}_2) \tag{73}$$

and initial conditions

$$W(0) = 1, \quad W(1) = (y-3)S_1(\mathbb{X}_2),$$
$$W(2) = (y-1)(y-2)S_2(\mathbb{X}_2) - 2(y-3)S_{11}(\mathbb{X}_2)\,.$$

We come back to Proposition 35, and we take the Pascal staircase (59). Then for $d = r-2$, the function $W(d,\mathbb{D})$ is the function $V_r(\mathbb{X}_2;0)$. We thus have to specialize y into $1,2,3$ successively. Apart from initial values, only $y=3$ contributes, and we get, for $d \geq 3$,

$$W(d,\mathbb{D}) = 3^{d+1}S_n(\mathbb{X}_2) - 2\cdot 3^d S_{1,d-1}(\mathbb{X}_2)\,.$$

This proves Proposition 35, checking the cases $r = 2,3,4$ directly.

We now pass to the specialization $F_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2})$. It is rather straightforward to prove the following lemma (cf. [19]).

Lemma 36 *The resultant* $R(\mathbb{X}_2, \mathbb{D}+\mathbb{B}_{r-2})$ *divides* $F_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2})$.

We set

$$U_r(\mathbb{X}_2; \mathbb{B}_{r-2}) := \frac{F_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2})}{R(\mathbb{X}_2, \mathbb{D}+\mathbb{B}_{r-2})}. \tag{74}$$

Note that each variable $b \in \mathbb{B}_{r-2}$ appears at most with degree 3 in $F_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2})$, and hence at most with degree 1 in $U_r(\mathbb{X}_2; \mathbb{B}_{r-2})$. We have the following precise recursive relation which follows from the observation that the coefficient of b_{r-2}^3 in $F_r(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-2})$ is equal to $F_{r-1}(\mathbb{X}_2 - \mathbb{D} - \mathbb{B}_{r-3})$.

Lemma 37 *For* $r \geq 2$*, we have*

$$U_r(\mathbb{X}_2; \mathbb{B}_{r-2}) = \sum_{i=0}^{r-2} U_{r-i}(\mathbb{X}_2; 0)\ S_i(-\mathbb{B}_{r-2}). \tag{75}$$

Let π be the endomorphism of the $\mathbf{C}$-vector space of functions of x_1, x_2, defined by

$$\pi\big(f(x_1,x_2)\big) = \frac{x_1 f(x_1,x_2) - x_2 f(x_2,x_1)}{x_1 - x_2}.$$

For any $i, j \in \mathbf{N}$, we have

$$\pi(x_1^j x_2^i) = S_{i,j}(\mathbb{X}_2). \tag{76}$$

The proof of the following proposition will make use of multi-Schur functions (see the end of Section 2).

Proposition 38 *For* $r \geq 2$*, we have*

$$F_r(\mathbb{X}_2 - \mathbb{D}) = -3^{r-2} R(\mathbb{X}_2, \mathbb{D})(x_1x_2)^{r-2}\big(3S_{r-2}(\mathbb{X}_2) - 2S_{1,r-3}(\mathbb{X}_2)\big). \tag{77}$$

Proof. The identity is true for $r = 2$. To prove the assertion for $r \geq 3$, we compute in two different ways the action of π on the multi-Schur function

$$S_{r,r;r}(\mathbb{X}_2 + \boxed{2x_1} + \boxed{3x_1} - \mathbb{D}; x_1 - \mathbb{D}). \tag{78}$$

Firstly, expanding (78), we have

$$\begin{aligned}
&\pi\big(S_{r,r;r}(\mathbb{X}_2 + \boxed{2x_1} + \boxed{3x_1} - \mathbb{D}; x_1 - \mathbb{D})\big) \\
&= \pi\Big(\sum_{j_1 \leq j_2 \leq r} S_{j_1,j_2}(\boxed{2x_1} + \boxed{3x_1})\ S_{r-j_2, r-j_1; r}(\mathbb{X}_2 - \mathbb{D}; x_1 - \mathbb{D})\Big) \\
&= \pi\Big(\sum_{j_1 \leq j_2 \leq r} S_{j_1,j_2}(\boxed{2} + \boxed{3})\ S_{r-j_2, r-j_1; r+j_1+j_2}(\mathbb{X}_2 - \mathbb{D}; x_1 - \mathbb{D})\Big) \\
&= \sum_{j_1 \leq j_2 \leq r} S_{j_1,j_2}(\boxed{2} + \boxed{3})\ S_{r-j_2, r-j_1, r+j_1+j_2}(\mathbb{X}_2 - \mathbb{D}) \\
&= F_r(\mathbb{X}_2 - \mathbb{D}).
\end{aligned}$$

Secondly, using Lemma 7, we subtract x_1 from the arguments in the first two rows of the determinant (78) without changing its value. We get the determinant

$$\begin{vmatrix} S_r(x_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D}) & S_{r+1}(x_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D}) & S_{r+2}(-\mathbb{D}) \\ S_{r-1}(x_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D}) & S_r(x_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D}) & S_{r+1}(-\mathbb{D}) \\ S_{r-2}(\mathbb{X}_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D}) & S_{r-1}(\mathbb{X}_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D}) & S_r(x_1-\mathbb{D}) \end{vmatrix}.$$

Since the elements in the first two rows of the third column are zero, this determinant is equal to

$$S_{r,r}(x_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D})\cdot S_r(x_1-\mathbb{D})\,.$$

Since

$$x_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D}=x_2+\boxed{3x_1}-\boxed{2x_2}-\boxed{x_1+x_2}$$

and the following two factorizations hold:

$$S_{r,r}(x_2+\boxed{3x_1}-\boxed{2x_2}-\boxed{x_1+x_2})=-3^{r-2}(x_2-2x_1)(x_1x_2)^{r-1}(3x_1-2x_2)$$

and

$$S_r(x_1-\mathbb{D})=x_1^{r-2}x_2(x_1-2x_2)\,,$$

we infer that

$$\begin{aligned} &S_{r,r;r}(\mathbb{X}_2+\boxed{2x_1}+\boxed{3x_1}-\mathbb{D};x_1-\mathbb{D}) \\ &\qquad=-3^{r-2}R(\mathbb{X}_2,\mathbb{D})(x_1x_2)^{r-2}x_1^{r-3}(3x_1-2x_2)\,. \end{aligned} \tag{79}$$

By (76), the result of applying π to (79) is

$$-3^{r-2}R(\mathbb{X}_2,\mathbb{D})(x_1x_2)^{r-2}\big(3S_{r-2}(\mathbb{X}_2)-2S_{1,r-3}(\mathbb{X}_2)\big)\,.$$

Comparison of both computations of π applied to (78) yields the proposition.

In terms of U_r, we rewrite Proposition 38 into

Corollary 39 *For $r\ge 2$, we have*

$$U_r(\mathbb{X}_2;0)=-3^{r-2}\big(3S_{r-2}(\mathbb{X}_2)-2S_{1,r-3}(\mathbb{X}_2)\big)\,. \tag{80}$$

These are the essential computations with Schur functions leading to the proof of Theorem 33.

9 Towards the Thom polynomial of $III_{3,3}(r)$

The singularity $III_{3,3}(r)$ has codimension $4r+2$. So, the partitions that we need to consider have weight $4r+2$. Moreover, all diagrams contain the partition $(r+1,r+1)$, have at most 4 rows and the length of the second row is at most r. Let $\mathbf{D}_r$ denote the set of all such diagrams. By $\mathbf{D}_{r,2},\mathbf{D}_{r,3} and \mathbf{D}_{r,4}$ we shall denote the subsets of $\mathbf{D}_r$, that consist of diagrams with 2,3 and 4 rows, respectively.

Set $\mathcal{T}_r := \mathcal{T}_r^{III_{3,3}}$. Then, the part of $\mathcal{T}_r$ corresponding to the partitions in $\mathbf{D}_{r,4}$ is given by $\Phi_4(\mathcal{T}_{r-1})$.

The Thom polynomial $\mathcal{T}_r$ must satisfy the following system of equations: (35) for $i = 0, 1, 2, 3, 4$, (36), (37), (38), (39), (40) together with the normalizing equation (46).

For a partition $I \in \mathbf{D}_r$, we have

$$\begin{aligned} S_I(-\mathbb{B}_{r-1}) =& S_I(x - \mathbb{B}_{r-1} - \boxed{2x}) \\ =& S_I(x - \mathbb{B}_{r-1} - \boxed{3x}) \\ =& S_I(x - \mathbb{B}_{r-1} - \boxed{4x}) = 0\,. \end{aligned}$$

Hence Eqs. (35) for $i = 0, 1, 2, 3, 4$ are satisfied automatically by any linear combination of Schur functions indexed by partitions in $\mathbf{D}_r$. Moreover, Eq.(36) implies Eq.(38) by the substitution $b_{r-1} = \boxed{x_1 + x_2}$. Hence we can replace the former set of equations by a smaller set of equations consisting of Eqs. : (36), (37), (39), (40) and (46). Note that in these equations, the alphabets we need to consider, are suitable for the factorization property (9) associated with a pair of alphabets of cardinalities $r + 1$ and 2.

In [26], we give an algorithm based on ACE (cf. [41]) which solves the latter system of equations. Using this algorithm, we get the (unique) $\mathcal{T}_r$ for $r = 2, \dots, 8$, expanded in the Schur function basis. In the next example, we give $\mathcal{T}_r$ for $r = 2, 3$, and in Section 11, we give $\mathcal{T}_4, \dots, \mathcal{T}_8$.

Example 40 *We have*

$$\begin{aligned} \mathcal{T}_2 =& 4S_{37} + 16S_{46} + 28S_{55} \\ &+20S_{145} + 6S_{136} + 7S_{235} + 3S_{244} \\ &+2S_{1135} + 3S_{1234} + 6S_{1144} + S_{2233}\,; \end{aligned}$$

$$\begin{aligned} \mathcal{T}_3 =& 8S_{4,10} + 40S_{59} + 88S_{68} + 120S_{77} \\ &+ 12S_{149} + 52S_{158} + 100S_{167} \\ &+ 14S_{248} + 50S_{257} + 20S_{266} \\ &+ 15S_{347} + 10S_{356} + \Phi_4(\mathcal{T}_2)\,. \end{aligned}$$

In [25], the author proposes a conjecture about the recursion for the coefficients in the Schur function expansion of $\mathcal{T}_r$. This recursion is checked for $2 \le r \le 8$, with the help of an algorithm in [26].

10 On the Thom polynomial of $I_{2,3}(r)$

Set $\mathcal{T}_r := \mathcal{T}_r^{I_{2,3}}$. The singularity $I_{2,3}(r)$ has codimension $4r{+}1$. So, the partitions that we need to consider have weight $4r + 1$. Moreover, all diagrams contain the partition $(r + 1, r + 1)$ and have at most 4 rows. Then, the part of $\mathcal{T}_r$ corresponding to the partitions with 4 rows is given by $\Phi_4(\mathcal{T}_{r-1})$.

The Thom polynomial $\mathcal{T}_r$ must satisfy the following system of equations: (35) for $i = 0, 1, 2, 3$, (36), (38), (39) together with the normalizing equation (43).

An algorithm analogous to the one in [26], allows us to get the (unique) solutions $\mathcal{T}_r$ of this system of equations for $r = 1, \dots, 7$, expanded in Schur function basis. In the next example, we give $\mathcal{T}_r$ for $r = 1, 2, 3$, and in Section 12, we give $\mathcal{T}_4, \dots, \mathcal{T}_7$.

Example 41 *We have*

$$\mathcal{T}_1 = 2S_{122} + 4S_{23};$$

$$\mathcal{T}_2 = 32S_{36} + 24S_{45} + 24S_{135} + 12S_{144} + 12S_{234} + 3S_{333} + \Phi_4(\mathcal{T}_1)\,;$$

$$\begin{aligned}\mathcal{T}_3 = {} & 208S_{49} + 208S_{58} + 112S_{67} \\ & + 168S_{148} + 152S_{157} + 56S_{166} \\ & + 100S_{247} + 76S_{256} \\ & + 50S_{346} + 24S_{355} \\ & + 18S_{445} + \Phi_4(\mathcal{T}_2)\,.\end{aligned}$$

11 Appendix 1: $\mathcal{T}_r^{III_{3,3}}$, $r = 4, \ldots, 8$

Let $\mathcal{T}_r = \mathcal{T}_r^{III_{3,3}}$. We have

$$\begin{aligned}\mathcal{T}_4 = {} & 16S_{5,13} + 96S_{6,12} + 256S_{7,11} + 416S_{8,10} + 496S_{9,9} \\ & + 24S_{1,5,12} + 128S_{1,6,11} + 304S_{1,7,10} + 448S_{189} \\ & + 28S_{2,5,11} + 128S_{2,6,10} + 264S_{279} + 100S_{288} \\ & + 30S_{3,5,10} + 112S_{369} + 70S_{378} \\ & + 31S_{459} + 25S_{468} + 10S_{477} + \Phi_4(\mathcal{T}_3)\,;\end{aligned}$$

$$\begin{aligned}\mathcal{T}_5 = {} & 32S_{6,16} + 224S_{7,15} + 704S_{8,14} + 1344S_{9,13} + 1824S_{10,12} + 2016S_{11,11} \\ & + 48S_{1,6,15} + 304S_{1,7,14} + 864S_{1,8,13} + 1504S_{1,9,12} + 1904S_{1,10,11} \\ & + 56S_{2,6,14} + 312S_{2,7,13} + 784S_{2,8,12} + 1232S_{2,9,11} + 448S_{2,10,10} \\ & + 60S_{3,6,13} + 284S_{3,7,12} + 616S_{3,8,11} + 364S_{3,9,10} \\ & + 62S_{4,6,12} + 238S_{4,7,11} + 182S_{4,8,10} + 70S_{499} \\ & + 63S_{5,6,11} + 56S_{5,7,10} + 35S_{589} + \Phi_4(\mathcal{T}_4)\,;\end{aligned}$$

$$\begin{aligned}\mathcal{T}_6 = {} & 64S_{7,19} + 512S_{8,18} + 1856S_{9,17} + 4096S_{10,16} + 6336S_{11,15} + 7680S_{12,14} + 8128S_{13,13} \\ & + 96S_{1,7,18} + 704S_{1,8,17} + 2336S_{1,9,16} + 4736S_{1,10,15} + 6816S_{1,11,14} + 7872S_{1,12,13} \\ & + 112S_{2,7,17} + 736S_{2,8,16} + 2192S_{2,9,15} + 4032S_{2,10,14} + 5392S_{2,11,12} + 1904S_{2,12,12} \\ & + 120S_{3,7,16} + 688S_{3,8,15} + 1800S_{3,9,14} + 2976S_{3,10,13} + 1680S_{3,11,12} \\ & + 124S_{4,7,15} + 600S_{4,8,14} + 1348S_{4,9,13} + 980S_{4,10,12} + 364S_{4,11,11} \\ & + 126S_{5,7,14} + 492S_{5,8,13} + 420S_{5,9,12} + 252S_{5,10,11} \\ & + 127S_{6,7,13} + 119S_{6,8,12} + 91S_{6,9,11} + 35S_{6,10,10} + \Phi_4(\mathcal{T}_5)\,;\end{aligned}$$

$$
\begin{aligned}
\mathcal{T}_7 &= 128S_{8,22} + 1152S_{9,21} + 4736S_{10,20} + 11904S_{11,19} + 20864S_{12,18} \\
&+ 28032S_{13,17} + 31616S_{14,16} + 32640S_{15,15} \\
&+ 192S_{1,8,21} + 1600S_{1,9,20} + 6080S_{1,10,19} + 14144S_{1,11,18} + 23104S_{1,12,17} \\
&+ 29376S_{1,13,16} + 32064S_{1,14,15} \\
&+ 224S_{2,8,20} + 1696S_{2,9,19} + 5856S_{2,10,18} + 12448S_{2,11,17} + 18848S_{2,12,16} \\
&+ 22752S_{2,13,15} + 7872S_{2,14,14} \\
&+ 240S_{3,8,19} + 1616S_{3,9,18} + 4976S_{3,10,17} + 9552S_{3,11,16} + 13392S_{3,12,15} + 7296S_{3,13,14} \\
&+ 248S_{4,8,18} + 1448S_{4,9,17} + 3896S_{4,10,16} + 6696S_{4,11,15} + 4656S_{4,12,14} + 1680S_{4,13,13} \\
&+ 252S_{5,8,17} + 1236S_{5,9,16} + 2844S_{5,10,15} + 2328S_{5,11,14} + 1344S_{5,12,13} \\
&+ 254S_{6,8,16} + 1002S_{6,9,15} + 912S_{6,10,14} + 672S_{6,11,13} + 252S_{6,12,12} \\
&+ 255S_{7,8,15} + 246S_{7,9,14} + 210S_{7,10,13} + 126S_{7,11,12} + \Phi_4(\mathcal{T}_6)\,;
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_8 &= 256S_{9,25} + 2560S_{10,24} + 11776S_{11,23} + 33280S_{12,22} + 65536S_{13,21} + 97792S_{14,20} \\
&+ 119296S_{15,19} + 128512S_{16,18} + 130816S_{17,17} \\
&+ 384S_{1,9,24} + 3584S_{1,10,23} + 15360S_{1,11,22} + 40448S_{1,12,21} + 74496S_{1,13,20} \\
&+ 104960S_{1,14,19} + 122880S_{1,15,18} + 129536S_{1,16,17} \\
&+ 448S_{2,9,23} + 3840S_{2,10,22} + 15104S_{2,11,21} + 36608S_{2,12,20} + 62592S_{2,13,19} \\
&+ 83200S_{2,14,18} + 93952S_{2,15,17} + 32064S_{2,16,16} \\
&+ 480S_{3,9,22} + 3712S_{3,10,21} + 13184S_{3,11,20} + 29056S_{3,12,19} \\
&+ 45888S_{3,13,18} + 57728S_{3,14,17} + 30624S_{3,15,16} \\
&+ 496S_{4,9,20} + 3392S_{4,10,20} + 10688S_{4,11,19} + 21184S_{4,12,18} \\
&+ 30880S_{4,13,17} + 20688S_{4,14,16} + 7296S_{4,15,15} \\
&+ 504S_{5,9,20} + 2976S_{5,10,19} + 8160S_{5,11,18} + 14432S_{5,12,17} + 11352S_{5,13,16} + 6336S_{5,14,15} \\
&+ 508S_{6,9,19} + 2512S_{6,10,18} + 5872S_{6,11,17} + 5172S_{6,12,16} + 3672S_{6,13,15} + 1344S_{6,14,14} \\
&+ 510S_{7,9,18} + 2024S_{7,10,17} + 1914S_{7,11,16} + 1584S_{7,12,15} + 924S_{7,13,14} \\
&+ 511S_{8,9,17} + 501S_{8,10,16} + 456S_{8,11,15} + 336S_{8,12,14} + 126S_{8,13,13} + \Phi_4(\mathcal{T}_7)\,.
\end{aligned}
$$

12 Appendix 2: $\mathcal{T}_r^{I_{2,3}}$, $r = 4, \ldots, 7$

Let $\mathcal{T}_r = \mathcal{T}_r^{I_{2,3}}$. We have

$$
\begin{aligned}
\mathcal{T}_4 &= 1280S_{5,12} + 1024S_{7,10} + 1408S_{6,11} + 480S_{89} \\
&+ 1056S_{1,5,11} + 1120S_{1,6,10} + 736S_{179} + 240S_{188} \\
&+ 656S_{2,5,10} + 656S_{269} + 368S_{278} \\
&+ 360S_{359} + 328S_{368} + 124S_{377} \\
&+ 180S_{458} + 134S_{467} \\
&+ 75S_{557} + 36S_{566} + \Phi_4(\mathcal{T}_3)\,;
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_5 = {} & 7744S_{6,15} + 8832S_{7,14} + 7168S_{8,13} + 4544S_{9,12} + 1984S_{10,11} \\
& + 6432S_{1,6,14} + 7232S_{1,7,13} + 5632S_{1,8,12} + 3232S_{1,9,11} + 992S_{1,10,10} \\
& + 4048S_{2,6,13} + 4448S_{2,7,12} + 3264S_{2,8,11} + 1616S_{2,9,10} \\
& + 2280S_{3,6,12} + 2416S_{3,7,11} + 1632S_{3,8,10} + 560S_{3,9,9} \\
& + 1204S_{4,6,11} + 1208S_{4,7,10} + 692S_{489} \\
& + 602S_{5,6,10} + 542S_{579} + 206S_{588} \\
& + 270S_{669} + 201S_{678} + \Phi_4(\mathcal{T}_4)\,;
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_6 = {} & 46592S_{7,18} + 53888S_{8,17} + 45824S_{9,16} + 32640S_{10,15} + 19200S_{11,14} + 8064S_{12,13} \\
& + 38784S_{1,7,17} + 44608S_{1,8,16} + 37248S_{1,9,15} + 25408S_{1,10,14} + 13568S_{1,11,13} + 4032S_{1,12,12} \\
& + 24512S_{2,7,16} + 27936S_{2,8,15} + 22720S_{2,9,14} + 14624S_{2,10,13} + 6784S_{2,11,12} \\
& + 13920S_{3,7,15} + 15632S_{3,8,14} + 12256S_{3,9,13} + 7312S_{3,10,12} + 2384S_{3,11,11} \\
& + 7472S_{4,7,14} + 8200S_{4,8,13} + 6128S_{4,9,12} + 3152S_{4,10,11} \\
& + 3864S_{5,7,13} + 4100S_{5,8,12} + 2812S_{5,9,11} + 980S_{5,10,10} \\
& + 1932S_{6,7,12} + 1924S_{6,8,11} + 1108S_{6,9,10} \\
& + 903S_{7,7,11} + 813S_{7,8,10} + 309S_{7,9,9} + \Phi_4(\mathcal{T}_5)\,;
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{T}_7 = {} & 279808S_{8,21} + 325376S_{9,20} + 282368S_{10,19} + 212224S_{11,18} \\
& + 140544S_{12,17} + 79104S_{13,16} + 32512S_{14,15} \\
& + 233088S_{1,8,20} + 270464S_{1,9,19} + 232832S_{1,10,18} + 171392S_{1,11,17} \\
& + 108672S_{1,12,16} + 55680S_{1,13,15} + 16256S_{1,14,14} \\
& + 147520S_{2,8,19} + 170560S_{2,9,18} + 145088S_{2,10,17} \\
& + 103872S_{2,11,16} + 62272S_{2,12,15} + 27840S_{2,13,14} \\
& + 84000S_{3,8,18} + 96544S_{3,9,17} + 80736S_{3,10,16} \\
& + 55776S_{3,11,15} + 31136S_{3,12,14} + 9856S_{3,13,13} \\
& + 45328S_{4,8,17} + 51600S_{4,9,16} + 42160S_{4,10,15} + 27888S_{4,11,14} + 13536S_{4,12,13} \\
& + 23688S_{5,8,16} + 26568S_{5,9,15} + 21080S_{5,10,14} + 12928S_{5,11,13} + 4304S_{5,12,12} \\
& + 12100S_{6,8,15} + 13284S_{6,9,14} + 10032S_{6,10,13} + 5232S_{6,11,12} \\
& + 6050S_{7,8,14} + 6388S_{7,9,13} + 4400S_{7,10,12} + 1540S_{7,11,11} \\
& + 2898S_{8,8,13} + 2886S_{8,9,12} + 1662S_{8,10,11} + \Phi_4(\mathcal{T}_6)\,.
\end{aligned}
$$

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