INVARIANTS OF HYPERSURFACES AND LOGARITHMIC DIFFERENTIAL FORMS

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1. Introduction

The main goal of this note is to give an overview of several methods to compute certain invariants of hypersurfaces in projective manifolds.

The case of Hodge numbers of a smooth hypersurface in a complex projective space was studied in the classical paper of Griffiths ([26]). The first formulae for Hodge numbers of resolutions of singular varieties of dimension three were given by Clemens (the case of nodal double solids – see [6]) and Werner (nodal hypersurfaces in $\mathbb{P}^4$ studied in [48]). Those seminal results admit various generalizations. We focus our interest on two kinds of problems closely related to the above-mentioned facts. First, we want to study hypersurfaces in (or double coverings of) more general projective manifolds. Unfortunately, in most cases under consideration one needs some Bott–type vanishing assumptions in order to obtain applicable formulae. The other natural question we want to consider, is the behaviour of Hodge structure of a resolution if we allow certain higher singularities (i.e. other than ordinary double points) on the studied varieties. Then we are interested in invariants of a fixed resolution of singularities. Moreover, we consider only resolutions that are given by a sequence of blow–ups with smooth centers.

The main tool we use in the paper are differential forms with logarithmic poles along a divisor, i.e. differential forms ω such that ω and $d\omega$ have at most simple poles. In the case of a simple normal crossing divisor those forms are well behaved. They form a locally free sheaf that appears in exact sequences given by the Poincare residue and the restriction map. For the convenience of the reader we collect basic information on logarithmic differential forms in Section 2. Section 3 is devoted to the study of behavior of differential forms under a blow–up with a smooth center. The next section contains an overview of basic properties of Hodge numbers.

The first case where behaviour of Hodge numbers of a resolution becomes very subtle are threefolds. For three-dimensional varieties (resp. their resolutions) one has two Hodge numbers that are difficult to study/compute. The numbers in question can be related using the Euler characteristic. In Section 5. we discuss a method to study the difference between the Euler characteristic of a smooth model and the degree of the Fulton–Johnson class, i.e. the Milnor number.

Section 6 contains a discussion of infinitesimal deformations of double coverings of algebraic manifolds. The original motivation was that for a Calabi–Yau threefold one of the Hodge numbers (i.e. $h^{1,2}$ of the manifold in question) equals the dimension of the Kuranishi space. In Section 7 we discuss the defect formulae by Clemens and Werner, and study Hodge

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numbers of nodal hypersurfaces and nodal double coverings. Last two sections contain an overview of the most general results on Hodge numbers of resolutions of hypersurfaces with A-D-E singularities.

2. Logarithmic differential forms and logarithmic vector fields

Let Y be a reduced divisor on a smooth algebraic manifold X, and let $\Omega_X^j(Y)$ stand for the sheaf of differential j–forms on X with at most simple poles along Y.

Definition 2.1. [14] A differential j–form with logarithmic poles along Y on an open subset $V \subset X$ is a meromorphic j–form ω on V regular on $V \setminus Y$ and such that both ω and $d\omega$ have at most simple poles along Y.

Differential j–forms with logarithmic poles along Y form a sheaf denoted by $\Omega_X^j(\log Y)$. For any open subset $V \subset X$ we have

$$\Gamma(V, \Omega_X^j(\log Y)) = \{\omega \in \Omega_X^j(Y) : d\omega \in \Omega_X^{j+1}(Y)\}.$$

A normal crossing divisor Y in X, is a reduced divisor which is locally defined by an equation of the form $f = f_1 \cdot \dots \cdot f_p$, where $f_1, \dots, f_n$ are local coordinates for X, $p \le n$.

If Y is a normal crossing divisor then $\Omega_X^j(\log Y)$ is a locally free sheaf. In this case a form $\omega \in \Omega_X^j(\log Y)$ can be written locally in the following way

$$\omega = \sum_{1 \le k_1 < \dots < k_j \le n} f_{k_1 \dots k_j} \delta_{k_1} \wedge \dots \wedge \delta_{k_j},$$

where $\delta_i = \begin{cases} \frac{df_i}{f_i}, & \text{if} \quad i \le j \\ df_i, & \text{if} \quad i > j \end{cases}$. In particular, we have

$$\Omega_X^j(\log Y) = \bigwedge^j \Omega_X^1(\log Y).$$

If Y is a smooth divisor, then we have the following exact sequences ([21, Prop. 2.3])

$$\begin{aligned} &0 \longrightarrow \Omega_X^1 \longrightarrow \Omega_X^1(\log Y) \longrightarrow \mathcal{O}_Y \longrightarrow 0, \\ &0 \longrightarrow \Omega_X^j \longrightarrow \Omega_X^j(\log Y) \longrightarrow \Omega_Y^{j-1} \longrightarrow 0, \\ &0 \longrightarrow \Omega_X^j(\log Y)(-Y) \longrightarrow \Omega_X^j \longrightarrow \Omega_Y^j \longrightarrow 0. \end{aligned} \tag{2.1}$$

The map $\Omega_X^j(\log Y) \longrightarrow \Omega_Y^{j-1}$ in the exact sequence (2.1) is the Poincare residue

$$\Omega_X^j(\log Y) \ni \omega \wedge \tfrac{df}{f} \mapsto \omega | Y \in \Omega_Y^{j-1},$$

where $f = 0$ is a local equation of Y in X.

In particular, for $\dim X = 4$, we get the following exact sequence (see [40, p. 444])

$$0 \longrightarrow \Omega_X^3 \longrightarrow \Omega_X^3(\log Y) \longrightarrow \Omega_Y^2 \longrightarrow 0\,, \tag{2.2}$$

and the following resolution of the sheaf $\Omega_X^3(\log Y)$ (see [40, p. 445])

$$0 \longrightarrow \Omega_X^3(\log Y) \longrightarrow \Omega_X^3(Y) \longrightarrow K_X(2Y)/K_X(Y) \longrightarrow 0\,. \tag{2.3}$$

A more detailed exposition of other properties of logarithmic forms can be found in [21].

The dual sheaf to the sheaf of differential one–forms $\Omega^1_X(\log Y)$ is the sheaf $\Theta_X(\log Y)$ of logarithmic vector fields along Y, that is defined by the following exact sequence (cf. [18, (2.1)])

$$0 \longrightarrow \Theta_X(\log Y) \longrightarrow \Theta_X \longrightarrow \mathcal{N}_{Y|X} \longrightarrow 0. \tag{2.4}$$

The sheaf $\Theta_X(\log Y)$ is the kernel of the natural restriction map $\Theta_X \longrightarrow \mathcal{N}_{Y|X}$. Consequently, it is the subsheaf of the tangent bundle Θ_X consisting of the vector fields that carry the ideal sheaf of Y into itself.

3. Blow–up

In this section we study a fixed resolution of a singular variety by a sequence of blow–ups with smooth centers. Let $C \subset X$ be a smooth subvariety of codimension $k = (n+1-d)$. and let $\sigma : \tilde{X} \longrightarrow X$ the blow–up of X along C with the exceptional divisor E. We put $\mathcal{N} := \mathcal{N}_{C|Y}$ (resp. $\mathcal{N}^\vee$) to denote the normal bundle of C in Y (resp. its dual). Moreover, $S^l\mathcal{N}$ stands for the l–th symmetric power of $\mathcal{N}$.

Our strategy in most proofs in next sections will be to consider separately the impact of each blow–up on the Hodge numbers of the studied varieties. In further sections we will sketch certain proofs and omit some computations. Below we collect basic technical facts that are necessary to work out the details (see [1, Thm. I.9.1] and [27, Exc. III.8.4])

Proposition 3.1. *We have*

$$\begin{array}{ll}
\sigma_*\mathcal{O}_{\tilde{X}} \cong \mathcal{O}_X, & \\
R^i\sigma_*\mathcal{O}_{\tilde{X}} = 0 & \textit{for } i > 0, \\
\mathcal{O}_{\tilde{X}}(E) \otimes \mathcal{O}_E \cong \mathcal{O}_E(-1). & \\
\sigma_*(\mathcal{O}_E(l)) \cong S^l\mathcal{N}^\vee, & \textit{for } l \geq 0, \\
\sigma_*(\mathcal{O}_E(l)) = 0, & \textit{for } l < 0, \\
R^i\sigma_*(\mathcal{O}_E(l)) = 0, & \textit{for } i \neq 0, k-1, \\
R^{k-1}\sigma_*(\mathcal{O}_E(l)) = 0, & \textit{for } l > -k, \\
R^{k-1}\sigma_*(\mathcal{O}_E(l)) \cong S^{-l-k}\mathcal{N} \otimes \bigwedge^k \mathcal{N}, & \textit{for } l \leq -k.
\end{array}$$

Moreover the following "relative Euler sequence"

$$0 \longrightarrow \Omega^p_{E/C} \longrightarrow \sigma^*(\bigwedge^p \mathcal{N}^\vee) \otimes \mathcal{O}_E(-p) \longrightarrow \Omega^{p-1}_{E/C} \longrightarrow 0$$

is exact.

Proposition 3.2. *For a non–negative integer m we have*

(1) $\sigma_*\mathcal{O}_{\tilde{X}}(mE) \cong \mathcal{O}_X$,

(2) $R^{k-1}\sigma_*\mathcal{O}_{\tilde{X}}(mE) \cong \bigoplus_{j=0}^{m-k} S^j(\mathcal{N}) \otimes \bigwedge^k \mathcal{N}$,

(3) $R^i\sigma_*\mathcal{O}_{\tilde{X}}(mE) = 0$, *for* $i \neq 0, k-1$.

4. Hodge numbers

The Hodge number $h^{p,q}(Y)$ $(0 \leq p, q \leq \dim Y)$ of a compact complex manifold is defined as the dimension of the Dolbeault cohomology

$$h^{p,q}(Y) = \dim_{\mathbb{C}} H^{p,q}(Y), \quad H^{p,q}(Y) = H^q(\Omega^p_Y).$$

Hodge numbers are usually collected in the Hodge diamond

$$
\begin{array}{ccccccccc}
 & & & & h^{0,0} & & & & \\
 & & & h^{0,1} & & h^{1,0} & & & \\
 & & h^{0,2} & & h^{1,1} & & h^{2,0} & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
h^{0,n} & & h^{1,n-1} & & \dots & & h^{n-1,1} & & h^{n,0} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & h^{n,n-2} & & h^{n-1,n-1} & & h^{n-2,n} & & \\
 & & & h^{n,n-1} & & h^{n-1,n} & & & \\
 & & & & h^{n,n} & & & &
\end{array}
$$

Moreover, Hodge numbers of a projective manifold Y satisfy the following symmetries

$$
\begin{aligned}
h^{p,q} &= h^{q,p}, && \text{Hodge duality} \\
h^{p,q} &= h^{n-p,n-q}, && \text{Serre duality}
\end{aligned}
$$

Furthermore, by the Hodge decomposition one has the following equality

$$\sum_{i=0}^{k} h^{i,k-i} = b_k,$$

where $b_k(Y) := \dim_{\mathbb{C}} H^k(Y, \mathbb{C})$ is the k–th Betti number.

If Y is a smooth ample divisor in a projective manifold X, then by Lefschetz's hyperplane section theorem we have isomorphisms

$$H^{p,q}(Y) \cong H^{p,q}(X), \text{ for } p+q \le \dim Y - 1.$$

Consequently, the Hodge diamond of an n–dimensional complete intersection Y in a projective space $\mathbb{P}^n$ looks as follows

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 0 & & 1 & & 0 & & \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
h^{0,n} & & h^{1,n-1} & & \dots & & h^{n-1,1} & & h^{n,0} \\
 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
 & & 0 & & 1 & & 0 & & \\
 & & & 0 & & 0 & & & \\
 & & & & 1 & & & &
\end{array}
$$

Moreover, the Euler characteristic of Y can be computed using the Gauss–Bonnet formula. For a degree-d hypersurface in $\mathbb{P}^{n+1}$ one obtains

$$e(Y) = \sum_{k=0}^{n} (-1)^k d^{k+1} \binom{n+2}{n-k}, \text{ where } d = \dim Y.$$

We can also compute the geometric genus

$$p_g = h^{n,0}(Y) = \binom{d-1}{n+1}.$$

As a consequence, we get the Hodge diamond of any (smooth) hypersurface of dimension less or equal 3

$\dim Y = 1$	$\dim Y = 2$	$\dim Y = 3$
$h^{00} = h^{11} = 1$	$h^{00} = h^{22} = 1$	$h^{00} = h^{11} = h^{22} = h^{33} = 1$
$h^{01} = h^{10} = \binom{d-1}{2}$	$h^{01} = h^{10} = h^{03} = h^{30} = 0$	$h^{01} = h^{10} = h^{02} = h^{20} = h^{04} = h^{40} =$
		$= h^{05} = h^{50} = 0$
	$h^{02} = h^{20} = \binom{d-1}{3}$	$h^{03} = h^{30} = \binom{d-1}{4}$
	$h^{11} = h^{11} = \frac{2d^3-6d^2+7d}{3}$	$h^{12} = h^{21} = \frac{11d^4-50d^3+85d^2-70d+24}{24}$

The Hodge decomposition of higher dimensional hypersurfaces in projective spaces was given by Griffiths ([26]).

Theorem 4.1. *If $Y = \{F = 0\}$ is a smooth degree d hypersurface in $\mathbb{P}^{n+1}$, then*

$$H_0^{p,n-p}(Y) \cong (\mathbb{C}[X_0, \dots, X_{n+1}]/\mathrm{J}_F)_{d(p+1)-(n+2)},$$

where $H_0^{p,q}$ denotes the primitive cohomology, J_F is the jacobian ideal of Y generated by partial derivatives of F.

Recall, that the primitive cohomology $H_0^{p,q}$ is the kernel of the map

$$H^{p,q}(Y) \longrightarrow H^{p+1,q+1}(Y)$$

defined by multiplication with a class of a hyperplane. Consequently, in the above theorem

$$\begin{array}{ll} h^{p,q}(Y) = h_0^{p,q}(Y) & \text{unless } n \text{ is even and } \quad p = q = \tfrac{n}{2} \\ h^{p,p}(Y) = h_0^{p,p}(Y) + 1 & \text{if } n = 2p. \end{array}$$

5. Euler characteristic of a smooth model of a singular hypersurface

Let Y be a hypersurface of dimension n in a smooth algebraic manifold X. If Y is smooth then the topological Euler characteristic of Y can be computed using the adjunction formula as

$$\mathrm{e}(Y) = \deg \frac{c(\Theta_X) \cap [Y]}{c(\mathcal{N}_{Y|X})} = \int_X \sum_{k=0}^{n} (-1)^k c_k(X)[Y]^{n-k},$$

where $[Y]$ is its cohomology class.

We are interested in the case when Y is singular. In this situation the topological Euler characteristic of the hypersurface in question is not determined by its cohomology class $[Y]$. The number

$$\tilde{\mathrm{e}}(Y) = \deg \frac{c(\Theta_X) \cap [Y]}{c(\mathcal{N}_{Y|X})} = \int_X \sum_{k=0}^{n} (-1)^k c_k(X)[Y]^{n-k}$$

is the degree of the Fulton–Johnson class c^{FJ} (see [23, Examp. 4.2.6] or [24]), while the Euler characteristic $\mathrm{e}(Y)$ is the degree of the Schwartz-MacPherson class c^{SM} (see [31]). The difference (up to a sign convention) of these classes is called the Milnor class (see [39]). In the case of an isolated singularity, the degree of the Milnor class agrees with the classical Milnor number studied in [35]. It equals the codimension of the Jacobian ideal. In the case of higher dimensional singularities, it agrees with the generalization of the Milnor number studied by Parusinski in [37] (see also [38, 39]).

In [9] we gave a method for computing the difference between the degree of the Fulton–Johnson class $\tilde{\mathrm{e}}(Y)$ and the Euler characteristic $\mathrm{e}(\tilde{Y})$ of a non–singular model $\tilde{Y}$ of Y (in the paper [9] we work in a more general setup of a complete intersection). We shall consider a non–singular model satisfying the following property: there is a sequence of blow-ups with smooth centers $\sigma : \tilde{X} \longrightarrow X$ such that $\tilde{Y} \subset \tilde{X}$ is the strict transform of Y.

If Y is a smooth manifold its Euler characteristic can be computed as an alternating sum of holomorphic Euler characteristics of sheaves of differential forms

$$\mathrm{e}(Y) = \sum_i (-1)^i \chi(\Omega^i_Y).$$

From the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-Y) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

and the additivity of the holomorphic Euler characteristic we get

$$\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-Y)).$$

Similarly, from the exact sequences (2.1) tensored with powers of $\mathcal{O}_X(-Y)$ we get

$$\begin{aligned}
&\chi(\Omega^1_Y) = \chi(\Omega^1_X) - \chi(\Omega^1_X(\log Y)(-Y)),\\
&\chi(\Omega^1_X(\log Y)(-Y)) = \chi(\Omega^1_X(\log Y)) + \chi(\mathcal{O}_Y(-Y)),\\
&\chi(\mathcal{O}_Y(-Y)) = \chi(\mathcal{O}_X(-Y)) - \chi(\mathcal{O}_X(-2Y)),
\end{aligned}$$

and consequently

$$\chi(\Omega^1_Y) = \chi(\Omega^1_X) - \chi(\Omega^1_X(-Y)) - \chi(\mathcal{O}_X(-Y)) + \chi(\mathcal{O}_X(-2Y)).$$

More generally, for any locally free sheaf $\mathcal{F}$ on X and any $p = 0, \ldots, n$ we have

$$\chi(\Omega^p_Y \otimes \mathcal{F}) = \sum_{q=0}^{p} (-1)^q \left[\chi(\Omega^{p-q}_X(-qY) \otimes \mathcal{F}) - \chi(\Omega^{p-q}_X(-(q+1)Y) \otimes \mathcal{F})\right].$$

As the above formulae make sense for any divisor Y and (by the Riemann-Roch theorem) depend only on the class of Y, they give the degree of the Fulton–Johnson class $\tilde{\mathrm{e}}(Y)$. Our goal is to compute the difference $(\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y))$. In order to do this we have to study

$$\chi(\Omega^p_{\tilde{X}}(-(q\tilde{Y})) - \chi(\Omega^p_X(-(qY))),$$

so it is enough to compute the numbers

$$D_p(\mathcal{L}, m) := \chi(\Omega^p_{\tilde{X}} \otimes \sigma^*\mathcal{L}^{-1} \otimes \mathcal{O}_{\tilde{X}}(mE)) - \chi(\Omega^p_X \otimes \mathcal{L}^{-1})$$

for an effective line bundle $\mathcal{L}$ on X and non–negative integers m, p.

Exact formulae for D_p, $p \leq 2$ are given in [9, Thm.5]. Applying those formulae we obtain

Theorem 5.1 ([9, Thm. 6])**.** *Let Y be a surface in a smooth threefold X, $\sigma : \tilde{X} \longrightarrow X$ be a blow–up of a smooth irreducible variety $C \subset Y$. Denote by $\tilde{Y}$ the strict transform of Y and by m the multiplicity of Y at a generic point of C. Then*

$$\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y) = \begin{cases} -m^3 + 2m^2, & \textit{if } \dim C = 0 \\ (3m^2 - 2m - 1)YC + (-m^3 + 1)c_1(\mathcal{N}) + & \textit{if } \dim C = 1 \\ +(-m^2 + m)c_1(C). & \end{cases}$$

Theorem 5.2 ([9, Thm. 7])**.** *Let Y be a threefold in a smooth fourfold X, $\sigma : \tilde{X} \longrightarrow X$ be a blow–up of a smooth irreducible variety $C \subset Y$. Denote by $\tilde{Y}$ the strict transform of Y and by m the multiplicity of Y at a generic point of C. Then*

$$\tilde{\mathrm{e}}(\tilde{Y}) - \tilde{\mathrm{e}}(Y) = \begin{cases} m^4 - 3m^3 + 2m^2 + 2m, & \textit{if } \dim C = 0 \\ (-m^3 + 2m^2)c_1(C) + (-m^4 + m^3 + & \textit{if } \dim C = 1 \\ +m^2 - m)c_1(\mathcal{N}) + (4m^3 - 6m^2 + 2)YC, & \\ (-m^4 + m^3 + 2m^2)\, c_2(\mathcal{N}) + & \textit{if } \dim C = 2 \\ +\,(m^2 - m)\, c_2(C) + (6m^2 - 3m - 1)\, Y^2C + & \\ +\,(-4m^3 + 2m)\, Yc_1(\mathcal{N}) + (-3m^2 + 2m + 1)Yc_1(C) + & \\ +\,(m^4 - m^2)\, c_1^2(\mathcal{N}) + (m^3 - m)\, c_1(C)c_1(\mathcal{N}). & \end{cases}$$

6. Deformations of double coverings

An *infinitesimal deformation* of X is a scheme X' flat over the ring of dual numbers $\mathbb{D} = \mathbb{C}[t]/[t^2]$ and such that $X' \otimes_{\mathbb{D}} \mathbb{C} \cong X$. If the variety X is smooth, then the space of infinitesimal deformations is isomorphic to the cohomology group $H^1\Theta_X$ of the tangent bundle Θ_X.

Let $\pi : X \longrightarrow Y$ be a double cover of a *smooth* algebraic variety branched along a smooth divisor D. The cover π is not determined by D itself, we have also to fix a line bundle $\mathcal{L}$ on Y s.t. $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus \mathcal{L}^{-1}$. This $\mathcal{L}$ satisfies $\mathcal{L}^{\otimes 2} \cong \mathcal{O}_Y(D)$. Since the map π is finite, we have $H^i(\Theta_X) \cong H^i(\pi_*\Theta_X)$. From [21, Lem. 3.16] we get $\pi_*\Theta_X \cong \Theta_Y \otimes \mathcal{L}^{-1} \oplus \Theta_Y(\log D)$ and so

$$H^1\Theta_X \cong H^1(\Theta_Y(\log D)) \oplus H^1(\Theta_Y \otimes \mathcal{L}^{-1}).$$

Consequently, we obtain the following proposition describing the deformations of a double covering with smooth branched divisor

Proposition 6.1 ([13, Prop. 2.2])**.**

(a) $H^1(\Theta_Y(\log D)) \cong \mathcal{C}o\mathcal{K}er(H^0\Theta_Y \longrightarrow H^0\mathcal{N}_{D|Y}) \oplus \mathcal{K}er(H^1\Theta_Y \longrightarrow H^1\mathcal{N}_{D|Y})$,

(b) $H^1(\Theta_Y(\log D))$ *is isomorphic to the space* $T^1_{X\to Y}$ *of infinitesimal deformations of* X *which are double covers of deformations of* Y*,*

(c) $\mathcal{C}o\mathcal{K}er(H^0\Theta_Y \longrightarrow H^0\mathcal{N}_{D|Y})$ *is isomorphic to the space* $T^1_{X/Y}$ *of infinitesimal deformations of* X *which are double covers of* Y*.*

Corollary 6.2 ([13, Cor. 2.3])**.**

(a) *Every deformation of* X *is a double cover of a deformation of* Y *iff* $H^1(\Theta_Y\otimes\mathcal{L}^{-1}) = 0$.

(b) *Every deformation of* X *is a double cover of* Y *iff* $H^1(\Theta_Y \otimes \mathcal{L}^{-1}) = 0$ *and the map* $H^1\Theta_Y \longrightarrow H^1\mathcal{N}_{D|Y}$ *is injective (e.g.* Y *is rigid).*

The situation becomes more complicated when we allow singularities of the branch divisor D. Then the double cover X is singular. We shall consider a resolution of singularities of X obtained by a special embedded resolution of D.

For any birational morphism $\sigma : \tilde{Y} \longrightarrow Y$ we have $\sigma^*D = \tilde{D} + \sum_j n_jE_j$ (where $\tilde{D}$ is the strict transform of D, E_j are the σ–exceptional divisors and $n_j \geq 0$). Therefore, the divisor

$$D^* = \tilde{D} + \sum_{2\not| n_j} E_j = \sigma^*D - 2\sum_j \left\lfloor \frac{n_j}{2} \right\rfloor E_j$$

is reduced and even. In fact, it is the only reduced and even divisor satisfying

$$\tilde{D} \leq D^* \leq \sigma^*D.$$

Let $\tilde{X} \xrightarrow{\tilde{\pi}} \tilde{Y}$ be the double cover branched along D^* defined by $\sigma^*\mathcal{L}\otimes\mathcal{O}_Y(-\sum_j\lfloor\frac{n_j}{2}\rfloor E_j)$. We can find a birational morphism $\tilde{X} \xrightarrow{\rho} X$ that fits into the following commutative diagram

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\rho} & X \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ \tilde{Y} & \xrightarrow{\sigma} & Y \end{array}$$

It follows from the Hironaka desingularization theorem that we can find a sequence of blow–ups with smooth centers $\sigma : \tilde{Y} \longrightarrow Y$ such that D^* is a smooth divisor. Obviously, such a sequence gives a resolution of singularities of the double cover.

Assume that $\sigma : \tilde{Y} \longrightarrow Y$ is a sequence $\sigma = \sigma_{n-1}\circ\cdots\circ\sigma_0$ of blow–ups $\sigma_i : Y_{i+1} \longrightarrow Y_i$ of smooth subvarieties $C_i \subset Y_i$ such that D^* is smooth, $Y_0 = Y$, $Y_n = \tilde{Y}$. Let m_i be the integer such that $D^*_{i+1} = \sigma_i^*D_i^* - m_iE_i$, where $E_i \subset Y_{i+1}$ is the exceptional divisor of σ_i.

Theorem 6.3 ([13, Thm. 4.1])**.** $H^1(\Theta_{\tilde{Y}}(\log D^*))$ *is isomorphic to the space of simultaneous deformations of* $D \subset Y$ *which have simultaneous resolution i.e. which can be lifted to deformations of* $C_i \subset D_i^* \subset Y_i$ *in such a way that the multiplicity of the deformation of* D_i^* *along the deformation of* C_i *is at least* m_i*.*

Definition 6.4. We call an infinitesimal deformation of D in Y *equisingular* if it satisfies the assertion of the above theorem.

Theorem 6.3 is particularly useful when we have an explicit description of infinitesimal deformations of Y, for instance when Y is rigid.

Corollary 6.5 ([13, Cor. 4.3])**.** *If the variety* Y *is rigid, then the space of equisingular deformations of* D *in* Y *is isomorphic to* $H^1(\Theta_{\tilde{Y}}(\log D^*))$.

We can compute $H^1(\Theta_{\tilde{Y}}(\log D^*))$ explicitely in local coordinates. Let $\mathcal{I}(C_i)$ stand for the ideal sheaf of C_i in Y_i, and let $\tilde{\mathcal{I}}_i^{m_i}$ be (for a nonnegative integer m_i) the push–forward $(\sigma_{i-1} \circ \cdots \circ \sigma_0)_*(\mathcal{I}(C_i)^{m_i})$ of the m_i–th power of $\mathcal{I}(C_i)$ to Y. Denote by $\mathcal{J}_i$ the image of the homomorphism $\Theta_{Y_i} \otimes \mathcal{O}_{D_i^*} \longrightarrow \mathcal{N}_{D_i^*|Y_i}$, and by $\tilde{\mathcal{J}}_i$ its pushforward $(\sigma_{i-1} \circ \cdots \circ \sigma_0)_*(\mathcal{J}_i)$ to Y. Let $\mathbf{J}$ stand for the image of the map $H^0(\Theta_Y) \longrightarrow H^0\mathcal{N}_{D|Y}$ induced by the exact sequence (2.4).

Theorem 6.6 ([13, Thm. 4.6])**.** *Under the above assumptions we have*

$$H^1(\Theta_{\tilde{Y}}(\log D^*)) \cong \bigcap_{i=0}^{n-1} \left(H^0\left(\left(\tilde{\mathcal{I}}_i^{m_i} \otimes \mathcal{N}_{D|Y} \right) + \tilde{\mathcal{J}}_i \right) \right) \Big/ \mathbf{J}.$$

In the most interesting case $Y = \mathbb{P}^N$ this formula can be written in a form more suitable for computations with a computer algebra system. Indeed, define the *equisingular ideal* of D in $\mathbb{P}^N$ (w.r.t. σ) as

$$\mathrm{I_{eq}}(D) = \bigcap_{i=0}^{n-1} \left(\mathrm{I}(\tilde{\mathrm{C}}_\mathrm{i})^{(\mathrm{m_i})} + \mathrm{J^i_F} \right),$$

where $\tilde{C}_i$ is the image of C_i in $\mathbb{P}^n$, and J^i_F is the homogeneous ideal associated to $\mathcal{J}_i$.

Theorem 6.7 ([13, Thm. 4.7])**.** *The space of equisingular deformations of D is isomorphic to the space of degree–d forms in the quotient of the equisingular ideal modulo the Jacobian ideal*

$$H^1(\Theta_{\tilde{Y}}(\log D^*)) \cong \left(\mathrm{I}_{eq}(D)/\mathrm{J}_F\right)_d.$$

The remaining part of the deformations space $H^1(\Theta_Y \otimes \mathcal{L}^{-1})$ coming from the deformations that fail to be a double cover is much more difficult to understand. However, in many situations it is easy to compute its dimension. There is a special case where the formula is particularly simple.

Proposition 6.8 ([13, Prop. 5.1])**.** *If $K_Y = \mathcal{L}^{-1}$ and $\sigma : \tilde{Y} \longrightarrow Y$ is a sequence of blow–ups satisfying the condition $\frac{1}{2}D^* + K_{\tilde{Y}} = \sigma^*(\frac{1}{2}D + K_Y)$, then we have the equality*

$$h^1(\Theta_{\tilde{Y}} \otimes \tilde{\mathcal{L}}^{-1}) = h^1(\Theta_Y \otimes \mathcal{L}^{-1}) + \sum_{\operatorname{codim} C_i = 2} h^0(K_{C_i}).$$

If $\dim Y = 3$, then the above assumptions correspond to a construction of a Calabi–Yau threefold. In this case the second summand coincides with the sum of genera of the blown–up (double and triple) curves.

If we specialize further to the case $Y = \mathbb{P}^3$ and $D = \sum_{i=1}^8 D_i$, where $D_1, \ldots, D_8$ are eigth planes satisfying the following conditions

- no six intersect,
- no four contain a common line,

the above construction gives a Calabi–Yau threefold called a double octic ([10, 34]). For a generic choice of the eight planes the singularities of the octic surface are given by 28 double lines with threefold intersections at 56 triple points. In order to obtain a Calabi–Yau smooth model it suffices to blow–up only the double lines, the triple points do not require any special treatment. At each triple point exactly one of the intersecting planes is blown–up (the one that does not contain the double line through that point that was blown–up first).

Consequently the resulting Calabi–Yau threefold is a double covering of the projective space $\mathbb{P}^3$ blown–up 28 times at a line branched along a disjoint sum of 8 planes blown–up 56 times at a point and so its Euler characteristic equals

$$2(4 + 28 \times 2) - (8 \times 3 + 56) = 40.$$

For special arrangements we have to take into account the number and types of singularities of D ([10, Thm. 2.1]). In this special case the equisingular ideals becomes

$$\mathrm{I}_{\mathrm{eq}} = \bigcap_{C} \left(\mathrm{I}_C^{mult_C D} + \mathrm{J}_{\mathrm{F}} \right),$$

the intersection being taken over all multiple curves and points of the arrangement D, and

$$\mathrm{J}_{\mathrm{F}} := \left(\frac{\partial F}{\partial z_0}, \dots, \frac{\partial F}{\partial z_3} \right)$$

is the Jacobian ideal of D.

7. Clemens' and Werner's defect formulae

The first examples of singular hypersurfaces are the ones that have ordinary double points (nodes) as the only singularities. We shall call such hypersurfaces *nodal*. A three-dimensional node admits two kinds of resolutions: the big one (blow–up of the singular point) and small resolutions (blow–ups along analytic smooth surfaces through the node in question). A small resolution replaces a singular point with a smooth rational curve. However, as we blow–up along a local analytic submanifold, the resulting manifold may fail to be projective (the delicate problem of an existence of a projective small resolution of a nodal variety is treated in details in Chapter III of Werner's thesis [48]). There are two different small resolutions of a node; each of which corresponds to a ruling of the projectivised tangent cone. A big resolution may be obtained from a small resolution by blowing–up the exceptional rational curve. In particular, the Hodge numbers of a small resolution are uniquely determined.

A double covering of a smooth algebraic variety branched along a nodal hypersurface is also nodal. In the seminal paper [6] Betti numbers of so-called double solids; i.e. double coverings X of $\mathbb{P}^3$ branched along a nodal surface D of an even degree d, are studied. Clemens proves that certain Betti numbers of a double solid depend not only on the number of nodes but on their position as well. The latter is encoded in the so–called defect. Clemens defines the defect of a nodal double solid as the difference between the second and the fourth Betti number of the singular threefold. Let $S := \operatorname{sing}(D)$ be the set of nodes. Moreover, let us put V to denote the vector space of degree–$(\frac{3}{2}d - 4)$ homogeneous polynomials on $\mathbb{P}^3$, and by V_S the subspace of V that consists of polynomials vanishing at S. Then (see [6]) one has the equality

$$\delta = \dim V_S - (\dim V - \mu), \qquad \text{where } \mu = \#S,$$

i.e. the defect equals the number of dependent conditions that vanishing on S imposes on the homogeneous polynomials of degree $(\frac{3}{2}d - 4)$.

The following formula for the Hodge numbers of the big resolution $\tilde{X}$ of the nodal double solid X is given in [6]:

$$\begin{aligned} h^{1,1}(\tilde{X}) &= 1+\mu+\delta, \\ h^{1,2}(\tilde{X}) &= \binom{3d/2-1}{3} - 4\binom{d/2}{3} - \mu + \delta. \end{aligned}$$

Clemens uses topological arguments in his proof. Using basic properties of logarithmic differential forms we can give a simple algebraic proof of the following direct generalization

Theorem 7.1. *Let Y be a smooth projective three–dimensional variety, and let $D \subset Y$ be an ample nodal hypersurface. Moreover, assume that D is even, i.e. there exists a line bundle $\mathcal{L}$ on Y such that $D = \mathcal{L}^{\otimes 2}$ in $\mathrm{Pic}(Y)$, and the following equality holds*

$$H^2(\Omega^1_Y \otimes \mathcal{L}^{-1}) = 0.$$

If X is the double covering of Y branched along D and defined by $\mathcal{L}$, and $\sigma : \tilde{X} \longrightarrow X$ is the big resolution, then one has the following formula

$$h^{1,1}(\tilde{X}) = h^{1,1}(Y) + \mu + \delta,$$

where

$$\delta = h^0(\mathcal{L}^{\otimes 3} \otimes K_Y \otimes \mathcal{I}_S) - (h^0(\mathcal{L}^{\otimes 3} \otimes K_Y) - \mu),$$

and $\mathcal{I}_S$ stands for the ideal of S.

Proof. Let $\pi : \tilde{X} \longrightarrow \tilde{Y}$ be the double covering of $\tilde{Y}$ branched along the strict transform $\tilde{D}$ of D. By [21, Lem. 3.16(d)] we have

$$\pi_* \Omega^1_{\tilde{X}} = \Omega^1_{\tilde{Y}} \oplus \Omega^1_{\tilde{Y}}(\log \tilde{D}) \otimes \tilde{\mathcal{L}}^{-1},$$

where $\tilde{\mathcal{L}} = \sigma^*\mathcal{L} \otimes \mathcal{O}_{\tilde{X}}(-E)$, E is the sum of exceptional divisors of σ. Since the map π is finite it suffices to prove

$$h^1(\Omega^1_{\tilde{Y}}(\log \tilde{D}) \otimes \tilde{\mathcal{L}}^{-1}) = \delta.$$

By the first exact sequence of (2.1) we have

$$0 \longrightarrow \Omega^1_{\tilde{Y}} \otimes \tilde{\mathcal{L}}^{-1} \longrightarrow \Omega^1_{\tilde{Y}}(\log \tilde{D}) \otimes \tilde{\mathcal{L}}^{-1} \longrightarrow \mathcal{O}_{\tilde{D}} \otimes \tilde{\mathcal{L}}^{-1} \longrightarrow 0.$$

Using Proposition 3.1 we show that $\sigma_*(\Omega^1_{\tilde{Y}} \otimes \tilde{\mathcal{L}}^{-1}) = \Omega^1_Y \otimes \mathcal{L}^{-1}$ and $R^i\sigma_*(\Omega^1_{\tilde{Y}} \otimes \tilde{\mathcal{L}}^{-1}) = 0$ for $i \geq 1$. By the (degenerate case of) Leray spectral sequence and Nakano vanishing $H^i(\Omega^1_{\tilde{Y}} \otimes \tilde{\mathcal{L}}^{-1}) = H^i(\Omega^1_Y \otimes \mathcal{L}^{-1}) = 0$, for $i = 1, 2$ and so it remains to show the equality

$$\delta = h^1(\mathcal{O}_{\tilde{D}} \otimes \tilde{\mathcal{L}}^{-1}). \tag{7.1}$$

Since $\tilde{\mathcal{L}}^{-1} = \sigma^*\mathcal{L}^{-1} \otimes \mathcal{O}_{\tilde{X}}(E)$ from Proposition 3.1 and projection formula we get

$$\sigma_*\tilde{\mathcal{L}}^{-1} = \mathcal{L}^{-1} \qquad \text{and} \qquad R^i\sigma_*\tilde{\mathcal{L}}^{-1} = 0, \quad \text{for} \quad i > 0.$$

So by the Leray spectral sequence $H^i(\tilde{\mathcal{L}}^{-1}) = H^i(\mathcal{L}^{-1}) = 0$ for $i = 1, 2$. Using the cohomology exact sequence associated to

$$0 \longrightarrow \tilde{\mathcal{L}}^{-3} \longrightarrow \tilde{\mathcal{L}}^{-1} \longrightarrow \mathcal{O}_{\tilde{D}} \otimes \tilde{\mathcal{L}}^{-1} \longrightarrow 0$$

we get

$$H^1(\mathcal{O}_{\tilde{D}} \otimes \tilde{\mathcal{L}}^{-1}) \cong H^2(\tilde{\mathcal{L}}^{-3}).$$

Again, using Proposition 3.1 and projection formula we get

$$\sigma_*\tilde{\mathcal{L}}^{-3} = \mathcal{L}^{-3}, \qquad R^1\sigma_*\tilde{\mathcal{L}}^{-3} = 0, \qquad R^2\sigma_*\tilde{\mathcal{L}}^{-3} = \mathcal{L}^{-3}\otimes\mathcal{O}_S \quad \text{(a sky–scraper sheaf).}$$

Now, the Leray spectral sequence implies

$$h^2\tilde{\mathcal{L}}^{-3} = \mu - (h^3(\mathcal{L}^{-3}) - h^3(\tilde{\mathcal{L}}^3)).$$

By Serre duality

$$h^3(\mathcal{L}^{-3}) = h^0(\mathcal{L}^{\otimes 3}\otimes K_Y)$$

while

$$h^3(\tilde{\mathcal{L}}^{-3}) = h^0(\tilde{\mathcal{L}}^{\otimes 3}\otimes K_{\tilde{Y}}) = h^0(\sigma^*(\mathcal{L}^{\otimes 3}\otimes K_Y)\otimes\mathcal{O}_{\tilde{Y}}(-E)) = h^0(\mathcal{L}^{\otimes 3}\otimes K_Y\otimes\mathcal{I}_S)$$

and the theorem follows. □

Nodal hypersurfaces were studied by J. Werner in his PhD thesis ([48]). Using some topological arguments he was able to deduce an analogue of Clemens' defect formula for nodal hypersurfaces in the projective space $\mathbb{P}^4(\mathbb{C})$ (see Example 8.1). Using a similar line of arguments as in the proof of Thm 7.1 one can obtain the following generalization of Werner's result.

We have the following formulae for the Hodge numbers of the big resolution of a nodal threefold hypersurface.

Theorem 7.2 ([13, Thm. 2]). *Let Y be a nodal hypersurface in a smooth projective four-dimensional manifold satisfying the conditions*

A1: *the line bundle $\mathcal{M} := \mathcal{O}_X(Y)$ is ample,*
A2: $H^2\Omega^1_X = 0$,
A3: $H^3(\Omega^1_X\otimes\mathcal{M}^{-1}) = 0$.

Then

$$\begin{aligned} h^1\Omega^1_{\tilde{Y}} &= h^1\Omega^1_X + \delta \\ h^2\Omega^1_{\tilde{Y}} &= h^0(\mathcal{O}_X(2Y + K_X)) + h^3\mathcal{O}_X - h^0(\mathcal{L}_0\otimes K_X) - \\ &\quad - h^3\Omega^1_X - h^4(\Omega^1_X\otimes\mathcal{L}_0^{-1}) - \mu + \delta \end{aligned}$$

where μ is the number of nodes and δ is a non–negative integer called defect equal to the number of dependent equations that vanishing at nodes of S imposes on the global sections of the line bundle $(\mathcal{M}^{\otimes 2}\otimes K_X)$ on X.

One can show, that if $\tilde{Y}$ (resp. $\hat{Y}$) is the big (resp. a small resolution) of a nodal threefold Y with μ nodes, then their Hodge numbers are related by the equalities

$$\begin{aligned} h^{1,1}(\tilde{Y}) &= h^{1,1}(\hat{Y}) + \mu\,, \\ h^{1,2}(\tilde{Y}) &= h^{1,2}(\hat{Y})\,. \end{aligned}$$

In [48] Werner gave also a sufficient and necessary condition for projectivity of a given small resolution. A necessary condition for a nodal threefold hypersurfaces to admit a projective small resolution is the existence of a Weil but not $\mathbb{Q}$–Cartier divisor. Recall that a variety X is called $\mathbb{Q}$–factorial if every Weil divisor on X is $\mathbb{Q}$–Cartier. An easy observation is that a nodal hypersurface in the projective space $\mathbb{P}^4$ (or a nodal double covering of the projective space $\mathbb{P}^3$) is $\mathbb{Q}$–factorial iff its defect is zero.

The simplest example of a nodal hypersurface of degree d in $\mathbb{P}^4$ is given by a degree–d polynomial of the form

$$p_k q_{d-k} + r_l s_{d-l},$$

where $p_k, q_{d-k}, r_l, s_{d-l}$ are generic homogeneous polynomials of degrees $k, d-k, l, d-l$ respectively $(1 \le k, l \le d-1)$. One easily verifies that $F = 0$ is degree-d nodal variety with exactly $k(d-k)l(d-l)$ nodes given by $p_k = q_{d-k} = r_l = s_{d-l} = 0$. As the Hilbert function of the singular locus of X equals

$$\frac{(1-t^k)(1-t^l)(1-t^{d-k})(1-t^{d-l})}{(1-t)^5},$$

a simple computation shows that the defect of Y is always one.

The above construction is a special case of the following

Theorem 7.3 ([29, Thm. 2.1])**.** *Let $D \subset \mathbb{P}^N$ be a smooth surface that is a scheme–theoretic base locus of a linear system of hypersurfaces of degree d. Then the generic complete intersection Y of $N-3$ hypersurfaces of degree d containing D is a nodal threefold.*

The above examples cover the case of complete intersection surfaces in $\mathbb{P}^4$. In the case when D is a plane, we obtain a degree–d non–factorial hypersurface with $(d-1)^2$ nodes. Cheltsov ([5]) proved that the above number of nodes is minimal possible: every degree–d nodal hypersurface in $\mathbb{P}^4$ with at most $(d-1)^2 - 1$ singular points is factorial.

For a general case we can use the last formula from Thm. 5.2 to compute the number of nodes and obtain

$$\mu = c_2(\mathcal{N}) + Y^2 D - Y c_1(\mathcal{N}).$$

If we consider degree–d surfaces, where $d \le 6$, that are considered in [36], we get ten non $\mathbb{Q}$—factorial quintics with the following number of nodes

	$\deg(D)$	$K_D H$	μ
$\mathbb{P}^2$	1	-3	16
$\mathbb{P}^1 \times X^1$	2	-4	24
$d_{1,3}$	3	-3	24
$\mathbb{F}_1$	3	-5	34
$D_{2,2}$	4	-4	36
Veronese	4	-6	46
$\mathbb{P}_C(E)$	5	-5	50
Castelnuovo	5	-3	40
$D_{2,3}$	6	0	36
Bordiga	6	-2	46

For a more detailed account of the above construction see [12].

The question which number of nodes can be realized on a hypersuface of degree d is open. Even the maximal number $\mu_n(d)$ of nodes on a degree–d hypersurface in $\mathbb{P}^n$ remains unknown. The best known upper bound is Varchenko's spectral bound

$$\mu_n(d) \le \mathrm{Ar}_n(D),$$

where $\mathrm{Ar}_n(D)$ is the Arnold number

$$\mathrm{Ar}_n(D) := \#\{(k_0, \ldots, k_n) : k_i \in \{1, \ldots, d-1\}, \sum_{i=0}^{n} k_i = \lfloor \frac{nd}{2} \rfloor + 1\}.$$

This bound is sharp in the case of a cubic. For surfaces in $\mathbb{P}^3$ the exact values are known for $d \leq 6$

$$\mu_3(3) = 4,\ \mu_3(4) = 16,\ \mu_3(5) = 31,\ \mu_3(6) = 65.$$

The upper bound for the octic surface is $\mu_3(8) \leq 174$, whereas the best known example has 168 nodes and was constructed by Endrass ([20]). In higher dimensions much less is known. For a quintic hypersurface in $\mathbb{P}^4$ there is the upper bound $\mu_4(5) \leq 135$, and the best known example is due to van Straten ([47]). It is a quintic with 130 nodes.

The pairs of integers μ, δ that can be realized as a number of nodes and the defect of a nodal quintic threefold or a nodal double octic were studied by Borcea ([3]). He uses deformation arguments to prove the following result.

Theorem 7.4 ([3])**.** *Let X be a nodal quintic threefold (resp. a double solid ramified along a nodal octic surface) with ν nodes and defect δ. Then for all but δ integers $d \in \{0, 1, \ldots, \mu\}$, there exists a nodal quintic (resp. nodal double octic) with d nodes.*

The defect of the Endrass double octic is 19, so all integers smaller than 169 with at most 19 exceptions are realized as the number of nodes of an octic surface. In his thesis Werner computed defect for some examples getting the pairs $(108, 0)$, $(123, 3)$, $(136, 7)$ and $(144, 9)$, so all integers up–to 108 can be realized, there are at most three gaps up–to 123 etc.

The defect of van Straten's example is 29 so there are at most 29 gaps in the region below 130. There exist examples with $\mu = 50, \delta = 1$ (degeneracy locus of a generic (5×5) matrix of linear forms in $x_0, \ldots, x_4$) and $\mu = 100, \delta = 3$ (dependency locus of two generic sections in the Horrocks–Mumford bundle). Consequently there is at most one gap up–to 50 and at most 3 gaps up–to 100.

8. The case of A-D-E singularities

In this section we deal with hypersurfaces with certain higher singularities. Let Y be a hypersurface in a smooth four-dimensional projective variety X. Moreover, we assume that $\mathrm{sing}(Y)$ consists of A-D-E points. In general, A-D-E points can be defined in various ways (see [19]). The following definitions/characterizations of this class are of use for us:

According to [19, Char. C 9], a point $P \in \mathrm{sing}(Y)$ is A-D-E iff we can choose (analytic) coordinates $x_{1,P}, \ldots, x_{4,P}$ centered at P such that the germ of Y at P is given by the semiquasihomogenous equation

$$\mathfrak{n}(x_{1,P}, x_{2,P}, x_{3,P}) + x_{4,P}^2 + F(x_{1,P}, x_{2,P}, x_{3,P}, x_{4,P}) = 0\,, \tag{8.1}$$

where $\mathfrak{n}(x_{1,P}, x_{2,P}, x_{3,P})$ is the normal form of the equation of a two-dimensional A-D-E singularity and $F(x_{1,P}, x_{2,P}, x_{3,P}, x_{4,P})$ is a polynomial of order strictly greater than 1 with respect to the weights $\mathrm{w}_{\mathfrak{n}}(x_{1,P})$, $\mathrm{w}_{\mathfrak{n}}(x_{2,P})$, $\mathrm{w}_{\mathfrak{n}}(x_{3,P})$, $\mathrm{w}_{\mathfrak{n}}(x_{4,P})$ given in the table below:

(8.2)

	$\mathfrak{n}(x_1, x_2, x_3)$	$(\mathrm{w}_{\mathfrak{n}}(x_1), \ldots, \mathrm{w}_{\mathfrak{n}}(x_4))$
A_m, $m \geq 1$	$x_1^{m+1} + x_2^2 + x_3^2$	$(\frac{1}{m+1}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$
D_m, $m \geq 4$	$x_1 \cdot (x_2^2 + x_1^{m-2}) + x_3^2$	$(\frac{1}{m-1}, \frac{m-2}{2(m-1)}, \frac{1}{2}, \frac{1}{2})$
E_6	$x_1^4 + x_2^3 + x_3^2$	$(\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$
E_7	$x_1^3 \cdot x_2 + x_2^3 + x_3^2$	$(\frac{2}{9}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$
E_8	$x_1^5 + x_2^3 + x_3^2$	$(\frac{1}{5}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2})$

In particular the singularities of Y are absolutely isolated, i.e. we have the big resolution $\sigma : \tilde{Y} \to Y$ of the threefold Y ([19, p.137]) obtained as the composition

$$\sigma = \sigma_n \circ \ldots \circ \sigma_1 \, : \, \tilde{Y} \to Y =: \tilde{Y}^0 \, , \tag{8.3}$$

where $\tilde{Y} := \tilde{Y}^n$ is smooth and $\sigma_j : \tilde{Y}^j \to \tilde{Y}^{j-1}$, for $j = 1, \ldots, n$, is the blow-up with the center $\mathrm{sing}(\tilde{Y}^{j-1}) \neq \emptyset$, that consists of isolated points. By direct computation, the singularities of $\tilde{Y}^j$ are double points for each $j \leq n-1$. The number of singular points (different from P) which are infinitely near P is as follows

(8.4)

A_m, $m \geq 1$	D_m, $m \geq 4$	E_6	E_7	E_8
$\lceil m/2 \rceil - 1$	$2 \cdot \lfloor m/2 \rfloor - 1$	3	6	7

By [46, Thm 1] the above property of the big resolution characterizes A-D-E singularities: if $P \in Y$ is an absolutely isolated double point (i.e. P is an isolated double point and all singularities infinitely near P are isolated double points), then it is an A-D-E singularity.

Let $\tilde{X}^0 := X$ and let $\tilde{X}^j$ stand for the fourfold obtained from $\tilde{X}^{j-1}$ by blowing it up along $\mathrm{sing}(\tilde{Y}^{j-1})$, $j = 1, \ldots, n$. We put $\tilde{X} := \tilde{X}^n$. The composition of the blow-ups in question is denoted by $\sigma : \tilde{X} \to X$. Moreover, let $\sum_l k_l E_l := K_{\tilde{X}/X}$ and let $\pi^* Y = \tilde{Y} + \sum_l m_l E_l$, where E_l are (reduced) components of the exceptional locus of σ. In order to generalize the notion of defect one defines the following sheaf ([11, Def.2.1])

$$\mathfrak{I}_Y := \sigma_* (\mathcal{O}_{\tilde{X}}((k_l - 2m_l)E_l)) \tag{8.5}$$

Then the *defect* of the hypersurface Y is defined as the integer

$$\delta_Y = h^0(K_X(2Y) \otimes \mathfrak{I}_Y) - (h^0(K_X(2Y)) - \mu_Y), \tag{8.6}$$

where μ_Y stands for the number of singularities and infinitely near singularities of Y. The motivation for the choice of the sheaf $\mathfrak{I}_Y$ in the above definition becomes clear, if one applies the projection formula to the map σ and the sheaf

$$K_{\tilde{X}} + 2\tilde{Y} \sim \sigma^*(K_X + 2Y) + \sum (k_l - 2m_l) E_l \, ,$$

to obtain the equality $h^0(K_{\tilde{X}}(2\tilde{Y})) = h^0(K_X(2Y) \otimes \mathfrak{I}_Y)$.

One can follow a more direct approach (see [43, § 3]) and consider the space $\mathfrak{V}_Y$ of sections $H \in H^0(K_X(2Y) \otimes \mathcal{I}_{\mathrm{sing}(Y)})$ such that

- if $P \in Y$ is an A_m point, with $m \geq 1$ then $\frac{\partial^j H}{\partial x_{1,P}^j}(P) = 0$ for $j \leq \lceil m/2 \rceil - 1$,
- for every D_m singularity of $P \in Y$, where $m \geq 4$, one has
$$\frac{\partial H}{\partial x_{2,P}}(P) = \frac{\partial^j H}{\partial x_{1,P}^j}(P) = 0 \text{ for } j \leq \lfloor m/2 \rfloor - 1\,,$$
- if P is an E_m point, where $m = 6, 7, 8$, then $\frac{\partial H}{\partial x_{2,P}}(P) = \frac{\partial^j H}{\partial x_{1,P}^j}(P) = 0$ for $j \leq m - 5$,

where $x_{1,P}, \ldots, x_{4,P}$ are analytic local coordinates centered at the point P such that the hypersurface Y is given near P by the semiquasihomogenous equation (8.1). By [43, Lemma 3.3] (see also [ibid., (4.2)]), the defect of Y can be expressed as
$$\delta_Y = \dim(\mathfrak{V}_Y) - h^0(K_X(2Y)) + \mu_Y.$$
In particular (for A-D-E singularities) we have $k_l \leq 2m_l$ and the sheaf $\mathfrak{I}_Y$ is indeed a sheaf of ideals.

Using the properties of logarithmic differential forms (see Sect. 2 and Sect. 3) one obtains the following formulae for Hodge numbers:

Theorem 8.1. [11, Thm 2.2] *Let X be a smooth projective fourfold, and let $Y \subset X$ be a hypersurface with A-D-E singularities. If*
$$h^2(\Omega^1_X) = h^3(\Omega^1_X(-Y)) = h^3(\mathcal{O}_X(-Y)) = h^2(\mathcal{O}_X(-Y)) = 0\,,$$
then
$$\begin{aligned} h^{1,1}(\tilde{Y}) &= h^{1,1}(X) + (\chi(\Omega^1_X(-Y)) - h^4(\Omega^1_X(-Y))) - (\chi(\mathcal{O}_X(-2Y)) - h^4(\mathcal{O}_X(-2Y))) + \\ &\quad -2h^1(\mathcal{O}_X(-Y)) + \mu_Y + \delta_Y\,, \\ h^{1,2}(\tilde{Y}) &= h^{4,1}(X) + h^{0,2}(X) + h^0(K_X(2Y)) - h^{3,1}(X) - h^4(\Omega^1_X(-Y)) - h^0(K_X(Y)) + \\ &\quad -\mu_Y + \delta_Y, \end{aligned}$$
where δ_Y (resp. μ_Y) is the defect (resp. the number of singularities and infinitely near singularities) of Y.

Let us comment on the proof of the above result. The reasoning consists of three steps:
(*Step 1:*) the Serre duality is applied to study Ω^3_X (and its twists) instead of Ω^1_X (resp. its twists),
(*Step 2:*) one uses the Leray spectral sequence to compare the cohomologies of various twisted sheaves of differentials on X and on the blow-up $\tilde{X}$,
(*Step 3:*) the Hodge numbers of the big resolution $\tilde{Y}$ are computed via Poincare residue on the blow-up $\tilde{X}$.

The assumption on singularities of Y implies vanishing of certain higher direct image sheaves, which in particular yields the equality
$$h^1(\Omega^3_{\tilde{X}}(\tilde{Y})) = h^1(\Omega^3_X(Y)).$$
Once one knows that $h^1(\Omega^3_{\tilde{X}}(\tilde{Y}))$, $h^2(\Omega^3_{\tilde{X}})$ vanish, one can see that the exact sequence of cohomology associated to the Poincare residue (resp. to the resolution of the sheaf of logarithmic differentials – see (2.1)) breaks into shorter exact sequences. The advantage of replacing one–forms with three–forms becomes apparent when we recall the resolution (2.3) of the sheaf of logarithmic three–forms. Details of the proof can be found in [11]. It should be

pointed out, however, that the above formulae cease to hold once we weaken the assumptions (see [11, Example 3.6]).

In order to see how Thm 8.1 works, let us consider the following example

Example 8.1. Let $Y \subset \mathbb{P}_4$ be a degree-d hypersurface with A-D-E singularities, where $d \geq 3$. Recall that $\mathfrak{V}_Y$ was defined (see the paragraph preceeding 8.1) as the space of degree-$(2d-5)$ polynomials that vanish along $\mathrm{sing}(Y)$ and such that some of their partial derivatives vanish in every A_m (resp. D_m, E_m) point of Y. Thm 8.1 implies the equalities

$$\begin{aligned} h^{1,1}(\tilde{Y}) &= 1 + 2 \cdot \mu_Y + \dim(\mathfrak{V}_Y) - \binom{2d-1}{4}, \\ h^{1,2}(\tilde{Y}) &= \dim(\mathfrak{V}_Y) - 5 \cdot \binom{d}{4}. \end{aligned}$$

In particular, for a nodal hypersurface we regain Werner's formula [48, Satz on p. 27].

In general, big (divisorial) resolutions perturb the canonical class: if one starts from a (singular) hypersurface with trivial canonical class (e.g. Calabi-Yau threefold), one obtains a smooth threefold without that property. That is why small resolutions are of interest. As we already mentioned in Sect. 7 small resolutions exist for three–dimensional nodal hypersurfaces. One can show that analogous resolutions can be also constructed for certain higher three-dimensional singularities. If we suppose that there exists a proper holomorphic map $\hat{\sigma} : \hat{Y} \to Y$ such that $\hat{Y}$ is smooth , $\hat{\sigma}|_{\hat{Y} \setminus \hat{\sigma}^{-1}(\mathrm{sing}(Y))}$ is an isomorphism onto the image and the exceptional set

$$\hat{E} := \hat{\sigma}^{-1}(\mathrm{sing}(Y))$$

is a curve, then $\hat{\sigma}$ (and sometimes $\hat{Y}$) is called a small resolution of Y. By [22, Thm 1.3] (see also [45]) it suffices to assume that Y has Gorenstein singularities to show that the exceptional set $\hat{E}$ consists of smooth rational curves meeting transversally. Let us put $\tilde{E}$ to denote the exceptional divisor of the big resolution. Then, the following simple lemma can be applied to use Thm 8.1 to compute Hodge numbers of small resolutions

Proposition 8.2. [43, Prop. 6.1] *If $h^1(\mathcal{O}_Y) = 0$, $h^3(\tilde{E}, \mathbb{C}) = 0$ and $\hat{Y}$ is Kähler, then*

$$h^{2,2}(\tilde{Y}) = h^{2,2}(\hat{Y}) + h^4(\tilde{E}, \mathbb{C}).$$

9. Further generalizations

A large class of ambient varieties X where the result from the previous section can be applied consists of toric varieties. Indeed one has Bott-type formulae (see e.g. [32]) for cohomology of various twisted sheaves of differentials, even if the considered ambient space is singular. Having that in mind we assume now that X is a *four-dimensional normal complex variety*, so the canonical (Weil) divisor K_X is well-defined (up to the linear equivalence). We have one-to-one correspondence between the linear equivalence classes of Weil divisors and isomorphism classes of rank-1 reflexive sheaves on X:

$$D \to \mathcal{O}_X(D).$$

We put $\overline{\Omega}^3_X := \mathrm{j}_* \Omega^3_{\mathrm{reg}(X)}$, where $\mathrm{j} : \mathrm{reg}(X) \to X$ stands for the inclusion, to denote the Zariski sheaf of germs of 3-forms. In order to obtain a direct generalization of Thm 8.1 one has to assume that $Y \subset X$ is a a hypersurface with A-D-E singularities such that

$$\operatorname{sing}(X) \cap Y = \emptyset\,. \tag{9.1}$$

In particular, the above assumptions assure that the defect δ_Y of Y is well-defined. One has the following theorem:

Theorem 9.1. [11, Thm 3.2] *Let $Y \subset X$ satisfy the assumptions of this section. If*

$$h^1(\mathcal{O}_X(Y+K_X)) = h^2(\overline{\Omega}^3_X) = h^1(\overline{\Omega}^3_X(Y)) = 0 \quad \text{and} \quad h^2(\mathcal{O}_X(-Y)) = h^3(\mathcal{O}_X(-Y)) = 0\,,$$

then the following equalities hold

$$\begin{aligned} h^{1,1}(\tilde{Y}) = {} & h^3(\overline{\Omega}^3_X) + (\chi(\overline{\Omega}^3_X(Y)) - h^0(\overline{\Omega}^3_X(Y))) + (\chi(\mathcal{O}_X(Y+K_X)) - h^0(\mathcal{O}_X(Y+K_X))) + \\ & + h^1(\mathcal{O}_X) - h^4(\overline{\Omega}^3_X) - (\chi(\mathcal{O}_X(2Y+K_X)) - h^0(\mathcal{O}_X(2Y+K_X))) - h^1(\mathcal{O}_X(-Y)) \\ & + \mu_Y + \delta_Y\,, \\ h^{1,2}(\tilde{Y}) = {} & h^0(\overline{\Omega}^3_X) + h^2(\mathcal{O}_X) + h^0(\mathcal{O}_X(2Y+K_X)) - h^1(\overline{\Omega}^3_X) - h^0(\mathcal{O}_X(Y+K_X)) + \\ & - h^0(\overline{\Omega}^3_X(Y)) - \mu_Y + \delta_Y\,. \end{aligned}$$

Here, we no longer assume X to be Cohen-Macaulay, so we cannot apply the Serre duality. The proof consists of steps 2, 3 of the proof of Thm 8.1. Indeed, the assumption (9.1) enables us to work with logarithmic 3-forms as in the smooth case.
Again, as in the previous section the first three vanishings are essential. The others are needed to control the Hodge numbers $h^{1,0}(\tilde{Y})$, $h^{2,0}(\tilde{Y})$.

In particular, in toric case, many summands in Thm 9.1 vanish and one arrives at the following result

Corollary 9.2. [11, Cor 3.4] *Let X be a complete toric fourfold, and let $Y \subset X$ be a hypersurface with A-D-E singularities such that* $\operatorname{sing}(X) \cap Y = \emptyset$. *If $\mathcal{O}_X(Y)$ is ample, then*

$$\begin{aligned} h^{1,1}(\tilde{Y}) &= h^3(\overline{\Omega}^3_X) + \mu_Y + \delta_Y\,, \\ h^{1,2}(\tilde{Y}) &= h^0(\mathcal{O}_X(2Y+K_X)) - h^0(\mathcal{O}_X(Y+K_X)) - h^0(\overline{\Omega}^3_X(Y)) - \mu_Y + \delta_Y\,. \end{aligned}$$

It is an interesting exercise to see what generalizations of the classical Clemens formula for double solids one can derive from Cor 9.2. Such generalizations can be found in [43, § 5].

In view of recent progress concerning study of behaviour of reflexive differential forms on singular varieties (see e.g. [25]), one should ask what is natural set-up for generalization of Thm 9.1. In particular, it seems natural to ask to what extent the assumption (9.1) can be weakened. In general, a residue map does not have to exist (see [25, § 11.B]). However, [ibid., Thm 11.7] suggests that a generalization (possibly with extra correction summands) can be obtained provided (X, Y) is a dlt pair.

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