A NOTE ON THE CHOW GROUPS OF PROJECTIVE DETERMINANTAL VARIETIES

(Appendix to "A cascade of determinantal Calabi-Yau threefolds"
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Abstract. We investigate the Chow groups of projective determinantal varieties and those of their strata of matrices of fixed rank, using Chern class computations.

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1. Introduction

In the present note we shall consider the following types of determinantal varieties.
(i) (generic) Let W,V be two vector spaces over an arbitrary field K with $m = \dim W \geq n = \dim V$. For $r \geq 0$, set

$$D_r = D_r(\varphi) = \{x \in P : \text{rank } \varphi(x) \leq r\}\,, \tag{1.1}$$

where $\varphi : W_P \longrightarrow V_P \otimes \mathcal{O}(1)$ is the canonical morphism on $P = \mathbb{P}(\mathrm{Hom}(W,V))$.
(ii) (symmetric) Take the following specialization of (i): let $m = n$, $W = V^*$, $P = \mathbb{P}(\mathrm{Sym}^2(V))$, and $\varphi : V_P^* \longrightarrow V_P \otimes \mathcal{O}(1)$ be the canonical symmetric morphism on P. Define D_r by (1.1).
(iii) (partially symmetric) Consider the following specialization of (i): let $m > n$, $W^* \twoheadrightarrow V$, $P = \mathbb{P}(W^* \vee V)$ (in the notation of [LP]), and let

$$\varphi : W_P \longrightarrow V_P \otimes \mathcal{O}(1)$$

be the canonical partially symmetric morphism on P. Define D_r by (1.1). (One can also, for even r, consider the skew-symmetric analogs of (ii) and (iii).)
In all cases (i), (ii) and (iii), we get a sequence of *projective determinantal varieties*

$$\emptyset = D_0 \subset D_1 \subset D_2 \subset \cdots \subset D_{n-1} \subset D_n = D_{n+1} = \cdots .$$

The scheme D_r is defined locally by the vanishing of $(r+1) \times (r+1)$ minors of φ. The codimensions of the determinantal varieties D_r in the respective cases are: (i) $(m-r)(n-r)$; (ii) $(n-r)(n-r+1)/2$; (iii) $(m-n)(n-r) + (n-r)(n-r+1)/2$.

In the present note, we compute $A^1(D_r)$ for the above determinantal varieties, getting the answers: $\mathbb{Z}\oplus\mathbb{Z}$ in cases (i) and (iii), and $\mathbb{Z}$ in case (ii). We also discuss generators of the Chow groups $A_*(D_r \setminus D_{r-1})$ in case (i); for $r = 1$ and $r = n-1$, we give some linearly independent generators.

Background. The content of this note was obtained in the late 80's, and has not been written up to now. Due to a recent ask of G. and M. Kapustka, we have decided to write this material up because it is needed in their research.

In this note, we shall use notation, conventions, and some results from [P], [LP]. Our basic reference for Chow groups is [F].

2. $D_r \setminus D_{r-1}$ as fiber bundles

In this section, we follow basically [P].

(i) For $f \in \mathrm{Hom}(W,V)$, we set $K_f = \mathrm{Ker}(f)$, $C_f = \mathrm{Coker}(f)$. When f varies in $D_r \setminus D_{r-1}$, we get the vector bundles K and C of ranks $m-r$ and $n-r$ on $D_r \setminus D_{r-1}$. We consider the fibration

$$D_r \setminus D_{r-1} \longrightarrow G_{m-r}(W) \times G_r(V)$$

given by $f \mapsto (K_f,\ C_f)$. Its fiber is equal to the space of nonsingular $r \times r$ matrices over K. More explicitly, let

$$P' = \mathbb{P}(\mathrm{Hom}(Q_W,R_V)) \longrightarrow G_{m-r}(W) \times G_r(V)\,.$$

The bundle Q_W is the pullback on $G_{m-r}(W) \times G_r(V)$ of the tautological quotient rank r bundle on $G_{m-r}(W)$. Moreover, the bundle R_V is the pullback on $G_{m-r}(W) \times G_r(V)$ of the tautological subbundle on $G_r(V)$.

On P', there is the tautological morphism

$$\varphi' : (Q_W)_{P'} \longrightarrow (R_V)_{P'} \otimes \mathcal{O}_{P'}(1)\,,$$

and we have

$$P' \setminus D_{r-1}(\varphi') \cong D_r \setminus D_{r-1}\,. \tag{2.1}$$

(ii) For symmetric $f \in \mathrm{Hom}(V^*, V)$, we have $K_f \cong C_f^*$. We consider the fibration

$$D_r \setminus D_{r-1} \longrightarrow G_r(V)$$

given by $f \mapsto C_f$. Its fiber is equal to the space of nonsingular symmetric $r \times r$ matrices. To be more explicit, let

$$P' = \mathbb{P}(\mathrm{Sym}^2(R)) \longrightarrow G_r(V)\,,$$

where R is the tautological subbundle on $G_r(V)$. On P', there is the tautological symmetric morphism

$$\varphi' : R^*_{P'} \longrightarrow R_{P'} \otimes \mathcal{O}_{P'}(1)\,,$$

and we have $P' \setminus D_{r-1}(\varphi') \cong D_r \setminus D_{r-1}$.

(iii) For a partially symmetric $f \in \mathrm{Hom}(W, V)$, we have $K_f^* \twoheadrightarrow C_f$. Let Fl denote the flag variety parametrizing the pairs (A, B), where A is an

$(m-r)$-dimensional quotient of W^*, B is an $(n-r)$-dimensional quotient of V and we have $A \twoheadrightarrow B$. We consider the fibration

$$D_r \setminus D_{r-1} \longrightarrow Fl$$

given by $f \mapsto (K_f^*, C_f)$. Its fiber is equal to the space of nonsingular $r \times r$ symmetric matrices. More explicitly, let

$$P' = \mathbb{P}(\mathrm{Sym}^2(R)) \longrightarrow Fl\,,$$

The bundle R is here the tautological rank r subbundle on Fl. On P', there is the tautological symmetric morphism

$$\varphi' : R_{P'}^* \longrightarrow R_{P'} \otimes \mathcal{O}_{P'}(1)\,,$$

and we have $P' \setminus D_{r-1}(\varphi') \cong D_r \setminus D_{r-1}$.

3. Computations of $A^1(D_r)$

Let i' denote the embedding $D_{r-1}(\varphi')) \to P'$.

Lemma 3.1. *In each case (i), (ii) and (iii), we have the following exact sequence of the Chow groups:*

$$A_*(D_{r-1}(\varphi')) \xrightarrow{i'_*} A_*(P') \longrightarrow A_*(D_r \setminus D_{r-1}) \longrightarrow 0\,. \tag{3.1}$$

This follows by combining (2.1) and its analogues with [F], Sect.1.8 applied to the embedding $D_{r-1} \subset D_r$.

With the help of the Schur S- and Q-functions (cf., e.g., [P]), we now record

Lemma 3.2. *In case (i), the image* $Im(i'_*)$ *is generated by*

$$s_I(Q) - s_I(R \otimes L)\,,$$

where

$$Q = (Q_W)_{P'}, \quad R = (R_V)_{P'}, \quad L = \mathcal{O}_{P'}(1)\,,$$

and I runs over all partitions of positive weight.

In cases (ii) and (iii), by putting M to be the formal square root of L, the image $Im(i'_*)$ *is generated by*

$$Q_I(R \otimes M)\,,$$

where R denotes the pullback to P' of the corresponding tautological rank r subbundle (on $G_r(V)$ or Fl), and and I runs over all (strict) partitions of positive weight.

This follows from [P], Corollary 3.13 and its symmetric analog established also in [P].

Proposition 3.1. *In cases (i) and (iii), we have* $A^1(D_r) \cong \mathbb{Z} \oplus \mathbb{Z}$ *for any* $r \geq 1$. *In case (ii), we have* $A^1(D_r) \cong \mathbb{Z}$ *for any* $r \geq 1$.

Proof. Since $\mathrm{codim}(D_{r-1}, D_r) \geq 2$ for $r \geq 1$, it suffices to prove the same assertions for $A^1(D_r \setminus D_{r-1})$ instead of $A^1(D_r)$. Set, in all three cases, $h = c_1(L)$.
(i) By Lemmas 3.1 and 3.2 we see that $A^1(D_r \setminus D_{r-1})$ is generated (over $\mathbb{Z}$) by $s_1(Q)$, $s_1(R)$, and h, modulo the following single relation:

$$s_1(Q) = s_1(R \otimes L) = s_1(R) + h\,.$$

Thus the assertion follows.
(ii) We see that $A^1(D_r \setminus D_{r-1})$ is generated by $s_1(R)$ and h, modulo the following single relation:

$$Q_1(R \otimes M) = 2(s_1(R) + s_1(M)) = 2s_1(R) + h = 0\,,$$

which implies the assertion.
(iii) Since Fl is a Grassmann bundle on a Grassmannian, we have

$$A^1(Fl) \cong \mathbb{Z} \oplus \mathbb{Z} = \mathbb{Z}s_1(R) \oplus \mathbb{Z}x\,,$$

for some x. We see that $A^1(D_r \setminus D_{r-1})$ is generated by $s_1(R)$, x and h, modulo the following single relation:

$$Q_1(R \otimes M) = 2s_1(R) + h = 0\,.$$

Hence the assertion follows. □

Similarly, one shows that the Chow groups $A^1(D_r)$ (r even) of skew-symmetric and partially skew-symmetric projective determinantal varieties are of rank 1 and 2, respectively.

4. Remarks on other Chow groups of $D_r \setminus D_{r-1}$

We work here in the generic case (i).

Proposition 4.1. *For $r \geq 1$, we have the following inequalities:*

$$\binom{n}{r} \leq \mathit{rank}\ A_*(D_r \setminus D_{r-1}) \leq \binom{n}{r}(m - r + 1)\,. \tag{4.1}$$

Proof. To prove the first inequality, we invoke the following exact sequence of the Chow groups (cf. [F], Example 2.6.2):

$$A_k(D_r) \xrightarrow{\cdot\, h} A_{k-1}(D_r) \to A_k(C_{D_r}) \to 0\,, \tag{4.2}$$

where C_{D_r} is the affine cone over D_r. We recall the following result from [P], Proposition 4.2 (recall that we assume $m \geq n$):

$$\text{rank } A_*(C_{D_r}) = \binom{n}{r}\,. \tag{4.3}$$

The equality (4.3), combined with the surjection in the sequence (4.2), implies the first inequality in (4.1).

To prove the second inequality in (4.1), we show that the elements $s_I(R) \cdot h^j$, where $I \subset (n-r)^r$ and $j = 0, \ldots, m-r$, generate over $\mathbb{Q}$ the Chow group $A^k(D_r \setminus D_{r-1})$, where $k = |I| + j$. It follows from Schubert calculus (cf., e.g., [F], Chap.14) and the surjection in (3.1) that the group $A_*(D_r \setminus D_{r-1})$

is generated by $s_I(Q)$, $I \subset (r)^{m-r}$; $s_J(R)$, $J \subset (n-r)^r$; and powers of the class h. By Lemma 3.2, in $A_*(D_r \setminus D_{r-1})$ we have

$$s_I(Q) = s_I(R \otimes L)\,,$$

and we see that the group $A_*(D_r \setminus D_{r-1})$ is generated by $s_J(R)$ (with $J \subset (n-r)^r$) and powers of the class h.

If $I \not\subseteq (n-r)^r$, then $s_I(Q) = 0$. Thus, invoking the Lascoux formula for the Schur polynomial of the twisted vector bundle (cf., e.g., [F], Ex. A.9.1), we get for such I:

$$0 = \sum_{J \subset I,\ J \subset (n-r)^r} d_{IJ} \cdot s_J(R) \cdot h^{|I|-|J|}\,. \tag{4.4}$$

These relations allow us to express the powers $h^{m-r+1}, h^{m-r+2}, \ldots$ with the help of h^j, $0 \le j \le m-r$, and $s_I(R)$, $I \subset (n-r)^r$. □

Example 4.1. Let $r = 1$. By the proof of Proposition 4.1, we know that $s_i(R) \cdot h^j$, where $i = 0, 1, \ldots, n-1$ and $j = 0, 1, \ldots, m-1$, generate over $\mathbb{Q}$ the Chow group $A_*(D_1)$. But by the Segre embedding, we have $D_1 \cong \mathbb{P}(W) \times \mathbb{P}(V)$. Since rank $A_*(\mathbb{P}(W) \times \mathbb{P}(V)) = mn$, the displayed elements are, in fact, $\mathbb{Z}$-linearly independent generators of $A_*(D_1)$. This can be also seen from the relations given in the proof of Proposition 4.1.

Example 4.2. Let now $r = n-1$.[1] In this case, C is a line bundle and $G_r(V) \cong \mathbb{P}^{n-1}$. Set $c = c_1(C)$. From the long exact sequence of bundles relating K and $C(h)$ we get, for $a \ge m-n+2$,

$$\binom{m}{m-a} \cdot h^a - \binom{m}{m-a+1} \cdot h^{a-1} \cdot c = 0\,. \tag{4.5}$$

With the help of (4.5), we deduce that the elements:

$$h^i \cdot c^j \quad (0 \le i \le m-n,\ 0 \le j \le n-1), \qquad h^{m-n+1}, \qquad h^{m-n+1} \cdot c$$

are $\mathbb{Q}$-linearly independent generators of $A_*(D_{n-1} \setminus D_{n-2})$.

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[1] This computation was done in collaboration with S.A. Strømme.